CORRECTION TO: CONVERGENCE OF DZIUK'S SEMIDISCRETE FINITE ELEMENT METHOD FOR MEAN CURVATURE FLOW OF CLOSED SURFACES WITH HIGH-ORDER FINITE ELEMENTS*

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Abstract. This erratum corrects Lemma 3.3, Proposition 4.1 and Proposition 4.2 of [B. Li, SIAM J. Numer. Anal., 59 (2021), pp. 1592–1617]

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1. Introduction.

In [3, p. 1611], the author used the following formula:

$$\underline{D}_i\underline{D}_j u = \underline{D}_j\underline{D}_i u + Hn_i\underline{D}_j u - Hn_j\underline{D}_i u, \tag{1.1}$$

which is not correct and should be replaced by the following one (see [1, Lemma 2.6]):

$$\underline{D}_i\underline{D}_j u = \underline{D}_j\underline{D}_i u + n_iH_{jl}\underline{D}_l u - n_jH_{il}\underline{D}_l u. \tag{1.2}$$

However, with the formula in (1.2), the cancellations in Lemma 3,3, Proposition 4.1 and Proposition 4.2 do not hold, i.e.

$$\int_{\Gamma_h^\theta}[\mathrm{tr}(\nabla_{\Gamma_h^\theta}e_h^\theta)^2 - \mathrm{tr}(\nabla_{\Gamma_h^\theta}e_h^\theta\nabla_{\Gamma_h^\theta}e_h^\theta)] \neq 0, \tag{1.3}$$

$$\int_{\Gamma_h^*}[\mathrm{tr}(\nabla_{\Gamma_h^*}e_h^*)^2 - \mathrm{tr}(\nabla_{\Gamma_h^*}e_h^*\nabla_{\Gamma_h^*}e_h^*)] \neq 0. \tag{1.4}$$

According to these changes, the error equation now should be replaced by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}\|_{\mathbf{M}(\mathbf{x})}^2 + 2\int_0^1\int_{\Gamma_h^\theta}|(\nabla_{\Gamma_h^\theta}e_h^\theta)\widehat{n}_h^\theta|^2\mathrm{d}\theta \\
&\lesssim ch^{2k-2} + c\epsilon^{-1}\|\mathbf{e}\|_{\mathbf{M}(\mathbf{x})}^2 + 2\epsilon\int_0^1\int_{\Gamma_h^\theta}|(\nabla_{\Gamma_h^\theta}e_h^\theta)\widehat{n}_h^\theta|^2\mathrm{d}\theta \\
&\quad + \Big|\int_0^1\int_{\Gamma_h^\theta}[\mathrm{tr}(\nabla_{\Gamma_h^\theta}e_h^\theta)^2 - \mathrm{tr}(\nabla_{\Gamma_h^\theta}e_h^\theta\nabla_{\Gamma_h^\theta}e_h^\theta)]\mathrm{d}\theta\Big|,
\end{aligned} \tag{1.5}$$

where the last term is missing in [3, Eq. (3.42)].

In the next section, we show that the last term in (1.5) differs from zero by a harmless lower-order correction term and an error term which can be absorbed into the left hand side of the error equation (1.5). This additional remainder will not influence the stability of the error equation. Therefore, the coercivity of $(\mathbf{A}(\mathbf{x})\mathbf{x} - \mathbf{A}(\mathbf{x}^*)\mathbf{x}^*)\cdot(\mathbf{x}-\mathbf{x}^*)$ can still be proved, leading to the correctness of the main theorem (i.e., [3, Theorem 2.1]) again.

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2. Correction. By using the commutator formula (1.2), integration by parts and the Einstein summation convention, we derive as follows on a closed smooth surface Γ for any smooth function $u \in C^\infty(\Gamma)$:

$$\begin{aligned}
\int_\Gamma \underline{D}_i u_j \underline{D}_j u_i &= -\int_\Gamma u_j \underline{D}_i \underline{D}_j u_i + \int_\Gamma u_j \underline{D}_j u_i H n_i \\
&= -\int_\Gamma u_j (\underline{D}_j \underline{D}_i u_i + n_i H_{jk} \underline{D}_k u_i - n_j H_{ik} \underline{D}_k u_i) + \int_\Gamma u_j \underline{D}_j u_i H n_i \\
&= \int_\Gamma \underline{D}_j u_j \underline{D}_i u_i - \int_\Gamma u_j \underline{D}_i u_i H n_j + \int_\Gamma u_j \underline{D}_j u_i H n_i \\
&\quad - \int_\Gamma u_j n_i H_{jk} \underline{D}_k u_i + \int_\Gamma u_j n_j H_{ik} \underline{D}_k u_i. \qquad (2.1)
\end{aligned}$$

By a density argument, (2.1) holds for $u \in H^1(\Gamma)$ as well. Then, by using the basic geometric perturbation estimates, we have

$$\begin{aligned}
&\Big| \int_{\Gamma_h^*} [\mathrm{tr}(\nabla_{\Gamma_h^*} e_h^*)^2 - \mathrm{tr}(\nabla_{\Gamma_h^*} e_h^* \nabla_{\Gamma_h^*} e_h^*)] \Big| \\
&= \Big| -\int_{\Gamma_h^*} \underline{D}_i e_{h,j}^* \underline{D}_j e_{h,i}^* + \int_{\Gamma_h^*} \underline{D}_j e_{h,j}^* \underline{D}_i e_{h,i}^* \Big| \\
&\le \Big| -\int_{\Gamma_h^*} \underline{D}_i e_{h,j}^* \underline{D}_j e_{h,i}^* + \int_\Gamma \underline{D}_i e_{h,j}^{*,l} \underline{D}_j e_{h,i}^{*,l} + \int_{\Gamma_h^*} \underline{D}_j e_{h,j}^* \underline{D}_i e_{h,i}^* - \int_\Gamma \underline{D}_j e_{h,j}^{*,l} \underline{D}_i e_{h,i}^{*,l} \Big| \\
&\quad + \Big| \int_\Gamma \underline{D}_i e_{h,j}^{*,l} \underline{D}_j e_{h,i}^{*,l} - \int_\Gamma \underline{D}_j e_{h,j}^{*,l} \underline{D}_i e_{h,i}^{*,l} \Big| \\
&\lesssim h^k \|e_h^*\|_{H^1(\Gamma_h^*)}^2 + \Big| -\int_\Gamma e_{h,j}^{*,l} \underline{D}_i e_{h,i}^{*,l} H n_j + \int_\Gamma e_{h,j}^{*,l} \underline{D}_j e_{h,i}^{*,l} H n_i \\
&\quad - \int_\Gamma e_{h,j}^{*,l} n_i H_{jk} \underline{D}_k e_{h,i}^{*,l} + \int_\Gamma e_{h,j}^{*,l} n_j H_{ik} \underline{D}_k e_{h,i}^{*,l} \Big| \quad \text{((2.1) is used here)} \\
&\lesssim h^{k-2} \|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \Big| \int_\Gamma \underline{D}_i e_{h,j}^{*,l} n_j H e_{h,i}^{*,l} + \int_\Gamma e_{h,j}^{*,l} H \underline{D}_j e_{h,i}^{*,l} n_i \\
&\qquad \text{(inverse inequality is used)} \\
&\quad - \int_\Gamma e_{h,j}^{*,l} H_{jk} \underline{D}_k e_{h,i}^{*,l} n_i - \int_\Gamma \underline{D}_k e_{h,j}^{*,l} n_j H_{ik} e_{h,i}^{*,l} \Big| \qquad \text{(integration by parts is used)},
\end{aligned}$$

where the lower-order terms arising from the integration by parts is used are estimated by $\|e_h^*\|_{L^2(\Gamma_h^*)}^2$. Then, by changing n to $\hat{n}_h^{*,l}$ (normal vector of the interpolation surface Γ_h^*, lifted onto Γ) in the inequality above and using the geometric approximation relation $|\hat{n}_h^{*,l} - n| \lesssim h^k$ (as well as the inverse inequality) again, we further derive that

$$\begin{aligned}
&\Big| \int_{\Gamma_h^*} [\mathrm{tr}(\nabla_{\Gamma_h^*} e_h^*)^2 - \mathrm{tr}(\nabla_{\Gamma_h^*} e_h^* \nabla_{\Gamma_h^*} e_h^*)] \Big| \\
&\lesssim (1 + \epsilon^{-1}) \|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \epsilon \|(\nabla_{\Gamma_h^*} e_h^*) \widehat{n}_h^*\|_{L^2(\Gamma_h^*)}^2. \qquad (2.2)
\end{aligned}$$

The induction hypothesis in [3, Eq. (3.2)] reads

$$\|e_h^*\|_{W^{1,\infty}(\Gamma_h^*)} \lesssim h^2 \qquad (2.3)$$

and it holds that

$$
\begin{aligned}
&\Big|\int_0^1 \|(\nabla_{\Gamma_h^\theta} e_h^\theta)\widehat{n}_h^\theta\|_{L^2(\Gamma_h^\theta)}^2 \mathrm{d}\theta - \|(\nabla_{\Gamma_h^*} e_h^*)\widehat{n}_h^*\|_{L^2(\Gamma_h^*)}^2\Big| \\
&= \Big|\int_0^1\int_0^\theta \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\Gamma_h^\alpha} (\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \cdot (\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \mathrm{d}\alpha\mathrm{d}\theta\Big| \\
&\lesssim \int_0^1\int_0^\theta \|\nabla_{\Gamma_h^\alpha} e_h^\alpha\|_{L^\infty(\Gamma_h^\alpha)}\|\nabla_{\Gamma_h^\alpha} e_h^\alpha\|_{L^2(\Gamma_h^\alpha)}^2 \mathrm{d}\alpha\mathrm{d}\theta \\
&\lesssim \|\nabla_{\Gamma_h^*} e_h^*\|_{L^\infty(\Gamma_h^*)}\|\nabla_{\Gamma_h^*} e_h^*\|_{L^2(\Gamma_h^*)}^2 \\
&\lesssim \|e_h^*\|_{L^2(\Gamma_h^*)}^2, \qquad (2.4)
\end{aligned}
$$

where in the second inequality we have used the evolution equation of $\widehat{n}_h^\alpha$ on every curved triangle of the triangulated surface Γ_h^α (c.f. [2, p. 33]), i.e.

$$\partial_\alpha^\bullet \widehat{n}_{h,i}^\alpha = -\underline{D}_i e_{h,j}^\alpha \widehat{n}_{h,j}^\alpha. \qquad (2.5)$$

Thus, analogous to (2.2), we have the following estimate for the last term of (1.5), after integrating w.r.t. θ, with the help of fundamental theorem of calculus:

$$
\begin{aligned}
&\Big|\int_0^1\int_{\Gamma_h^\theta} [\mathrm{tr}(\nabla_{\Gamma_h^\theta} e_h^\theta)^2 - \mathrm{tr}(\nabla_{\Gamma_h^\theta} e_h^\theta \nabla_{\Gamma_h^\theta} e_h^\theta)]\mathrm{d}\theta\Big| \\
&\le \Big|\int_0^1\int_{\Gamma_h^*} [\mathrm{tr}(\nabla_{\Gamma_h^*} e_h^*)^2 - \mathrm{tr}(\nabla_{\Gamma_h^*} e_h^* \nabla_{\Gamma_h^*} e_h^*)]\mathrm{d}\theta\Big| \\
&\quad + \Big|\int_0^1\int_0^\theta \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\Gamma_h^\alpha} [\mathrm{tr}(\nabla_{\Gamma_h^\theta} e_h^\alpha)^2 - \mathrm{tr}(\nabla_{\Gamma_h^\alpha} e_h^\alpha \nabla_{\Gamma_h^\alpha} e_h^\alpha)]\mathrm{d}\alpha\mathrm{d}\theta\Big| \\
&\lesssim (1+\epsilon^{-1})\|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \epsilon\|(\nabla_{\Gamma_h^*} e_h^*)\widehat{n}_h^*\|_{L^2(\Gamma_h^*)}^2 + \|\nabla_{\Gamma_h^*} e_h^*\|_{L^\infty(\Gamma_h^*)}\|\nabla_{\Gamma_h^*} e_h^*\|_{L^2(\Gamma_h^*)}^2 \\
&\qquad\qquad\qquad \text{((2.2) is used here)} \\
&\lesssim (1+\epsilon^{-1})\|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \epsilon\int_0^1 \|(\nabla_{\Gamma_h^\theta} e_h^\theta)\widehat{n}_h^\theta\|_{L^2(\Gamma_h^\theta)}^2 \mathrm{d}\theta, \qquad (2.6)
\end{aligned}
$$

where we have used induction hypothesis (2.3) and (2.4) in deriving the last inequality. The second term on the right hand side of (2.6) can be absorbed by the left hand side of (1.5) by choosing a sufficiently small ϵ, and then the first term can be controlled by using Gronwall's inequality. Then the stability of the error equation [3, Eq. (3.42)-(3.43)] follows and therefore [3, Theorem 2.1] still holds.

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