Journal of Scientific Computing (2021) 88:70

Second-Order Convergence of the Linearly Extrapolated Crank–Nicolson Method for the Navier–Stokes Equations with H^1 Initial Data

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Abstract
This article concerns the numerical approximation of the two-dimensional nonstationary Navier–Stokes equations with H^1 initial data. By utilizing special locally refined temporal stepsizes, we prove that the linearly extrapolated Crank–Nicolson scheme, with the usual stabilized Taylor–Hood finite element method in space, can achieve second-order convergence in time and space. Numerical examples are provided to support the theoretical analysis.

Keywords Navier–Stokes equations · Linearly extrapolated Crank–Nicolson method · Locally refined stepsizes · Nonsmooth initial data · Error estimate

Mathematics Subject Classification 65M12 · 65M15 · 65M60 · 76D05

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with boundary $\partial\Omega$. We consider the time-dependent Navier–Stokes (NS) equations describing the dynamics of an incompressible, homogeneous, viscous fluid in the domain Ω up to a given time $T > 0$, i.e.,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = 0 & \text{in } \Omega \times (0, T], \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \partial\Omega \times [0, T], \\ u = u^0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

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where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ and $p = p(x,t)$ denote the fluid velocity and pressure, respectively, and $u^0 = u^0(x)$ is a given initial value of the fluid velocity.

As the fundamental mathematical equations to understand and predict the dynamics of incompressible fluid flow, the numerical solution of the NS equations has attracted much attention in the community of scientific computing and numerical analysis. In particular, if the solution of the NS equations is sufficiently smooth (with enough compatibility conditions), then optimal-order convergence of high-order numerical methods can be proved; see [4,6, 18,19,26,27].

For H^2 initial data, i.e., $u^0 \in H_0^1(\Omega)^2 \cap H^2(\Omega)^2$ and $\nabla \cdot u^0 = 0$ without additional compatibility conditions, Heywood and Rannacher [13–15] considered both semidiscrete and fully discrete finite element methods for the NS equations and proved second-order convergence in time for the implicit Crank–Nicolson scheme. Shen [20,21] proved optimal-order convergence of the first-order and second-order projection methods for decoupling velocity and pressure. He and Sun [12] proved second-order convergence of the Crank–Nicolson/Adams–Bashforth implicit-explicit scheme. Emmrich [5] proved second-order convergence of the two-step backward differentiation formula. Guo and He [8] proved second-order convergence of the linearly extrapolated Crank–Nicolson scheme. Tang and Huang [23] proved second-order convergence of the Crank–Nicolson leap-frog scheme. For the Crank–Nicolson methods mentioned above, the convergence of pressure was proved with sub-optimal order. Recently, Sonner and Richter [22] proved second-order convergence of pressure for the Crank–Nicolson method.

For H^1 initial data, i.e., $u^0 \in H_0^1(\Omega)^2$ and $\nabla \cdot u^0 = 0$ without additional compatibility conditions, only a few results were provided in the literature. As far as we know, Hill and Süli [16] proved second-order convergence of the semidiscrete finite element method. He derived first-order convergence of the Euler implicit/explicit scheme in [9] and 1.5th-order convergence of the Crank–Nicolson/Adams–Bashforth implicit-explicit scheme in [10].

The objective of this paper is to prove that, for H^1 initial data without additional compatibility conditions, the linearly extrapolated Crank–Nicolson scheme has second-order convergence by utilizing a class of locally refined stepsizes, with the semi-implicit Euler scheme at the first two time levels. The total computational cost would be equivalent to using a uniform stepsize. The proof is based on two technical lemmas (Lemma 3.2 and 3.3) established in Sect. 3.1 and the consistency error estimate presented in Sect. 3.2. For simplicity, we focus on the homogeneous NS equations (1.1) (i.e., the right-hand side is zero in the velocity equation) with a normalised viscosity. All the results can be carried over to the general case if we assume appropriate smoothness of f.

2 Preliminary Results for the Semidiscrete Finite Element Method

2.1 Functional Setting of the NS Equations

For $s \geq 0$ and $1 \leq p \leq \infty$, we denote by $W^{s,p}(\Omega)$ the conventional Sobolev space of functions on Ω, with abbreviations $H^s(\Omega) = W^{s,2}(\Omega)$, $L^2(\Omega) = H^0(\Omega)$ and $L^p(\Omega) = W^{0,p}(\Omega)$. As usual, we denote by $H_0^1(\Omega)$ the space of functions in $H^1(\Omega)$ with zero trace on the boundary $\partial\Omega$. For simplicity, the norms on the spaces $H^s(\Omega)$, $H^s(\Omega)^m$ and $H^s(\Omega)^{m\times m}$, with any integer $m \geq 1$, are all denoted by $\|\cdot\|_{H^s(\Omega)}$.

We introduce the following Hilbert spaces associated with the NS equations:

$$X = H_0^1(\Omega)^2,$$

$$Y = \{v \in L^2(\Omega)^2;\ \nabla \cdot v = 0,\ v \cdot n|_{\partial\Omega} = 0\},$$
$$M = L_0^2(\Omega) = \{q \in L^2(\Omega);\ \int_\Omega q \mathrm{d}x = 0\}.$$

Let $\mathring{X}$ be the divergence-free subspace of X, defined by

$$\mathring{X} = \{v \in X;\ \nabla \cdot v = 0\}.$$

In a convex polygon Ω, it is known that the steady-state Stokes equations

$$\begin{cases} -\Delta v + \nabla q = g & \text{in } \ \Omega, \\ \nabla \cdot v = 0 & \text{in } \ \Omega, \\ v = 0 & \text{on } \ \partial\Omega, \end{cases}$$

with $g \in L^2(\Omega)^2$, have a unique solution $(v, q) \in \big(\mathring{X} \cap H^2(\Omega)^2\big) \times H^1(\Omega)/\mathbb{R}$ satisfying the following estimate:

$$\|v\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)/\mathbb{R}} \leq c_1 \|g\|_{L^2(\Omega)}, \tag{2.1}$$

where $c_1 > 0$ is some positive constant depending on Ω. This result can be found in [17, Theorem 2] and [24, p. 33, Proposition 2.2].

Let $D(A) = \mathring{X} \cap H^2(\Omega)^2 \subset Y$ and define the Stokes operator

$$A = -P\Delta : D(A) \to Y,$$

where P is the L^2-orthogonal projection of $L^2(\Omega)^2$ onto Y. As a result of (2.1), the following inequalities hold; see [1,13]:

$$\|v\|_{L^2(\Omega)} \leq c_2 \|\nabla v\|_{L^2(\Omega)} \quad v \in X,$$
$$\|v\|_{H^2(\Omega)} \leq c_2 \|Av\|_{L^2(\Omega)} \quad v \in D(A),$$

where c_2 is some positive constant depending on Ω.

We recall the following result concerning the existence and uniqueness of a global strong solution to the Navier–Stokes problem (1.1) (cf. [16, Theorem 2.1]).

Theorem 2.1 *For any given $u^0 \in \mathring{X}$ there exists a unique solution to (1.1) such that*

$$u \in H^1(0, T; L^2(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2) \cap C([0, T]; \mathring{X}),$$
$$p \in L^2(0, T; H^1(\Omega)/\mathbb{R}).$$

The initial condition is satisfied in the sense that

$$\|u(\cdot, t) - u^0\|_{H^1(\Omega)} \to 0 \quad as \quad t \to 0.$$

We define a trilinear form on $X \times X \times X$ by

$$b(u, v, w) = \big((u \cdot \nabla)v, w\big) + \frac{1}{2}\big((\nabla \cdot u)v, w\big)$$
$$= \frac{1}{2}\big((u \cdot \nabla)v, w\big) - \frac{1}{2}\big((u \cdot \nabla)w, v\big) \quad \text{for } \ u, v, w \in X.$$

Then the solution of problem (1.1), as stated in Theorem 2.1, satisfies the following equations for all $(v, q) \in X \times M$ and $t \in (0, T]$:

$$\begin{cases} (\partial_t u, v) + b(u, u, v) + (\nabla u, \nabla v) - (p, \nabla \cdot v) = 0, \\ (\nabla \cdot u, q) = 0. \end{cases} \tag{2.2}$$

2.2 Semidiscrete Finite Element Approximation

Let $X_h \times M_h$ be a finite element subspace of $X \times M$ subject to a triangulation of Ω with mesh size $h > 0$, with the following three properties.

(1) Inverse inequality: there exists a constant $c_3 > 0$ (independent of h) such that

$$\|v_h\|_{W^{m,q}(\Omega)} \le c_3 h^{-(m-l)-(\frac{2}{p}-\frac{2}{q})}\|v_h\|_{W^{l,p}(\Omega)} \quad \forall v_h \in X_h, \tag{2.3}$$

for $0 \le l \le m \le 1$ and $1 \le p \le q \le \infty$.

(2) Inf-sup condition: there exists a constant $c_4 > 0$ (independent of h) such that

$$\|q_h\|_{L^2(\Omega)} \le c_4 \sup_{v_h \in X_h \backslash \{0\}} \frac{(\nabla \cdot v_h, q_h)}{\|\nabla v_h\|_{L^2(\Omega)}} \quad \forall q_h \in M_h. \tag{2.4}$$

(3) Fortin projection: there exists a linear projection $\Pi_h : H_0^1(\Omega)^2 \to X_h$ such that for $v \in H_0^1(\Omega)^2 \cap H^2(\Omega)^2$

$$\begin{aligned} &\|v - \Pi_h v\|_{H^m(\Omega)} \le c_5 h^{s-m}\|v\|_{H^s(\Omega)} && 0 \le m \le 1,\ \ 1 \le s \le 2, \\ &\|\Pi_h v\|_{W^{1,p}(\Omega)} \le c_5\|v\|_{W^{1,p}(\Omega)} && 1 \le p < \infty, \end{aligned} \tag{2.5}$$

where $c_5 > 0$ is a constant independent of h.

For example, the Taylor–Hood P2-P1 element space [7,25] has all these properties.

For the simplicity of notation, in the rest of this paper, we denote by c a generic positive constant that is independent of h.

Let $\mathring{X}_h$ be the discrete divergence-free subspace of X_h, defined by

$$\mathring{X}_h := \{v_h \in X_h; (\nabla \cdot v_h, q_h) = 0 \quad \forall q_h \in M_h\}.$$

Let $P_h : L^2(\Omega)^2 \to \mathring{X}_h$ be the L^2-orthogonal projection defined by

$$(P_h v, v_h) = (v, v_h) \quad \forall v_h \in \mathring{X}_h.$$

Equivalently, $P_h v$ can be found by solving the following coupled equations:

$$\begin{cases} (P_h v, v_h) - (\eta_h, \nabla \cdot v_h) = (v, v_h) & \forall v_h \in X_h, \\ (\nabla \cdot P_h v, q_h) = 0 & \forall q_h \in M_h. \end{cases}$$

Then the following inequalities are consequences of properties (2.3)–(2.5); see [3]:

$$\|\nabla P_h v\|_{L^2(\Omega)} \le c\|\nabla v\|_{L^2(\Omega)} \quad \forall v \in \mathring{X}, \tag{2.6}$$

$$\|v - P_h v\|_{L^2(\Omega)} + h\|\nabla(v - P_h v)\|_{L^2(\Omega)} \le ch^2\|v\|_{H^2(\Omega)} \quad \forall v \in \mathring{X} \cap H^2(\Omega)^2. \tag{2.7}$$

The semidiscrete finite element method for (2.2) reads: Find $\big(u_h(t), p_h(t)\big) \in X_h \times M_h$ such that

$$\begin{cases} (\partial_t u_h, v_h) + b(u_h, u_h, v_h) + (\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (\nabla \cdot u_h, q_h) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u_h(0) = P_h u^0, \end{cases} \tag{2.8}$$

holds for all $(v_h, q_h) \in X_h \times M_h$ and $t \in (0, T]$.

It is known that the semidiscrete finite element solution $u_h(t)$ satisfies the following regularity estimates; see [10].

Lemma 2.2 *(Regularity of semidiscrete finite element solution) Let $u^0 \in H_0^1(\Omega)^2$ and $\nabla \cdot u^0 = 0$, and assume that the finite element space $X_h \times M_h$ has properties (2.3)–(2.5). Then the semidiscrete finite element solution $u_h(t)$ determined by (2.8) satisfies the following regularity estimates:*

$$\|\partial_t^m u_h(t)\|_{H^1(\Omega)} \le Ct^{-m} \quad \forall t \in (0, T], \quad m = 1, 2, \tag{2.9}$$

$$\|u_h(t)\|_{L^2(\Omega)} + \|\nabla u_h(t)\|_{L^2(\Omega)} + t^{\frac{1}{2}}\|A_h u_h(t)\|_{L^2(\Omega)} \le C \quad \forall t \in (0, T], \tag{2.10}$$

where C is a general positive constant depending on $\|u^0\|_{H^1(\Omega)}$, Ω and T.

3 The Linearly Extrapolated Crank–Nicolson Scheme

In this section, we present the error estimate for the fully discrete finite element method with the linearly extrapolated Crank–Nicolson scheme in time. We consider a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $[0, T]$ with the following stepsizes:

$$\begin{aligned} \tau_1 &= \tau_2 = T\left(\frac{\tau}{T}\right)^{\frac{1}{1-\alpha}}, \\ \tau_n &= t_n - t_{n-1} \sim \left(\frac{t_{n-1}}{T}\right)^{\alpha}\tau \quad \text{for } n \ge 3, \end{aligned} \tag{3.1}$$

where τ is the maximal stepsize and $\frac{3}{4} < \alpha < 1$ is any fixed number.

Remark 3.1 The computational cost of using the stepsizes in (3.1) and using the uniform stepsize τ is equivalent. For example, for the stepsize choice $\tau_n = \left(\frac{t_{n-1}}{T}\right)^{\alpha}\tau$ we can estimate the number of total time levels as follows. We divide the time interval $[t_1, T]$ into dyadic subintervals $[2^{-j-1}T, 2^{-j}T]$, with $j = 0, 1, \ldots, J$, where J is the smallest integer satisfying $2^{-J}T \le t_1$. Since $t_1 = \tau_1 = T\left(\frac{\tau}{T}\right)^{\frac{1}{1-\alpha}}$, it follows that $J \le 1 + \frac{1}{(1-\alpha)\ln 2}\ln\left(\frac{T}{\tau}\right)$. Any time interval $[t_{n-1}, t_n] \subset [2^{-j-1}T, 2^{-j}T]$ would satisfy

$$\tau_n = \left(\frac{t_{n-1}}{T}\right)^{\alpha}\tau \ge 2^{-(j+1)\alpha}\tau.$$

Hence, the number of time levels in $[2^{-j-1}T, 2^{-j}T]$ is bounded by

$$N_j \le \frac{2^{-(j+1)}T}{2^{-(j+1)\alpha}\tau} = 2^{-(j+1)(1-\alpha)}\frac{T}{\tau}.$$

As a result, the number of total time levels in $[0, T]$ is bounded by

$$N \le \sum_{j=0}^{J} N_j \le \sum_{j=0}^{J} 2^{-(j+1)(1-\alpha)}\frac{T}{\tau} \le \frac{1}{2^{1-\alpha}-1}\frac{T}{\tau} \qquad \text{for } \alpha \in (0, 1).$$

Therefore, for any fixed $\alpha \in (0, 1)$, the number of total time levels is bounded by a constant multiple of T/τ. The number of total time levels is increasing as α increases and blows up as $\alpha \to 1$. But in practical computation we only need to choose a fixed $\alpha \in (0, 1)$ for a given problem. For example, in the numerical solution of the NS equations we only need to choose a fixed constant $\alpha \in (\frac{3}{4}, 1)$; see Theorem 3.1.

For any sequence of functions u_h^n, $n = 0, 1, \ldots, N$, we adopt the conventional notations:

$$\delta_\tau u_h^n := \frac{u_h^n - u_h^{n-1}}{\tau_n}, \qquad \overline{u}_h^{n-\frac{1}{2}} := \frac{u_h^n + u_h^{n-1}}{2} \qquad n \geq 1,$$

$$\widehat{u}_h^{n-\frac{1}{2}} := \Big(1 + \frac{r_n}{2}\Big)u_h^{n-1} - \frac{r_n}{2}u_h^{n-2} \quad \text{with } r_n = \frac{\tau_n}{\tau_{n-1}} \qquad n \geq 2.$$

The stepsizes in (3.1) guarantee that $r_n \leq c$ for some positive constant c.

Let $u_h^0 = P_h u^0 \in \mathring{X}_h$. For $(u_h^n, p_h^n) \in X_h \times M_h$, $n = 1, 2$, we compute the numerical solutions by the semi-implicit Euler method:

$$\begin{cases} (\delta_\tau u_h^n, v_h) + b(u_h^{n-1}, u_h^n, v_h) + (\nabla u_h^n, \nabla v_h) - (p_h^n, \nabla \cdot v_h) = 0 & \forall\, v_h \in X_h, \\ (\nabla \cdot u_h^n, q_h) = 0 & \forall\, q_h \in M_h. \end{cases} \tag{3.2}$$

For $n \geq 3$ and given functions

$$(u_h^{n-2}, p_h^{n-2}), (u_h^{n-1}, p_h^{n-1}) \in \mathring{X}_h \times M_h,$$

we consider the following linearly extrapolated Crank–Nicolson method: Find $(u_h^n, p_h^n) \in X_h \times M_h$ such that

$$\begin{cases} (\delta_\tau u_h^n, v_h) + b(\widehat{u}_h^{n-\frac{1}{2}}, \overline{u}_h^{n-\frac{1}{2}}, v_h) + (\nabla \overline{u}_h^{n-\frac{1}{2}}, \nabla v_h) - (p_h^{n-\frac{1}{2}}, \nabla \cdot v_h) = 0 & \forall\, v_h \in X_h, \\ (\nabla \cdot \overline{u}_h^{n-\frac{1}{2}}, q_h) = 0 & \forall\, q_h \in M_h. \end{cases} \tag{3.3}$$

The main result of this paper is presented in the following theorem.

Theorem 3.1 *Let $u^0 \in H_0^1(\Omega)^2$ and $\nabla \cdot u^0 = 0$, and assume that the finite element space has properties* (2.3)–(2.5) *(such as the Taylor–Hood element space). If the temporal stepsizes are chosen from* (3.1) *with some fixed α satisfying $3/4 < \alpha < 1$, then the fully discrete finite element solution u_h^n given by* (3.2)–(3.3) *has the following error bound:*

$$\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq C\tau^2 + Ct_n^{-\frac{1}{2}}h^2, \tag{3.4}$$

where C is a general positive constant depending on $\|u^0\|_{H^1(\Omega)}$, Ω, T, c_3 and c_5.

The proof of Theorem 3.1 is presented in the following subsections.

Remark 3.2 The Taylor–Hood P2-P1 elements can achieve at most third-order convergence when the solution is sufficiently smooth, but only have lower-order convergence when the regularity of the solution is not enough. For example, in (2.5) we only consider the approximation of the Fortin projection for $v \in H_0^1(\Omega)^2 \cap H^2(\Omega)^2$. If $v \in H_0^1(\Omega)^2 \cap H^3(\Omega)^2$ then (2.5) can also hold for $s = 3$.

3.1 Some Technical Inequalities

In this subsection, we present two technical lemmas to be used in the error estimate for the linearly extrapolated Crank–Nicolson method.

In a convex polygon, it is known that the following interpolation inequalities hold (cf. [2, p. 139, Theorem 5.8 and 5.9]):

$$\|\nabla v\|_{L^4(\Omega)} \le c\|\nabla v\|^{\frac{1}{2}}_{L^2(\Omega)}\|\Delta v\|^{\frac{1}{2}}_{L^2(\Omega)} \qquad \forall v \in H^1_0(\Omega)^2 \cap H^2(\Omega)^2, \tag{3.5}$$

$$\|v\|_{L^\infty(\Omega)} \le c\|v\|^{\frac{1}{2}}_{L^2(\Omega)}\|v\|^{\frac{1}{2}}_{H^2(\Omega)} \qquad \forall v \in H^1_0(\Omega)^2 \cap H^2(\Omega)^2. \tag{3.6}$$

For the discrete Stokes operator $A_h = -P_h\Delta_h : X_h \to \mathring{X}_h$ defined by

$$(A_h v_h, w_h) = -(\Delta_h v_h, w_h) = (\nabla v_h, \nabla w_h) \quad \forall\, v_h \in X_h, \quad w_h \in \mathring{X}_h.$$

We shall need the following discrete analogues of (3.5)–(3.6).

Lemma 3.2 *(Discrete Sobolev interpolation inequalities)*

$$\|\nabla v_h\|_{L^4(\Omega)} \le c\|\nabla v_h\|^{\frac{1}{2}}_{L^2(\Omega)}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)} \qquad \forall v_h \in \mathring{X}_h, \tag{3.7}$$

$$\|v_h\|_{L^\infty(\Omega)} \le c\|v_h\|^{\frac{1}{2}}_{L^2(\Omega)}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)} \qquad \forall v_h \in \mathring{X}_h. \tag{3.8}$$

Proof To obtain a bound of $\|\nabla v_h\|_{L^4(\Omega)}$, we let $v \in D(A) = \mathring{X} \cap H^2(\Omega)^2$ be the solution of

$$Av = A_h v_h \quad v_h \in \mathring{X}_h, \tag{3.9}$$

where (3.9) is equivalent to the linear Stokes equations for $(v, q) \in X \times M$

$$\begin{cases} -\Delta v + \nabla q = A_h v_h & \text{in } \Omega, \\ \nabla \cdot v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

According to the estimate (2.1), we know that the solution $v \in D(A)$ satisfies that

$$\|v\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)} \le c\|A_h v_h\|_{L^2(\Omega)}. \tag{3.11}$$

Note that v_h is the solution of the following equations:

$$\begin{cases} (\nabla v_h, \nabla w_h) - (q_h, \nabla \cdot w_h) = (A_h v_h, w_h) & \forall\, w_h \in X_h, \\ (\nabla \cdot v_h, \eta_h) = 0 & \forall\, \eta_h \in M_h. \end{cases}$$

As a result, v_h is the Stokes–Ritz projection of v, i.e., there exists $q_h \in M_h$ such that

$$\begin{cases} (\nabla(v - v_h), \nabla w_h) - (q - q_h, \nabla \cdot w_h) = 0 & \forall\, w_h \in X_h, \\ (\nabla \cdot (v - v_h), \eta_h) = 0 & \forall\, \eta_h \in M_h. \end{cases}$$

It is known that the Stokes–Ritz projection satisfies the following estimate; see [25]:

$$\|v - v_h\|_{H^m(\Omega)} \le ch^{s-m}(\|v\|_{H^s(\Omega)} + \|q\|_{H^{s-1}(\Omega)}) \quad 0 \le m \le 1,\ 1 \le s \le 2. \tag{3.12}$$

In view of (2.5) and (3.12), we derive that

$$\|v_h - \Pi_h v\|_{H^m(\Omega)} \le ch^{s-m}(\|v\|_{H^s(\Omega)} + \|q\|_{H^{s-1}(\Omega)}) \quad 0 \le m \le 1,\ 1 \le s \le 2. \tag{3.13}$$

Inequality (3.5) and (3.11) imply that

$$\|\nabla v\|_{L^4(\Omega)} \le c\|\nabla v\|^{\frac{1}{2}}_{L^2(\Omega)}\|v\|^{\frac{1}{2}}_{H^2(\Omega)} \le c\|\nabla v\|^{\frac{1}{2}}_{L^2(\Omega)}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)},$$

and therefore

$$\begin{aligned}\|\nabla \Pi_h v\|_{L^4(\Omega)} &\le c\|\nabla v\|_{L^4(\Omega)} && \big((2.5) \text{ is used}\big)\\ &\le c\|\nabla v\|^{\frac{1}{2}}_{L^2(\Omega)}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)}.\end{aligned} \tag{3.14}$$

Since

$$\begin{aligned}&\|\nabla(v_h-\Pi_h v)\|_{L^4(\Omega)} \\ &\le c\|\nabla(v_h-\Pi_h v)\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla(v_h-\Pi_h v)\|^{\frac{1}{2}}_{L^\infty(\Omega)}\\ &\le c\|\nabla(v_h-\Pi_h v)\|^{\frac{1}{2}}_{L^2(\Omega)}h^{-\frac{1}{2}}\|\nabla(v_h-\Pi_h v)\|^{\frac{1}{2}}_{L^2(\Omega)}\\ &\le c\big(\|\nabla v_h\|_{L^2(\Omega)}+\|\nabla v\|_{L^2(\Omega)}\big)^{\frac{1}{2}}\big(\|v\|_{H^2(\Omega)}+\|q\|_{H^1(\Omega)}\big)^{\frac{1}{2}} \quad \big((2.5) \text{ and } (3.13) \text{ are used}\big)\\ &\le c\big(\|\nabla v_h\|_{L^2(\Omega)}+\|\nabla v\|_{L^2(\Omega)}\big)^{\frac{1}{2}}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)} \quad \big((3.11) \text{ is used}\big),\end{aligned} \tag{3.15}$$

combining (3.14) and (3.15) yields that

$$\begin{aligned}\|\nabla v_h\|_{L^4(\Omega)} &\le \|\nabla \Pi_h v\|_{L^4(\Omega)} + \|\nabla(v_h-\Pi_h v)\|_{L^4(\Omega)}\\ &\le c\big(\|\nabla v_h\|_{L^2(\Omega)}+\|\nabla v\|_{L^2(\Omega)}\big)^{\frac{1}{2}}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)}.\end{aligned} \tag{3.16}$$

It remains to prove the following inequality

$$\|\nabla v\|_{L^2(\Omega)} \le c\|\nabla v_h\|_{L^2(\Omega)}. \tag{3.17}$$

Then substituting (3.17) into (3.16) yields the desired inequality (3.7). In fact, testing equation (3.10) by $v \in D(A)$ gives

$$\begin{aligned}\|\nabla v\|^2_{L^2(\Omega)} &= (A_h v_h, v) + (q, \nabla\cdot v)\\ &= (A_h v_h, P_h v) = (\nabla v_h, \nabla P_h v)\\ &\le c\|\nabla v_h\|_{L^2(\Omega)}\|\nabla P_h v\|_{L^2(\Omega)}\\ &\le c\|\nabla v_h\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)},\end{aligned}$$

where we have used (2.6) in the last inequality. This proves the first inequality of Lemma 3.2.

To prove the second inequality of Lemma 3.2, we first test (3.10) by w and obtain

$$\begin{aligned}(q, \nabla\cdot w) &= (\nabla v, \nabla w) - (A_h v_h, P_h w)\\ &= (\nabla v, \nabla w) - (\nabla v_h, \nabla P_h w)\\ &\le c\big(\|\nabla v\|_{L^2(\Omega)}+\|\nabla v_h\|_{L^2(\Omega)}\big)\|w\|_{H^1(\Omega)}\\ &\le c\|\nabla v_h\|_{L^2(\Omega)}\|w\|_{H^1(\Omega)} \quad \forall w \in X,\end{aligned}$$

where we have used (3.17) in the last inequality. Through the inf-sup condition, we derive that

$$\|q\|_{L^2(\Omega)} \le c\|\nabla v_h\|_{L^2(\Omega)}. \tag{3.18}$$

On the one hand, by using the inverse inequality and (3.13), we have

$$\begin{aligned}
\|v_h - \Pi_h v\|_{L^\infty(\Omega)} &\le ch^{-1}\|v_h - \Pi_h v\|_{L^2(\Omega)} \\
&= ch^{-1}\|v_h - \Pi_h v\|_{L^2(\Omega)}^{\frac{1}{2}}\|v_h - \Pi_h v\|_{L^2(\Omega)}^{\frac{1}{2}} \\
&\le ch^{\frac{1}{2}}\left(\|v\|_{H^1(\Omega)} + \|q\|_{L^2(\Omega)}\right)^{\frac{1}{2}}\left(\|v\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)}\right)^{\frac{1}{2}} \qquad (3.19)\\
&\le ch^{\frac{1}{2}}\|v_h\|_{H^1(\Omega)}^{\frac{1}{2}}\|A_h v_h\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \big((3.17), (3.18) \text{ and } (3.11) \text{ are used}\big)\\
&\le c\|v_h\|_{L^2(\Omega)}^{\frac{1}{2}}\|A_h v_h\|_{L^2(\Omega)}^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, it follows from the fact

$$\begin{aligned}
\|v\|_{L^2(\Omega)} &\le \|v - v_h\|_{L^2(\Omega)} + \|v_h\|_{L^2(\Omega)} \\
&\le ch(\|v\|_{H^1(\Omega)} + \|q\|_{L^2(\Omega)}) + \|v_h\|_{L^2(\Omega)} \qquad \big((3.12) \text{ is used}\big)\\
&\le ch\|v_h\|_{H^1(\Omega)} + \|v_h\|_{L^2(\Omega)} \qquad \big((3.17) \text{ and } (3.18) \text{ are used}\big)\\
&\le c\|v_h\|_{L^2(\Omega)},
\end{aligned}$$

and therefore

$$\begin{aligned}
\|\Pi_h v\|_{L^\infty(\Omega)} &\le \|v\|_{L^\infty(\Omega)} \\
&\le c\|v\|_{L^2(\Omega)}^{\frac{1}{2}}\|v\|_{H^2(\Omega)}^{\frac{1}{2}} \qquad \big((3.6) \text{ is used}\big) \qquad (3.20)\\
&\le c\|v_h\|_{L^2(\Omega)}^{\frac{1}{2}}\|A_h v_h\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \big((3.11) \text{ is used}\big).
\end{aligned}$$

Using the triangle inequality and combining (3.19) and (3.20) yield that

$$\begin{aligned}
\|v_h\|_{L^\infty(\Omega)} &\le \|\Pi_h v\|_{L^\infty(\Omega)} + \|v_h - \Pi_h v\|_{L^\infty(\Omega)} \\
&\le c\|v_h\|_{L^2(\Omega)}^{\frac{1}{2}}\|A_h v_h\|_{L^2(\Omega)}^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof of this Lemma. □

By the definition of the trilinear form, it is easy to see that

$$b(u_h, v_h, v_h) = 0. \qquad (3.21)$$

For $u_h, v_h, w_h \in X_h$, it is known that (cf. [15, p. 360, eq. (3.7)])

$$|b(u_h, v_h, w_h)| \le c\|u_h\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)}\|w_h\|_{H^1(\Omega)}. \qquad (3.22)$$

By using the interpolation inequalities (3.7)–(3.8), we prove the following result.

Lemma 3.3 *For $u_h, v_h, w_h \in \mathring{X}_h$, there holds*

$$|b(u_h, v_h, w_h)| \le c\|u_h\|_{L^2(\Omega)}\|v_h\|_{H^1(\Omega)}^{\frac{1}{2}}\|A_h v_h\|_{L^2(\Omega)}^{\frac{1}{2}}\|w_h\|_{H^1(\Omega)}. \qquad (3.23)$$

Proof According to the definition of the trilinear form and Lemma 3.2, we derive that

$$
\begin{aligned}
&|b(u_h, v_h, w_h)| \\
&\le \frac{1}{2}\big|\big((u_h \cdot \nabla)v_h, w_h\big)\big| + \frac{1}{2}\big|\big((u_h \cdot \nabla)w_h, v_h\big)\big| \\
&\le c\|u_h\|_{L^2(\Omega)}\|\nabla v_h\|_{L^4(\Omega)}\|w_h\|_{L^4(\Omega)} + c\|u_h\|_{L^2(\Omega)}\|v_h\|_{L^\infty(\Omega)}\|\nabla w_h\|_{L^2(\Omega)} \\
&\le c\|u_h\|_{L^2(\Omega)}\big(\|\nabla v_h\|^{\frac{1}{2}}_{L^2(\Omega)}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)} + \|v_h\|^{\frac{1}{2}}_{L^2(\Omega)}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)}\big)\|w_h\|_{H^1(\Omega)} \\
&\le c\|u_h\|_{L^2(\Omega)}\|v_h\|^{\frac{1}{2}}_{H^1(\Omega)}\|A_h v_h\|^{\frac{1}{2}}_{L^2(\Omega)}\|w_h\|_{H^1(\Omega)}.
\end{aligned}
$$

This proves the desired result. □

In addition to the two lemmas above, we also need to use the discrete Gronwall inequality, which is stated in the following lemma; see [11].

Lemma 3.4 *Let B and a_n, b_n, d_n, τ_n be nonnegative numbers such that*

$$a_m + \sum_{n=n_0+1}^{m} b_n\tau_n \le \sum_{n=n_0}^{m-1} a_n d_n \tau_n + B \quad \text{for } m \ge n_0 \ge 1.$$

Then

$$a_m + \sum_{n=n_0+1}^{m} b_n\tau_n \le B \exp\left(\sum_{n=n_0}^{m-1} d_n\tau_n\right) \quad \text{for } m \ge n_0.$$

3.2 Consistency

Under the assumptions of Theorem 3.1, Hill and Süli [16] proved the following result for the semidiscrete finite element approximation:

$$\max_{t\in(0,T]} \|u(t) - u_h(t)\|_{L^2(\Omega)} \le Ct^{-1/2}h^2. \tag{3.24}$$

Hence, we only need to present the estimate for the temporal discretization error

$$e_h^n := u_h(t_n) - u_h^n \quad n \ge 1.$$

In this subsection, we consider the consistency error for the linearly extrapolated Crank–Nicolson scheme (3.2)–(3.3), by comparing the fully discrete scheme (3.2)–(3.3) with the semidiscrete scheme (2.8). Here and after, we use the following notations:

$$
\begin{aligned}
\delta_\tau u_h(t_n) &= \frac{u_h(t_n) - u_h(t_{n-1})}{\tau_n} && n \ge 1, \\
\overline{u}_h(t_{n-\frac{1}{2}}) &= \frac{u_h(t_n) + u_h(t_{n-1})}{2} && n \ge 1, \\
\widehat{u}_h(t_{n-\frac{1}{2}}) &= \left(1 + \frac{r_n}{2}\right)u_h(t_{n-1}) - \frac{r_n}{2}u_h(t_{n-2}) && n \ge 2.
\end{aligned}
$$

Then the semidiscrete solution $u_h(t_n)$ given by (2.8) satisfies the following system for $n = 1, 2$:

$$\begin{cases} (\delta_\tau u_h(t_n), v_h) + b(u_h(t_{n-1}), u_h(t_n), v_h) + (\nabla u_h(t_n), \nabla v_h) \\ \qquad\qquad -(p_h(t_n), \nabla \cdot v_h) + (\varepsilon^n, v_h) = 0 \quad \forall v_h \in X_h, \\ \qquad\qquad (\nabla \cdot u_h(t_n), q_h) = 0 \quad \forall q_h \in M_h, \end{cases} \tag{3.25}$$

and the following system for $n \geq 3$:

$$\begin{cases} (\delta_\tau u_h(t_n), v_h) + b(\widehat{u}_h(t_{n-\frac{1}{2}}), \overline{u}_h(t_{n-\frac{1}{2}}), v_h) + \big(\nabla \overline{u}_h(t_{n-\frac{1}{2}}), \nabla v_h\big) \\ \qquad\qquad -(p_h(t_{n-\frac{1}{2}}), \nabla \cdot v_h) + (\varepsilon^n, v_h) = 0 \quad \forall v_h \in X_h, \\ \qquad\qquad (\nabla \cdot \overline{u}_h(t_{n-\frac{1}{2}}), q_h) = 0 \quad \forall q_h \in M_h, \end{cases} \tag{3.26}$$

where $\varepsilon^n \in X_h$ is the consistency error defined by

$$(\varepsilon^n, v_h) = \begin{cases} (\partial_t u_h(t_n) - \delta_\tau u_h(t_n), v_h) + b(u_h(t_n) - u_h(t_{n-1}), u_h(t_n), v_h) \quad \text{for } n = 1, 2, \\ \big(\partial_t u_h(t_{n-\frac{1}{2}}) - \delta_\tau u_h(t_n), v_h\big) + \big(\nabla(u_h(t_{n-\frac{1}{2}}) - \overline{u}_h(t_{n-\frac{1}{2}})), \nabla v_h\big) \\ \quad + b\big(u_h(t_{n-\frac{1}{2}}), u_h(t_{n-\frac{1}{2}}), v_h\big) - b\big(\widehat{u}_h(t_{n-\frac{1}{2}}), \overline{u}_h(t_{n-\frac{1}{2}}), v_h\big) \\ =: (\varepsilon_1^n, v_h) + (\varepsilon_2^n, v_h) + (\varepsilon_3^n, v_h) \qquad \text{for } n \geq 3. \end{cases} \tag{3.27}$$

The following lemma gives a proof that $r_n \leq c$ for $n \geq 2$, where c is a positive constant. It will be used in the consistency error estimate.

Lemma 3.5 *For $n \geq 2$, there holds $r_n \leq c$.*

Proof From the stepsizes choice in (3.1) we know that

$$\begin{aligned} r_2 &= \frac{\tau_2}{\tau_1} = 1 && n = 2, \\ r_3 &= \frac{\tau_3}{\tau_2} \sim \frac{\left(\frac{t_2}{T}\right)^\alpha \tau}{\tau_2} = \frac{(2\tau_2)^\alpha \tau}{T^\alpha \tau_2} = 2^\alpha < 2 && n = 3, \\ r_n &= \frac{\tau_n}{\tau_{n-1}} \sim \frac{\left(\frac{t_{n-1}}{T}\right)^\alpha \tau}{\left(\frac{t_{n-2}}{T}\right)^\alpha \tau} = \frac{t_{n-1}^\alpha \tau}{t_{n-2}^\alpha \tau} = \left(\frac{t_{n-2} + \tau_{n-1}}{t_{n-2}}\right)^\alpha \\ &= \left(1 + \frac{\tau_{n-1}}{t_{n-2}}\right)^\alpha \sim \left(1 + \frac{t_{n-2}^{\alpha-1}\tau}{T^\alpha}\right)^\alpha \\ &\leq 1 + \frac{t_{n-2}^{\alpha-1}\tau}{T^\alpha} \leq 1 + \frac{t_1^{\alpha-1}\tau}{T^\alpha} = 2 && n \geq 4. \end{aligned}$$

This proves the desired result. □

Lemma 3.6 *If $u^0 \in H_0^1(\Omega)^2$ and $\nabla \cdot u^0 = 0$ and the stepsizes in (3.1) are used, then the consistency error defined in (3.27) satisfies the following estimate:*

$$|(\varepsilon^n, v_h)| \leq C\tau_n^2 t_n^{-2} \|\nabla v_h\|_{L^2(\Omega)} \quad \forall\, v_h \in \mathring{X}_h. \tag{3.28}$$

Proof For $n = 1, 2$ we have

$$
\begin{aligned}
|(\varepsilon^n, v_h)| &= |(\partial_t u_h(t_n) - \delta_\tau u_h(t_n), v_h) + b(u_h(t_n) - u_h(t_{n-1}), u_h(t_n), v_h)| \\
&\le c|(\partial_t u_h(t_n) - \delta_\tau u_h(t_n), v_h)| \\
&\quad + c\|u_h(t_n) - u_h(t_{n-1})\|_{H^1(\Omega)}\|u_h(t_n)\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \qquad \big((3.22) \text{ is used}\big) \\
&\le c|(\partial_t u_h(t_n) - \delta_\tau u_h(t_n), v_h)| \qquad (3.29) \\
&\quad + c\|u_h(t_n) - u_h(t_{n-1})\|_{H^1(\Omega)}\|u_h(t_n)\|_{H^1(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\
&\le c \max_{t\in[0,t_2]} |(\partial_t u_h(t), v_h)| + c \max_{t\in[0,t_2]} s\|u_h(t)\|^2_{H^1(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\
&\le c \max_{t\in[0,t_2]} |(\partial_t u_h(t), v_h)| + C\|\nabla v_h\|_{L^2(\Omega)},
\end{aligned}
$$

where the last inequality uses the boundedness of $\|u_h(t)\|_{H^1(\Omega)}$ as shown in (2.10). By choosing $v_h \in \mathring{X}_h$ in (2.8), we have $(p_h, \nabla \cdot v_h) = 0$ and therefore

$$
(\partial_t u_h(t), v_h) + b(u_h(t), u_h(t), v_h) + (\nabla u_h(t), \nabla v_h) = 0 \quad \forall v_h \in \mathring{X}_h, \qquad (3.30)
$$

which implies that

$$
\begin{aligned}
|(\partial_t u_h(t), v_h)| &\le |b(u_h(t), u_h(t), v_h)| + |(\nabla u_h(t), \nabla v_h)| \\
&\le c\|u_h(t)\|_{H^1(\Omega)}\|u_h(t)\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} + c\|\nabla u_h(t)\|_{L^2(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\
&\le C\|\nabla v_h\|_{L^2(\Omega)}.
\end{aligned}
$$

Substituting this into (3.29) yields that

$$
|(\varepsilon^n, v_h)| \le C\|\nabla v_h\|_{L^2(\Omega)} \le C\tau_n^2 t_n^{-2}\|\nabla v_h\|_{L^2(\Omega)} \quad \text{for } v_h \in \mathring{X}_h \text{ and } n = 1, 2.
$$

In the case $n \ge 3$, we present estimates for $|(\varepsilon_j^n, v_h)|$, $j = 1, 2, 3$, respectively. First, we note that

$$
|(\varepsilon_1^n, v_h)| = |(\partial_t u_h(t_{n-\frac{1}{2}}) - \delta_\tau u_h(t_n), v_h)| \le c\tau_n^2 \max_{t\in[t_{n-1},t_n]} |(\partial_t^3 u_h(t), v_h)|. \qquad (3.31)
$$

By differentiating (3.30) in time twice, we obtain

$$
\begin{aligned}
&(\partial_t^3 u_h(t), v_h) + b(\partial_t^2 u_h(t), u_h(t), v_h) + 2b(\partial_t u_h(t), \partial_t u_h(t), v_h) \\
&\qquad + b(u_h(t), \partial_t^2 u_h(t), v_h) + (\nabla\partial_t^2 u_h(t), \nabla v_h) = 0 \qquad \forall v_h \in \mathring{X}_h,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|(\partial_t^3 u_h(t), v_h)| &\le c\|\partial_t^2 u_h(t)\|_{H^1(\Omega)}\|u_h(t)\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \\
&\quad + c\|\partial_t u_h(t)\|^2_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \\
&\quad + c\|\partial_t^2 u_h(t)\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \\
&\le Ct^{-2}\|\nabla v_h\|_{L^2(\Omega)},
\end{aligned}
$$

where we have used (2.9) (with $m = 1, 2$ therein) and (2.10). Substituting this into (3.31) yields that

$$
|(\varepsilon_1^n, v_h)| \le C\tau_n^2 t_{n-1}^{-2}\|\nabla v_h\|_{L^2(\Omega)} \quad \forall v_h \in \mathring{X}_h. \qquad (3.32)
$$

Second, by using the definitions of (ε_2^n, v_h) and (ε_3^n, v_h) for $v_h \in \mathring{X}_h$, we have

$$\begin{aligned}
|(\varepsilon_2^n, v_h)| \leq & c\|\nabla(u_h(t_{n-\frac{1}{2}}) - \overline{u}_h(t_{n-\frac{1}{2}}))\|_{L^2(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\
\leq & c\tau_n^2 \max_{t\in[t_{n-1},t_n]} \|\partial_t^2 u_h(t)\|_{H^1(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\
\leq & C\tau_n^2 t_{n-1}^{-2}\|\nabla v_h\|_{L^2(\Omega)},
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
|(\varepsilon_3^n, v_h)| &= \Big|b\big(u_h(t_{n-\frac{1}{2}}), u_h(t_{n-\frac{1}{2}}), v_h\big) - b\big(\widehat{u}_h(t_{n-\frac{1}{2}}), \overline{u}_h(t_{n-\frac{1}{2}}), v_h\big)\Big| \\
&= \Big|b\big(u_h(t_{n-\frac{1}{2}}) - \widehat{u}_h(t_{n-\frac{1}{2}}), u_h(t_{n-\frac{1}{2}}), v_h\big) \\
&\quad + b\big(\widehat{u}_h(t_{n-\frac{1}{2}}), u_h(t_{n-\frac{1}{2}}) - \overline{u}_h(t_{n-\frac{1}{2}}), v_h\big)\Big| \\
&\leq c\|u_h(t_{n-\frac{1}{2}}) - \widehat{u}_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)}\|u_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \\
&\quad + c\|\widehat{u}_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)}\|u_h(t_{n-\frac{1}{2}}) - \overline{u}_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \\
&\leq c\tau_n^2 \max_{t\in[t_{n-2},t_n]} \|\partial_t^2 u_h(t)\|_{H^1(\Omega)}\|u_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \\
&\quad + c\|\widehat{u}_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)}\tau_n^2 \max_{t\in[t_{n-1},t_n]} \|\partial_t^2 u_h(t)\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)} \\
&\leq C\tau_n^2 t_{n-2}^{-2}\|\nabla v_h\|_{L^2(\Omega)},
\end{aligned} \tag{3.34}$$

where in the last inequality we have used

$$\|\widehat{u}_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)} \leq \big(1 + \frac{r_n}{2}\big)\|u_h(t_{n-1})\|_{H^1(\Omega)} + \frac{r_n}{2}\|u_h(t_{n-2})\|_{H^1(\Omega)} \leq C,$$

which is a result of Lemma 3.5 and (2.10).

Since $t_{n-2} \sim t_{n-1} \sim t_n$ for $n \geq 3$, summing up the above three estimates (3.32)–(3.34), we obtain

$$|(\varepsilon^n, v_h)| \leq C\tau_n^2 t_n^{-2}\|\nabla v_h\|_{L^2(\Omega)} \quad \text{for} \quad v_h \in \mathring{X}_h \quad \text{and} \quad n \geq 3.$$

This proves the desired estimate in Lemma 3.6. □

3.3 Error Estimate

Let $e_h^n = u_h(t_n) - u_h^n$ and $\eta_h^n = p_h(t_n) - p_h^n$ be the error functions. Then subtracting (3.2) from (3.25) yields the following error equations for $n = 1, 2$:

$$\begin{cases}
(\delta_\tau e_h^n, v_h) + (\nabla e_h^n, \nabla v_h) + b(u_h(t_{n-1}), u_h(t_n), v_h) - b(u_h^{n-1}, u_h^n, v_h) \\
\qquad\qquad -(\eta_h^n, \nabla\cdot v_h) + (\varepsilon^n, v_h) = 0, \\
\qquad\qquad (\nabla\cdot e_h^n, q_h) = 0,
\end{cases} \tag{3.35}$$

for all $(v_h, q_h) \in X_h \times M_h$.
In the light of (3.21), we notice that

$$
\begin{aligned}
&\left|b\big(u_h(t_{n-1}), u_h(t_n), e_h^n\big) - b\big(u_h^{n-1}, u_h^n, e_h^n\big)\right| \\
&= \left|b\big(e_h^{n-1}, u_h(t_n), e_h^n\big) + b\big(u_h^{n-1}, e_h^n, e_h^n\big)\right| \\
&= \left|b\big(e_h^{n-1}, u_h(t_n), e_h^n\big)\right| \\
&\le c\|e_h^{n-1}\|_{L^2(\Omega)}\|u_h(t_n)\|_{H^1(\Omega)}^{\frac{1}{2}}\|A_h u_h(t_n)\|_{L^2(\Omega)}^{\frac{1}{2}}\|e_h^n\|_{H^1(\Omega)} \\
&\qquad\qquad\qquad\qquad\text{(here we have used Lemma 3.3)} \\
&\le C t_n^{-\frac{1}{4}}\|e_h^{n-1}\|_{L^2(\Omega)}\|\nabla e_h^n\|_{L^2(\Omega)},
\end{aligned}
\tag{3.36}
$$

where we have used (2.10) in the last inequality. Then, substituting $(v_h, q_h) = (e_h^n, \eta_h^n) \in \mathring{X}_h \times M_h \subset X_h \times M_h$ into the error equations (3.35) and using estimate (3.36), we obtain

$$
\begin{aligned}
&\frac{1}{2\tau_n}\big(\|e_h^n\|_{L^2(\Omega)}^2 - \|e_h^{n-1}\|_{L^2(\Omega)}^2 + \|e_h^n - e_h^{n-1}\|_{L^2(\Omega)}^2\big) + \|\nabla e_h^n\|_{L^2(\Omega)}^2 \\
&\le |(\varepsilon^n, e_h^n)| + C t_n^{-\frac{1}{4}}\|e_h^{n-1}\|_{L^2(\Omega)}\|\nabla e_h^n\|_{L^2(\Omega)} \\
&\le C\tau_n^2 t_n^{-2}\|\nabla e_h^n\|_{L^2(\Omega)} + C t_n^{-\frac{1}{4}}\|e_h^{n-1}\|_{L^2(\Omega)}\|\nabla e_h^n\|_{L^2(\Omega)} \\
&\le C\tau_n^4 t_n^{-4} + C t_n^{-\frac{1}{2}}\|e_h^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla e_h^n\|_{L^2(\Omega)}^2 \qquad \text{for } n = 1, 2,
\end{aligned}
$$

where we have used Lemma 3.6 in obtaining the second to last inequality. The last term of the inequality above can be absorbed by the left-hand side. As a result, we have

$$
\begin{aligned}
\|e_h^n\|_{L^2(\Omega)}^2 + \tau_n\|\nabla e_h^n\|_{L^2(\Omega)}^2 &\le C\tau_n^5 t_n^{-4} + (1 + C\tau_n t_n^{-\frac{1}{2}})\|e_h^{n-1}\|_{L^2(\Omega)}^2 \\
&\le C\tau_n + (1 + C\tau_n t_n^{-\frac{1}{2}})\|e_h^{n-1}\|_{L^2(\Omega)}^2 \qquad \text{for } n = 1, 2.
\end{aligned}
$$

Since $\|e_h^0\|_{L^2(\Omega)} = 0$, it follows that

$$
\begin{aligned}
&\|e_h^1\|_{L^2(\Omega)}^2 + \tau_1\|\nabla e_h^1\|_{L^2(\Omega)}^2 \le C\tau_1, \\
&\|e_h^2\|_{L^2(\Omega)}^2 + \tau_2\|\nabla e_h^2\|_{L^2(\Omega)}^2 \le C\tau_2 + (1 + C\tau_2^{\frac{1}{2}})\|e_h^1\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{3.37}
$$

When $3/4 < \alpha < 1$, we have

$$
\tau_1 = \tau_2 = T\left(\frac{\tau}{T}\right)^{\frac{1}{1-\alpha}} \le c\tau^4.
$$

Substituting this into (3.37) yields that

$$
\|e_h^1\|_{L^2(\Omega)} + \|e_h^2\|_{L^2(\Omega)} \le C\tau^2. \tag{3.38}
$$

For $n \ge 3$, subtracting (3.3) from (3.26) yields the following error equations:

$$
\left\{
\begin{aligned}
&(\delta_\tau e_h^n, v_h) + (\nabla \overline{e}_h^{n-\frac{1}{2}}, \nabla v_h) + b(\widehat{u}_h(t_{n-\frac{1}{2}}), \overline{u}_h(t_{n-\frac{1}{2}}), v_h) - b(\widehat{u}_h^{n-\frac{1}{2}}, \overline{u}_h^{n-\frac{1}{2}}, v_h) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - (\eta_h^{n-\frac{1}{2}}, \nabla \cdot v_h) + (\varepsilon^n, v_h) = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\nabla \cdot \overline{e}_h^{n-\frac{1}{2}}, q_h) = 0,
\end{aligned}
\right.
\tag{3.39}
$$

for all $(v_h, q_h) \in X_h \times M_h$.

In view of (3.21), it can easily be seen that

$$
\begin{aligned}
&\big|b\big(\widehat{u}_h(t_{n-\frac{1}{2}}), \overline{u}_h(t_{n-\frac{1}{2}}), \overline{e}_h^{n-\frac{1}{2}}\big) - b\big(\widehat{u}_h^{n-\frac{1}{2}}, \overline{u}_h^{n-\frac{1}{2}}, \overline{e}_h^{n-\frac{1}{2}}\big)\big| \\
&= \big|b\big(\widehat{e}_h^{n-\frac{1}{2}}, \overline{u}_h(t_{n-\frac{1}{2}}), \overline{e}_h^{n-\frac{1}{2}}\big) + b\big(\widehat{u}_h^{n-\frac{1}{2}}, \overline{e}_h^{n-\frac{1}{2}}, \overline{e}_h^{n-\frac{1}{2}}\big)\big| \\
&= \big|b\big(\widehat{e}_h^{n-\frac{1}{2}}, \overline{u}_h(t_{n-\frac{1}{2}}), \overline{e}_h^{n-\frac{1}{2}}\big)\big| \qquad (3.40)\\
&\le c\|\widehat{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}\|\overline{u}_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)}^{\frac{1}{2}}\|A_h\overline{u}_h(t_{n-\frac{1}{2}})\|_{L^2(\Omega)}^{\frac{1}{2}}\|\overline{e}_h^{n-\frac{1}{2}}\|_{H^1(\Omega)} \\
&\qquad\qquad \text{(here we have used Lemma 3.3)} \\
&\le Ct_{n-1}^{-\frac{1}{4}}\|\widehat{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}\|\nabla\overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)},
\end{aligned}
$$

where in the last inequality we have used

$$
\begin{aligned}
&\|A_h\overline{u}_h(t_{n-\frac{1}{2}})\|_{L^2(\Omega)} \le \frac{1}{2}\|A_h u_h(t_{n-1})\|_{L^2(\Omega)} + \frac{1}{2}\|A_h u_h(t_n)\|_{L^2(\Omega)} \le Ct_{n-1}^{-\frac{1}{2}}, \\
&\|\overline{u}_h(t_{n-\frac{1}{2}})\|_{H^1(\Omega)} \le \frac{1}{2}\|u_h(t_{n-1})\|_{H^1} + \frac{1}{2}\|u_h(t_n)\|_{H^1} \le C,
\end{aligned}
$$

which are consequences of (2.10).

Substituting $(v_h, q_h) = (\overline{e}_h^{n-\frac{1}{2}}, \eta_h^{n-\frac{1}{2}}) \in \mathring{X}_h \times M_h \subset X_h \times M_h$ into the error equations (3.39) and using estimate (3.40), we obtain

$$
\begin{aligned}
&\frac{1}{2\tau_n}\big(\|e_h^n\|_{L^2(\Omega)}^2 - \|e_h^{n-1}\|_{L^2(\Omega)}^2\big) + \|\nabla\overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}^2 \\
&\le |(\varepsilon^n, \overline{e}_h^{n-\frac{1}{2}})| + Ct_{n-1}^{-\frac{1}{4}}\|\widehat{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}\|\nabla\overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)} \\
&\le C\tau_n^2 t_n^{-2}\|\nabla\overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)} + Ct_{n-1}^{-\frac{1}{4}}\|\widehat{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}\|\nabla\overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)} \\
&\le C\tau_n^4 t_n^{-4} + Ct_{n-1}^{-\frac{1}{2}}\|\widehat{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla\overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}^2.
\end{aligned}
$$

The last term of the inequality above can be absorbed by the left-hand side. As a result, we have

$$
\begin{aligned}
&\frac{1}{2\tau_n}\big(\|e_h^n\|_{L^2(\Omega)}^2 - \|e_h^{n-1}\|_{L^2(\Omega)}^2\big) + \frac{1}{2}\|\nabla\overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}^2 \\
&\le C\tau_n^4 t_n^{-4} + Ct_{n-1}^{-\frac{1}{2}}\big(\|e_h^{n-1}\|_{L^2(\Omega)}^2 + \|e_h^{n-2}\|_{L^2(\Omega)}^2\big) \quad \text{for } n \ge 3.
\end{aligned} \tag{3.41}
$$

When $4\alpha - 4 > -1$ (or equivalently $\alpha > 3/4$), we have

$$
\sum_{n=3}^{N}\tau_n t_n^{4\alpha-4} \le \int_0^T t^{4\alpha-4}\mathrm{d}t \le c. \tag{3.42}
$$

Hence, summing up (3.41) times $2\tau_n$ for $n = 3, \ldots, m$ yields

$$
\begin{aligned}
&\|e_h^m\|_{L^2(\Omega)}^2 + \sum_{n=3}^{m} \tau_n \|\nabla \overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}^2 \\
&\le \|e_h^2\|_{L^2(\Omega)}^2 + C\tau^4 \sum_{n=3}^{m} \tau_n t_n^{4\alpha-4} + C \sum_{n=3}^{m} \tau_n t_{n-1}^{-\frac{1}{2}} \big(\|e_h^{n-1}\|_{L^2(\Omega)}^2 + \|e_h^{n-2}\|_{L^2(\Omega)}^2\big) \\
&\le C\tau^4 + C \sum_{n=3}^{m} \tau_n t_{n-1}^{-\frac{1}{2}} \big(\|e_h^{n-1}\|_{L^2(\Omega)}^2 + \|e_h^{n-2}\|_{L^2(\Omega)}^2\big),
\end{aligned}
$$

where we have used (3.38) and (3.42) in deriving the last inequality. Since this inequality holds for all $3 \le m \le N$, by applying Gronwall inequality (i.e. Lemma 3.4), we obtain

$$
\max_{3\le n\le N} \|e_h^n\|_{L^2(\Omega)}^2 + \sum_{n=3}^{N} \tau_n \|\nabla \overline{e}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)}^2 \le C\tau^4. \tag{3.43}
$$

Combining (3.38) and (3.43), we have

$$
\max_{1\le n\le N} \|e_h^n\|_{L^2(\Omega)} \le C\tau^2.
$$

This result and (3.24) imply the desired error bound in Theorem 3.1.

4 Numerical Examples

In this section, we present numerical experiments to support the theoretical analysis in Theorem 3.1. In Example 4.1 we present numerical results to illustrate that the number of total time levels N using the variable stepsize in (3.1) is equivalent to the number of total time levels using a uniform stepsize. In Example 4.2 and 4.3 we present numerical results to illustrate the convergence rates of numerical method by solving problem (1.1) in the unit square $\Omega = (0, 1) \times (0, 1)$ up to $T = 0.1$. The Taylor–Hood P2-P1 finite element space is used for spatial discretization, and the method (3.2)–(3.3) for temporal discretization.

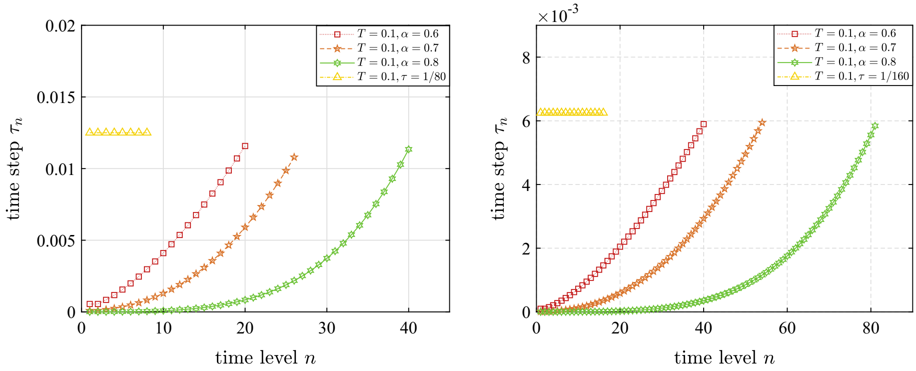
For the stepsizes in (3.1), we simply choose $\tau_n = \big(\frac{t_{n-1}}{T}\big)^{\alpha}\tau$ for $n \ge 3$ in all numerical simulations. All the computations are performed by FreeFEM++; see www.freefem.org.

Example 4.1 In Table 1, we present the number of total time levels N using the stepsizes (3.1) corresponding to different parameters, including $T = 0.1, 0.5, 1.0, 10, 100$, $\alpha = 0.6, 0.7, 0.8$ and $\tau = 1/80, 1/160$. We can see that when $\alpha = 0.6$, the total number of time levels $N \le 2.6(T/\tau)$; when $\alpha = 0.7$, $N \le 3.4(T/\tau)$; when $\alpha = 0.8$, $N \le 5.1(T/\tau)$. This is consistent with the conclusion we proved in Remark 3.1.

In Figures 1 and 2, we present the evolution of the stepsize τ_n with different parameters $\alpha = 0.6, 0.7, 0.8$, and different maximal stepsizes $\tau = 1/80, 1/160$, for both $T = 0.1$ and $T = 1.0$. Figures 1 and 2 illustrate how the variable stepsize in (3.1) increases from $T(\frac{\tau}{T})^{\frac{1}{1-\alpha}}$ to τ, while Table 1 shows that the number of total time levels satisfies $N \le C(T/\tau)$.

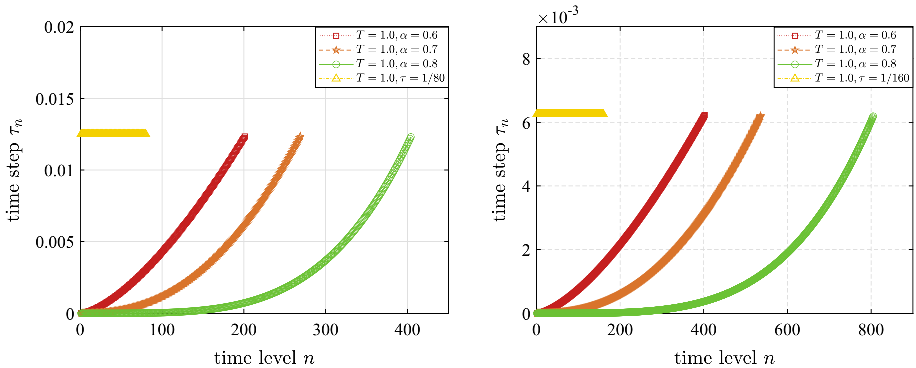
Table 1 The number of time levels N

α	T / τ	0.1	0.5	1.0	10	100	$N\tau/T$
0.6	1/80	20	101	201	2003	20005	2.6
	1/160	40	201	402	4004	40005	2.6
0.7	1/80	26	135	269	2672	26674	3.4
	1/160	54	269	536	5339	53342	3.4
0.8	1/80	40	203	404	4009	40013	5.1
	1/160	81	404	805	8010	80015	5.1



(a) $T = 0.1, \tau = 1/80$ **(b)** $T = 0.1, \tau = 1/160$

Fig. 1 The evolution of τ_n at $T = 0.1$



(a) $T = 1.0, \tau = 1/80$ **(b)** $T = 1.0, \tau = 1/160$

Fig. 2 The evolution of τ_n at $T = 1.0$

Example 4.2 We consider an example with initial value in $H_0^1(\Omega)^2$ but not in $H^2(\Omega)^2$, i.e., $u^0 = (u_1^0(x, y), u_2^0(x, y))$ with

$$u_1^0(x, y) = \frac{5}{2}\pi \sin^{\frac{5}{2}}(\pi x) \sin^{\frac{3}{2}}(\pi y) \cos(\pi y),$$

$$u_2^0(x, y) = -\frac{5}{2}\pi \sin^{\frac{3}{2}}(\pi x) \cos(\pi x) \sin^{\frac{5}{2}}(\pi y).$$

Table 2 Temporal discretization errors using variable stepsize with $\alpha = 0.8$

τ / h	1/320	1/640	1/1280	1/2560	convergence rate
1/16	5.494E-05	1.102E-05	2.805E-06	6.783E-07	≈ 2.05
1/32	5.496E-05	1.099E-05	2.807E-06	6.785E-07	≈ 2.05
1/64	5.496E-05	1.099E-05	2.806E-06	6.785E-07	≈ 2.05

Table 3 Spatial discretization errors using variable stepsize with $\alpha = 0.8$

h / τ	1/4	1/8	1/16	1/32	convergence rate
1/80	8.406E-03	1.626E-03	3.105E-04	6.834E-05	≈ 2.18
1/160	8.651E-03	1.679E-03	3.226E-04	7.122E-05	≈ 2.18
1/320	8.724E-03	1.696E-03	3.264E-04	7.219E-05	≈ 2.18

The initial value satisfies

$$u^0 \in H^{2-\epsilon}(\Omega)^2 \cap H_0^1(\Omega)^2 \ \ \forall \epsilon \in (0,1), \quad \nabla \cdot u^0 = 0 \ \text{ in } \Omega \quad \text{and} \quad u^0 = 0 \ \text{ on } \partial\Omega.$$

The temporal discretization errors $\|u_{h,\mathrm{ref}}^N - u_h^N\|_{L^2(\Omega)}$ and convergence rates are presented in Table 2, where the reference solution $u_{h,\mathrm{ref}}^N$ is computed by using a sufficiently small stepsize with $\tau = 1/10240$. The spatial discretization errors $\|u_{h,\mathrm{ref}}^N - u_h^N\|_{L^2(\Omega)}$ and convergence rates are presented in Table 3, where the reference solution $u_{h,\mathrm{ref}}^N$ is computed by using a sufficiently small spatial mesh size with $h = 1/128$. The parameter in (3.1) is selected as $\alpha = 0.8$. From Tables 2 and 3, we see that the convergence rates in space and time are consistent with the theoretical result proved in Theorem 3.1.

Example 4.3 We present numerical results for an initial value $u^0 = (u_1^0(x,y), u_2^0(x,y))$ given by

$$u_1^0(x,y) = \frac{3}{2}\pi \sin^{\frac{3}{2}}(\pi x) \sin^{\frac{1}{2}}(\pi y) \cos(\pi y),$$
$$u_2^0(x,y) = -\frac{3}{2}\pi \sin^{\frac{1}{2}}(\pi x) \cos(\pi x) \sin^{\frac{3}{2}}(\pi y).$$

The initial value satisfies that

$$u^0 \in H^{1-\epsilon}(\Omega)^2 \ \ \forall \epsilon \in (0,1), \quad \nabla \cdot u^0 = 0 \ \text{ in } \Omega \quad \text{and} \quad u^0 = 0 \ \text{ on } \partial\Omega,$$

but $u_0 \notin H^1(\Omega)^2$. Hence, the initial value in this example is in the critical space that our assumption of Theorem 3.1 does not hold.

The temporal discretization errors $\|u_{h,\mathrm{ref}}^N - u_h^N\|_{L^2(\Omega)}$ and convergence rates are presented in Table 4, where the reference solution $u_{h,\mathrm{ref}}^N$ is computed by using a sufficiently small stepsize with $\tau = 1/10240$. The spatial discretization errors $\|u_{h,\mathrm{ref}}^N - u_h^N\|_{L^2(\Omega)}$ and convergence rates are presented in Table 5, where the reference solution $u_{h,\mathrm{ref}}^N$ is computed by using a sufficiently small spatial mesh size with $h = 1/128$. The parameter in (3.1) is also selected as $\alpha = 0.8$. From Tables 4 and 5, we see that the numerical solutions have second-order

Table 4 Temporal discretization errors using variable stepsize with $\alpha = 0.8$

τ / h	1/320	1/640	1/1280	1/2560	convergence rate
$1/64$	5.841E-05	1.187E-05	3.215E-06	7.210E-07	≈ 2.16
$1/128$	5.840E-05	1.170E-05	3.001E-06	7.212E-07	≈ 2.06
$1/256$	5.840E-05	1.168E-05	2.984E-06	7.245E-07	≈ 2.04

Table 5 Spatial discretization errors using variable stepsize with $\alpha = 0.8$

h / τ	1/4	1/8	1/16	1/32	convergence rate
$1/2560$	8.8477E-03	1.6699E-03	3.1670E-04	7.2398E-05	≈ 2.13
$1/5120$	8.8480E-03	1.6700E-03	3.1666E-04	7.2390E-05	≈ 2.13
$1/10240$	8.8480E-03	1.6700E-03	3.1667E-04	7.2391E-05	≈ 2.13

convergence in time and space. This shows that the theoretical result in Theorem 3.1 not only holds for H^1 initial data but also may be extended to rougher initial data.

5 Conclusion

We have presented error analysis for the linearly extrapolated Crank–Nicolson method for the NS equations with a specific locally refined temporal grid. We have proved second-order temporal convergence of the numerical method for H^1 initial data by utilizing the property of locally refined stepsizes in the consistency analysis and utilizing a technical lemma (Lemma 3.3) in the stability analysis. The numerical results are consistent with the theoretical analysis and indicate that the error analysis may be furthermore extended to rougher initial data.

Funding The research of Buyang Li and Shu Ma were partially funded by the internal grant ZZKQ at The Hong Kong Polytechnic University. The research of Na Wang was partially funded by the National Natural Science Foundation of China (NSFC Grant U1930402).

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