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RESEARCH ARTICLE

Electromagnetic scattering from a cavity embedded in an impedance ground plane

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Summary

This paper is concerned with time-harmonic electromagnetic scattering from a cavity embedded in an *impedance* ground plane. The fillings (which may be *inhomogeneous*) do not protrude the cavity and the space above the ground plane is empty. This problem is obviously different from those considered in previous work where either perfectly conducting boundary conditions were used or the cavity was assumed to be empty. By employing the Green's function method, we reduce the scattering problem to a boundary-value problem in a bounded domain (the cavity), with impedance boundary conditions on the cavity walls and an impedance-to-Dirichlet condition on the cavity aperture. Existence and uniqueness of the solution are proved for the weak formulation of the reduced problem. We also propose a numerical method to calculate the radar cross section (RCS), which is a parameter of physical interest. Numerical experiments show that the proposed model and numerical method are efficient for the calculation of RCS from cavities.

KEYWORDS:

electromagnetic cavity, impedance boundary condition, impedance-to-Dirichlet map, existence and uniqueness, radar cross section

1 | INTRODUCTION

Radar cross section (RCS) is a measure of how detectable a target is in radar systems. A larger RCS indicates that a target is more easily detected. Reducing the RCS from cavities is highly valuable in many applications since it dominates the target's overall RCS. Accurate prediction of the RCS from cavities relies on the direct electromagnetic scattering problems involving cavities. Well-known examples of cavities include cavity-backed antennas, jet engine inlet ducts, and cracks and gaps in the skin of aircrafts.

Electromagnetic scattering from cavities has attracted much attention in recent years. Existing literature mainly deals with cavities embedded in a perfectly conducting plane; e.g., see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Numerical methods for solving these cavity problems have also been studied extensively; e.g., see [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Some stability estimates on these cavity problems are given in [31, 32, 28].

When cavities are embedded in an imperfect conductor, it can be shown that the electric and magnetic fields at the surface of the conductor satisfy *impedance boundary conditions*, which are more prevalent in real applications, e.g., the detection of a target hidden in a hole on the ground plane, the detection of improvised explosive devices, etc. Although there are a wide range of applications, little mathematical analysis exists for the problem with filled cavities embedded in an impedance ground plane. As far as we know, the only mathematical treatment of the transient problem with overfilled cavities embedded in an impedance ground plane is reported in [33], where the (time-domain) wave equation was first discretized in time and then reduced to a modified Helmholtz equation $-\Delta u^{n+1} + \alpha^2 \varepsilon_r u^{n+1} = f^{n+1}$ in a bounded domain with an artificial boundary condition on a semicircle enclosing the overfilled cavity. The well-posedness of the associated variational formulation and convergence of finite element solutions at a fixed time step were proved.

The cavity problem can be viewed as a scattering problem by locally perturbed infinite planes. We mention that some works (on perfectly conductors or homogeneous media) are reported for problems involving locally perturbed infinite planes; e.g., see [34, 35, 36, 37, 38]. For the case of nonlocal perturbations of infinite planes, which is called rough surface scattering, we refer to [39, 40, 41, 42, 43, 44, 45, 46].

In this paper, we consider time-harmonic electromagnetic scattering from cavities embedded in an impedance ground plane and assume that the fillings (which may be inhomogeneous) do not protrude the ground plane and the upper half-space is empty. The current paper differs from the existing work on impedance ground plane in the following several aspects:

- By using the Green's function in the upper half plane, we reducd the scattering problem to a boundary-value problem of the Helmholtz equation $\nabla \cdot (a^{-1}\nabla u) + k_0^2 bu = 0$ in a bounded domain (cavity) with an artificial boundary condition on a part of the boundary. The reduced Helmholtz equation in the current paper and modified Helmholtz equation studied in [33] contain opposite signs in the Laplacian operator. This brings different mathematical difficulties in the analysis of well-posedness of the reduced problem.

- The artificial boundary condition in [33] contains a nonlocal integral operator (Steklov-Poincaré operator $\mathcal{T}_R$) acting on u^{n+1}, while our formulation yields a boundary integral operator (impedence-to-Dirichlet) acting on $a^{-1}\partial_y u + \rho u$; see equation (12). This difference motivates us to introduce an auxiliary variable $w = a^{-1}\partial_y u$ in the weak formulation and the corresponding finite element method. In particular, the weak formulation of the reduced problem is to find $(u, w) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$ simultaneously; see (14).

- Another contribution of this paper is the numerical evaluation of the impedence-to-Dirichlet boundary integral operator in (12) by the finite element method:

$$G_{ij}^{\Gamma} = -\int_{\Gamma}\int_{\Gamma} \left(\frac{1}{\pi} \int_0^{\infty} \frac{\cos((x - x_0)\xi)}{\rho - \sqrt{\xi^2 - k_0^2}} d\xi \right) \phi_i(x_0)\phi_j(x)\, \mathrm{d}x_0 \mathrm{d}x,$$

  where ϕ_i and ϕ_j are the basis functions of the finite element space. The improper integral with respect to ξ causes mathematical difficulties for the convergence of standard quadratures, as the kernel $\frac{\cos((x-x_0)\xi)}{\rho-\sqrt{\xi^2-k_0^2}}$ is not absolutely integrable with respect to ξ. To overcome this difficulty, we rewrite the matrix G_{ij}^{Γ} into an equivalent form

$$G_{ij}^{\Gamma} = -\frac{1}{\pi}\int_0^{k_0} \frac{g_{ij}(\xi)}{\rho + i\sqrt{k_0^2 - \xi^2}} \mathrm{d}\xi - \frac{1}{\pi}\int_{k_0}^{\infty} \frac{g_{ij}(\xi)}{\rho - \sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi,$$

  with a matrix $g_{ij}(\xi) = O(\xi^{-4})$ decaying sufficiently fast as $\xi \to \infty$ so that both integrals above can be evaluated sufficiently accurately by a quadrature.

Numerical examples are provided to show that the proposed model and numerical method are efficient for accurate prediction of the RCS from cavities.

2 | THE ELECTROMAGNETIC CAVITY PROBLEM

We consider a time-harmonic electromagnetic plane wave incident on a cavity embedded in an infinite impedance ground plane, and assume that no currents are present. The total electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ satisfy the following time-harmonic Maxwell's equations (time dependence $e^{-\mathrm{i}\omega t}$):

$$\begin{cases} \nabla \times \mathbf{E} - \mathrm{i}\omega\mu\mathbf{H} = \mathbf{0}, \\ \nabla \times \mathbf{H} + \mathrm{i}\omega\varepsilon\mathbf{E} = \mathbf{0}, \end{cases} \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, ω is the angular frequency and the physical parameters ε and μ denote, respectively, the permittivity (farads/meter) and the permeability (henrys/meter) of the medium. Throughout the paper, we assume that the medium is isotropic.

Let ε_0 and μ_0 denote the permittivity and the permeability of the free space. Let $\varepsilon_r^+ = \varepsilon^+/\varepsilon_0$ and $\mu_r^+ = \mu^+/\mu_0$ be the relative permittivity and the relative permeability of the medium in $\mathbb{R}^3_+ \cup \Omega$, respectively, where $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 | y > 0\}$ denotes the upper half-space and Ω denotes the cavity. Similarly, $\varepsilon_r^- = \varepsilon^-/\varepsilon_0$ and $\mu_r^- = \mu^-/\mu_0$ denote, respectively, the relative permittivity and relative permeability of the homogeneous medium in the complementary domain $\mathbb{R}^3 \setminus (\mathbb{R}^3_+ \cup \Omega)$. On the ground plane and the cavity wall, we have the following impedance boundary conditions (see (1.56) and (1.57) of [47])

$$\frac{1}{\mu_r^+}\mathbf{n} \times (\nabla \times \mathbf{E}) - \frac{\mathrm{i}k_0}{\eta}\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = 0, \tag{2}$$

and

$$\frac{1}{\varepsilon_r^+}\mathbf{n} \times (\nabla \times \mathbf{H}) - \mathrm{i}k_0\eta\mathbf{n} \times (\mathbf{n} \times \mathbf{H}) = 0, \tag{3}$$

where $\mathbf{n}$ is the unit normal pointing into the ground, $k_0 = \omega\sqrt{\varepsilon_0\mu_0} > 0$ is the free space wave number, and $\eta = \sqrt{\mu_r^-/\varepsilon_r^-}$ is the normalized intrinsic impedance of the homogeneous medium below the ground plane and the walls of the cavity.

We simplify the problem by using a two-dimensional model to approximate the three-dimensional problem. Assume that the fields, the associated medium and the cavity have no variation with respect to the z-axis. Let Ω denote the cross-section of the cavity, Γ the aperture of the cavity, S the wall of the cavity, and assume that Ω is a bounded Lipschitz domain. Denote the upper half-plane by $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. Let $\Gamma^C = \{y = 0\} \setminus \Gamma$. See Figure 1 for the problem geometry.

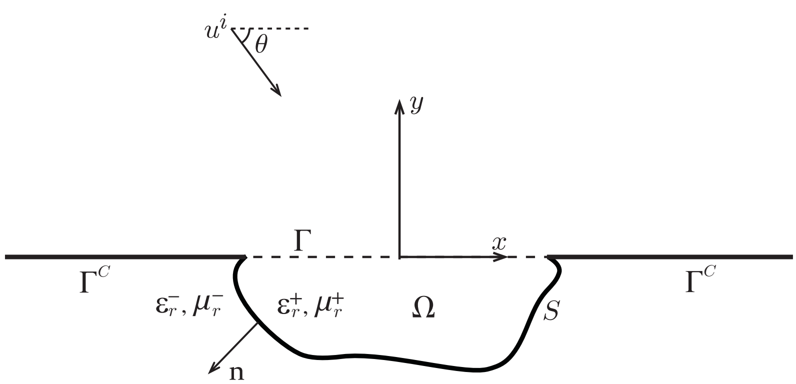

FIGURE 1 Cavity geometry

Two fundamental polarizations, the transverse magnetic (TM) and the transverse electric (TE), are often considered in the study of the propagation of the waves from the cavity.

- TM polarization: the magnetic field is transverse to the z-axis so that $\mathbf{E}$ and $\mathbf{H}$ are of the form $\mathbf{E} = (0, 0, E_z)$, $\mathbf{H} = (H_x, H_y, 0)$. By (1), (2) and $\mathbf{n} = (n_x, n_y, 0)$, we can show that E_z satisfies

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\mu_r^+} \nabla E_z \right) + k_0^2 \varepsilon_r^+ E_z = 0, \text{ in } \mathbb{R}_+^2 \cup \Omega, \\ \dfrac{1}{\mu_r^+} \dfrac{\partial E_z}{\partial n} - \dfrac{\mathrm{i} k_0}{\eta} E_z = 0, \qquad \text{on } \Gamma^C \cup S. \end{cases} \tag{4}$$

- TE polarization: the electric field is transverse to the z-axis so that $\mathbf{E}$ and $\mathbf{H}$ are of the form $\mathbf{E} = (E_x, E_y, 0)$, $\mathbf{H} = (0, 0, H_z)$. By (1), (3) and $\mathbf{n} = (n_x, n_y, 0)$, we can show that H_z satisfies

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\varepsilon_r^+} \nabla H_z \right) + k_0^2 \mu_r^+ H_z = 0, \text{ in } \mathbb{R}_+^2 \cup \Omega, \\ \dfrac{1}{\varepsilon_r^+} \dfrac{\partial H_z}{\partial n} - \mathrm{i} k_0 \eta H_z = 0, \qquad \text{on } \Gamma^C \cup S. \end{cases} \tag{5}$$

The problems (4) and (5) can be written in the following unified form

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{a} \nabla u \right) + k_0^2 b u = 0, \text{ in } \mathbb{R}_+^2 \cup \Omega, \\ \dfrac{1}{a} \dfrac{\partial u}{\partial n} - \rho u = 0, \qquad \text{on } \Gamma^C \cup S, \end{cases} \tag{6}$$

where u is the z-component of the unknown *total* electric or magnetic field, a and b are complex scalar functions of position with $\mathrm{Re}(a) \geq a_0 > 0$, $\mathrm{Im}(a) \geq 0$, $\mathrm{Re}(b) \geq b_0 > 0$, and $\mathrm{Im}(b) \geq 0$, $\rho \in \mathbb{C}$ is a constant with $\mathrm{Im}(\rho) > 0$ or $\rho = 0$. In this paper, we assume that the fillings do not protrude the cavity and the space above the ground plane is empty, i.e., $a = 1$ and $b = 1$ in $\mathbb{R}_+^2$.

We assume that the incident field u^i is given by

$$u^i = e^{\mathrm{i} k_0 (x \cos\theta - y \sin\theta)},$$

where $0 < \theta < \pi$ is the angle of incidence with respect to the positive x-axis. The total field

$$u = u^i + u^r + u^s,$$

where u^r is the reflected field by the infinite impedance ground plane,

$$u^r = -\frac{\rho - \mathrm{i} k_0 \sin\theta}{\rho + \mathrm{i} k_0 \sin\theta} e^{\mathrm{i} k_0 (x \cos\theta + y \sin\theta)},$$

and u^s is the unknown scattered field. We note that $u^i + u^r$ satisfies

$$\begin{cases} \Delta(u^i + u^r) + k_0^2 (u^i + u^r) = 0, \text{ in } \mathbb{R}_+^2, \\ \dfrac{\partial (u^i + u^r)}{\partial n} - \rho (u^i + u^r) = 0, \quad \text{on } \{y = 0\}. \end{cases}$$

The scattering problem reads: for a given incident plane wave u^i, determine the scattered field u^s in the cavity and the upper half-plane. To obtain a unique solution for the problem, some appropriate boundary conditions must be specified at the outer boundary for the scattered fields. Here, we use the Sommerfeld radiation condition [47]

$$\frac{\partial u^s}{\partial r} - \mathrm{i} k_0 u^s = o(r^{-1/2}), \quad u^s = O(r^{-1/2}) \tag{7}$$

uniformly as $r = \sqrt{x^2 + y^2} \to \infty$.

3 | INTERIOR PROBLEM IN THE CAVITY

The scattered field u^s satisfies

$$\begin{cases} \Delta u^s + k_0^2 u^s = 0, \text{ in } \mathbb{R}_+^2, \\ \dfrac{\partial u^s}{\partial n} - \rho u^s = 0, \quad \text{on } \Gamma^C, \\ u^s = u - g, \qquad \text{on } \Gamma, \end{cases} \tag{8}$$

where $g = u^i + u^r$. We use the Green's function method to derive an integral expression for u^s in $\mathbb{R}_+^2$, and by the field continuity conditions we obtain a transparent boundary condition on the aperture of the cavity, which reduces the unbounded domain problem (6) to the interior problem defined in the cavity.

Let $\mathbf{x} = (x, y) \in \mathbb{R}_+^2$ be the fixed source point, and $\mathbf{x_0} = (x_0, y_0)$. We introduce the impedance Green's function $G_\rho(\mathbf{x}, \mathbf{x_0})$, which is governed by the following boundary value problem

$$\begin{cases} \Delta_{\mathbf{x_0}} G_\rho(\mathbf{x}, \mathbf{x_0}) + k_0^2 G_\rho(\mathbf{x}, \mathbf{x_0}) = -\delta(\mathbf{x} - \mathbf{x_0}), \text{ in } \mathbb{R}_+^2, \\ \dfrac{\partial G_\rho(\mathbf{x}, \mathbf{x_0})}{\partial n(\mathbf{x_0})} - \rho G_\rho(\mathbf{x}, \mathbf{x_0}) = 0, \qquad \text{on } \{y_0 = 0\}, \end{cases} \tag{9}$$

and the radiation conditions [48, 49]. The solution of (9) is

$$\begin{aligned} G_\rho(\mathbf{x}, \mathbf{x_0}) = &\frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-\sqrt{\xi^2 - k_0^2}|y_0 - y|} \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi \\ &- \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\rho + \sqrt{\xi^2 - k_0^2}}{\rho - \sqrt{\xi^2 - k_0^2}} e^{-\sqrt{\xi^2 - k_0^2}(y_0 + y)} \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi. \end{aligned} \tag{10}$$

See [48, 49] for a derivation. The complex square root is characterized, for $\xi, k_0 \in \mathbb{R}$, by

$$\sqrt{\xi^2 - k_0^2} = \begin{cases} \sqrt{\xi^2 - k_0^2}, & \text{if } |\xi| \geq k_0, \\ -\mathrm{i}\sqrt{k_0^2 - \xi^2}, & \text{if } |\xi| < k_0. \end{cases}$$

We have the following remark for the impedance Green's function.

Remark 1. Note that

$$\int_{-\infty}^{\infty} e^{-\sqrt{\xi^2 - k_0^2}|y_0 - y|} \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi = \mathrm{i}\pi H_0^{(1)}(k_0|\mathbf{x} - \mathbf{x_0}|),$$

and

$$\int_{-\infty}^{\infty} e^{-\sqrt{\xi^2 - k_0^2}(y_0 + y)} \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi = \mathrm{i}\pi H_0^{(1)}(k_0|\mathbf{x} - \bar{\mathbf{x}}_\mathbf{0}|),$$

where $\bar{\mathbf{x}}_\mathbf{0} = (x_0, -y_0)$ and $H_0^{(1)}$ denotes the zeroth-order Hankel function of the first kind [50]. When $\rho = +\infty\mathrm{i}$, we have the Green's function of the half-plane Helmholtz operator with the Dirichlet boundary condition

$$\begin{aligned} G_{+\infty\mathrm{i}}(\mathbf{x}, \mathbf{x_0}) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(e^{-\sqrt{\xi^2 - k_0^2}|y_0 - y|} - e^{-\sqrt{\xi^2 - k_0^2}(y_0 + y)} \right) \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi \\ &= \frac{\mathrm{i}}{4} \left[H_0^{(1)}(k_0|\mathbf{x} - \mathbf{x_0}|) - H_0^{(1)}(k_0|\mathbf{x} - \bar{\mathbf{x}}_\mathbf{0}|) \right]. \end{aligned}$$

When $\rho = 0$, we have the Green's function of the half-plane Helmholtz operator with the Neumann boundary condition

$$\begin{aligned} G_0(\mathbf{x}, \mathbf{x_0}) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(e^{-\sqrt{\xi^2 - k_0^2}|y_0 - y|} + e^{-\sqrt{\xi^2 - k_0^2}(y_0 + y)} \right) \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi \\ &= \frac{\mathrm{i}}{4} \left[H_0^{(1)}(k_0|\mathbf{x} - \mathbf{x_0}|) + H_0^{(1)}(k_0|\mathbf{x} - \bar{\mathbf{x}}_\mathbf{0}|) \right]. \end{aligned}$$

Now we derive the interior problem in the cavity. By the second Green's scalar theorem, we have

$$u^s(\mathbf{x}) = \oint_{\Gamma_\infty} \left(G_\rho(\mathbf{x}, \mathbf{x_0}) \frac{\partial u^s}{\partial n}(\mathbf{x_0}) - u^s(\mathbf{x_0}) \frac{\partial G_\rho(\mathbf{x}, \mathbf{x_0})}{\partial n(\mathbf{x_0})} \right) \mathrm{d}s(\mathbf{x_0}), \qquad \mathbf{x} \in \mathbb{R}^2_+,$$

where Γ_∞ denotes the contour that encloses $\mathbb{R}^2_+$. The contour integral over Γ_∞ consists of a line integral along the horizontal axis from $-\infty$ to $+\infty$ and another line integral over the upper half-circle whose radius extends to infinity. Since both u^s and G_ρ satisfy the Sommerfeld radiation condition, the line integral over the upper half-circle vanishes. Since both u^s and G_ρ satisfy the impedance boundary condition on Γ^C, the line integral over Γ^C vanishes. Thus

$$\begin{aligned} u^s(\mathbf{x}) &= \int_{\Gamma} \left(G_\rho(\mathbf{x}, \mathbf{x_0}) \frac{\partial u^s}{\partial n}(\mathbf{x_0}) - u^s(\mathbf{x_0}) \frac{\partial G_\rho(\mathbf{x}, \mathbf{x_0})}{\partial n(\mathbf{x_0})} \right) \mathrm{d}s(\mathbf{x_0}) \\ &= \int_{\Gamma} G_\rho(\mathbf{x}, \mathbf{x_0}) \left(\frac{\partial u^s}{\partial n}(\mathbf{x_0}) - \rho u^s(\mathbf{x_0}) \right) \mathrm{d}s(\mathbf{x_0}), \qquad \mathbf{x} \in \mathbb{R}^2_+. \end{aligned} \tag{11}$$

Let $\mathbf{x} = (x, y) \to (x, 0) \in \Gamma$ from $\mathbb{R}^2_+$. We obtain the boundary integral representation (the single layer potential is continuous up to the boundary Γ)

$$u^s(\mathbf{x}) = \int_{\Gamma} G_\rho(\mathbf{x}, \mathbf{x_0}) \left(\frac{\partial u^s}{\partial n}(\mathbf{x_0}) - \rho u^s(\mathbf{x_0}) \right) \mathrm{d}s(\mathbf{x_0}), \qquad \mathbf{x} \in \Gamma.$$

By $u = g + u^s$, the impedance boundary conditions for g and G_ρ on Γ, and the field continuity conditions (see (10.17) and (10.53) of [47]),

$$u|_{y=0^+} = u|_{y=0^-}, \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0^+} = \frac{1}{a} \left.\frac{\partial u}{\partial y}\right|_{y=0^-},$$

we have the impedance-to-Dirichlet type nonlocal boundary condition

$$u(\mathbf{x}) = g(\mathbf{x}) - \int_{\Gamma} G_\rho(\mathbf{x}, \mathbf{x_0}) \left(\frac{1}{a} \frac{\partial u}{\partial y}(\mathbf{x_0}) + \rho u(\mathbf{x_0}) \right) \mathrm{d}s(\mathbf{x_0}), \qquad \mathbf{x} \in \Gamma.$$

Therefore, the problem (6)-(7) reduces to the following interior problem

$$\begin{cases} \nabla \cdot \left(\frac{1}{a} \nabla u \right) + k_0^2 b u = 0, & \text{in } \Omega, \\ \frac{1}{a} \frac{\partial u}{\partial n} - \rho u = 0, & \text{on } S, \\ u(\mathbf{x}) = g(\mathbf{x}) - \int_{\Gamma} G_\rho(\mathbf{x}, \mathbf{x_0}) \left(\frac{1}{a} \frac{\partial u}{\partial y}(\mathbf{x_0}) + \rho u(\mathbf{x_0}) \right) \mathrm{d}s(\mathbf{x_0}), & \text{on } \Gamma, \end{cases} \tag{12}$$

where

$$\begin{aligned} G_\rho(\mathbf{x}, \mathbf{x_0}) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\rho - \sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi \\ &= -\frac{1}{\pi} \int_0^{\infty} \frac{\cos((x_0 - x)\xi)}{\rho - \sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi, \quad \forall \mathbf{x}, \mathbf{x_0} \in \Gamma. \end{aligned}$$

We mention that mathematical analysis of the special case $\rho = 0$ of (12) has been presented in [2], where existence and uniqueness of solutions have been proved in a Hilbert space consisting of functions satisfying the homogeneous Neumann-to-Dirichlet condition on Γ. Here we prove the well-posedness of (12) with a different approach, by introducing a new variable $w = a^{-1}\partial_y u$ as unknown, which allows general inhomogeneous impedence-to-Dirichlet boundary condition on Γ.

4 | WELL-POSEDNESS OF THE REDUCED PROBLEM

For $s \in \mathbb{R}$, let $H^s(\mathbb{R})$ denote the space of tempered distributions w with Fourier transform $\widehat{w} \in L^2_{\mathrm{loc}}(\mathbb{R})$, equipped with the norm

$$\|w\|_{H^s(\mathbb{R})} := \left(\int_{-\infty}^{\infty} (1+|\xi|^2)^s |\widehat{w}(\xi)|^2 \mathrm{d}\xi \right)^{1/2}.$$

The Sobolev space $H^s(\Gamma)$ is defined by

$$H^s(\Gamma) := \{u \in (C_0^\infty(\Gamma))' : u = w|_\Gamma \text{ for some } w \in H^s(\mathbb{R})\},$$

equipped with the norm

$$\|u\|_{H^s(\Gamma)} = \inf\{\|w\|_{H^s(\mathbb{R})} : w \in H^s(\mathbb{R}), w|_\Gamma = u\}.$$

We denote by $\widetilde{H}^s(\Gamma)$ the space of functions $w \in H^s(\Gamma)$ whose zero extension

$$\widetilde{w}(x) = \begin{cases} w(x) & \text{if } x \in \Gamma, \\ 0 & \text{if } x \in \mathbb{R}\backslash\Gamma, \end{cases}$$

is in $H^s(\mathbb{R})$, equipped with the norm

$$\|w\|_{\widetilde{H}^s(\Gamma)} := \|\widetilde{w}\|_{H^s(\mathbb{R})}. \tag{13}$$

Then we have the following properties [51]:

$$\widetilde{H}^s(\Gamma) = (H^{-s}(\Gamma))', \qquad H^s(\Gamma) = (\widetilde{H}^{-s}(\Gamma))'.$$

where $\widetilde{w}$ denotes the extension of w by zero outside Γ. In what follows we mainly consider the cases $s = \pm 1/2$. In this case $\widetilde{H}^{\frac{1}{2}}(\Gamma) \simeq H_{00}^{\frac{1}{2}}(\Gamma)$, where the latter is often referred to as the Lions–Magenes Space (cf. [52, pp. 159–161]).

To simplify the notation, we define

$$m(\xi) := \frac{1}{\sqrt{\xi^2 - k_0^2 - \rho}}$$

and define the operator $\mathbf{G}_\rho : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ by

$$\mathbf{G}_\rho w(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} m(\xi) \widehat{\widetilde{w}}(\xi) e^{\mathrm{i}x\xi} \mathrm{d}\xi,$$

where

$$\widehat{\widetilde{w}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{w}(x) e^{-\mathrm{i}x\xi} \mathrm{d}x.$$

For any given $g \in H^{1/2}(\Gamma)$, we seek a solution $u \in H^1(\Omega)$ of the problem (12) in certain weak sense such that $a^{-1}\partial_y u + \rho u$ on Γ is in $\widetilde{H}^{-1/2}(\Gamma)$. Note that the conditions $u \in H^1(\Omega)$ and $a^{-1}\partial_y u + \rho u \in \widetilde{H}^{-1/2}(\Gamma)$ are equivalent to $u \in H^1(\Omega)$ and $a^{-1}\partial_y u \in \widetilde{H}^{-1/2}(\Gamma)$. To avoid technical difficulties in defining $a^{-1}\partial_y u$ for a given function $u \in H^1(\Omega)$, we shall introduce a new variable w to denote $a^{-1}\partial_y u$.

For $(u, w) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$ and $(v, \varphi) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$ we define the linear form

$$l(v, \varphi) = \int_\Gamma g\varphi \mathrm{d}x$$

and the bilinear form

$$\begin{aligned} B(u, w; v, \varphi) = &\int_\Omega \left(\frac{1}{a} \nabla u \cdot \nabla v - k_0^2 buv \right) \mathrm{d}x\mathrm{d}y - \int_S \rho uv \mathrm{d}s - \int_\Gamma wv \mathrm{d}x \\ &+ \int_\Gamma \mathbf{G}_\rho(w + \rho u)\, \varphi \mathrm{d}x + \int_\Gamma u\varphi \mathrm{d}x \end{aligned}$$

where ds denotes the length element on the contour S. The weak formulation of (12) is to find $(u, w) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$ such that

$$B(u, w; v, \varphi) = l(v, \varphi), \quad \forall\, (v, \varphi) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma). \tag{14}$$

Theorem 1 (Existence and uniqueness)**.** Suppose that $k_0 > 0$, ρ is a complex constant with either $\mathrm{Im}(\rho) > 0$ or $\rho = 0$, and $a, b \in L^\infty(\Omega)$ are complex scalar functions such that $\mathrm{Re}(a) \geq a_0 > 0$, $\mathrm{Im}(a) \geq 0$, $\mathrm{Re}(b) \geq b_0 > 0$, $\mathrm{Im}(b) \geq 0$. Then, there exists a unique solution $(u, w) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$ for the problem (14).

The proof of Theorem 1 relies on the following lemma, which is concerned with the mapping properties of the operator $\mathbf{G}_\rho$.

Lemma 1. *If* $\mathrm{Im}(\rho) > 0$ *or* $\rho = 0$*, then the operator* $\mathbf{G}_\rho : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is bounded and, for* $w, \varphi \in \widetilde{H}^{-1/2}(\Gamma)$*,*

$$\left| \int_\Gamma \mathbf{G}_\rho w(x) \overline{\varphi}(x) \mathrm{d}x \right| \leq C \|w\|_{\widetilde{H}^{-1/2}(\Gamma)} \|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)},$$

and

$$\mathrm{Re} \int_\Gamma \mathbf{G}_\rho w(x) \overline{w}(x) \mathrm{d}x \geq \alpha \|w\|^2_{\widetilde{H}^{-1/2}(\Gamma)} - \beta \|\widetilde{w}\|^2_{H^{-1}(\mathbb{R})},$$

where C*,* α *and* β *are some positive constants.*

Proof. Let $\xi_\rho = \sqrt{k_0^2 + |\rho|^2} + 1$. If $|\xi| \geq \xi_\rho$, then,

$$\frac{1}{(1 + |\xi|^2)^{1/2}} \lesssim \mathrm{Re}(m(\xi)) \leq |m(\xi)| \lesssim \frac{1}{(1 + |\xi|^2)^{1/2}}. \tag{15}$$

For any $w \in \widetilde{H}^{-1/2}(\Gamma)$, it follows from (15) that

$$\begin{aligned} \int_{|\xi| \geq \xi_\rho} |m(\xi)| |\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi &\lesssim \int_{|\xi| \geq \xi_\rho} \frac{|\widehat{\widetilde{w}}(\xi)|^2}{(1 + |\xi|^2)^{1/2}} \mathrm{d}\xi \\ &\lesssim \int_{-\infty}^{\infty} \frac{|\widehat{\widetilde{w}}(\xi)|^2}{(1 + |\xi|^2)^{1/2}} \mathrm{d}\xi = \|\widetilde{w}\|^2_{H^{-1/2}(\mathbb{R})}. \end{aligned} \tag{16}$$

If $|\xi| < \xi_\rho$ and $\mathrm{Im}(\rho) > 0$, then, $|m(\xi)| \lesssim 1$, which implies that

$$\begin{aligned} \int_{|\xi| < \xi_\rho} |m(\xi)| |\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi &\lesssim \int_{|\xi| < \xi_\rho} |\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi \\ &\lesssim \int_{-\infty}^{\infty} \frac{|\widehat{\widetilde{w}}(\xi)|^2}{1 + |\xi|^2} \mathrm{d}\xi = \|\widetilde{w}\|^2_{H^{-1}(\mathbb{R})} \lesssim \|\widetilde{w}\|^2_{H^{-1/2}(\mathbb{R})}. \end{aligned} \tag{17}$$

If $|\xi| < \xi_\rho$ and $\rho = 0$, then $|m(\xi)| = |\xi^2 - k_0^2|^{-1/2}$. Let $\psi \in C_0^\infty(\mathbb{R})$ be a nonnegative function so that $\widehat{\psi}$ is a Schwartz function and

$$\widehat{\psi}(0) = \int_{-\infty}^{\infty} \psi(x) \mathrm{d}x > 0.$$

Let $\psi_R(x) = 2R\psi(Rx)/\widehat{\psi}(0)$. Then, $\widehat{\psi}_R(\xi) = 2\widehat{\psi}(\xi/R)/\widehat{\psi}(0)$ and $\widehat{\psi}_R(0) = 2$. Let R be a sufficiently large constant such that ψ_R has compact support in $[-1, 1]$, and for $|\xi| < \xi_\rho$, $\widehat{\psi}_R(\xi) \geq 1$. Choose $\widehat{\chi}(\xi) \in C_0^\infty(\mathbb{R})$ such that $\widehat{\chi}(\xi) = 1$ for $|\xi| < \xi_\rho$ and

$\widehat{\chi}(\xi) = 0$ for $|\xi| > 2\xi_\rho$. Then, we have

$$
\begin{aligned}
\int\limits_{|\xi|<\xi_\rho} |m(\xi)||\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi &= \int\limits_{|\xi|<\xi_\rho} \frac{|\widehat{\chi}(\xi)\widehat{\widetilde{w}}(\xi)|^2}{\sqrt{|\xi^2 - k_0^2|}} \mathrm{d}\xi \\
&\leq \int\limits_{|\xi|<\xi_\rho} \frac{|\widehat{\chi}(\xi)\widehat{\psi}_R(\xi)\widehat{\widetilde{w}}(\xi)|^2}{\sqrt{|\xi^2 - k_0^2|}} \mathrm{d}\xi \\
&\lesssim \|\widehat{\chi}\widehat{\psi}_R\widehat{\widetilde{w}}\|^2_{L^6(\mathbb{R})} \Bigg(\int\limits_{|\xi|<\xi_\rho} |\xi^2 - k_0^2|^{-3/4} \mathrm{d}\xi \Bigg)^{2/3} \\
&\lesssim \|\widehat{\chi}\widehat{\psi}_R\widehat{\widetilde{w}}\|^2_{L^6(\mathbb{R})} \\
&\lesssim \|\chi * (\psi_R * \widetilde{w})\|^2_{L^{6/5}(\mathbb{R})} \quad (* \text{ denotes the convolution operator}) \\
&\lesssim \|\psi_R * \widetilde{w}\|^2_{L^{6/5}(\mathbb{R})} \quad (\text{since } \chi \text{ is a Schwartz function}) \\
&\lesssim \|\psi_R * \widetilde{w}\|^2_{L^2(\mathbb{R})} \quad (\text{since } \psi_R * \widetilde{w} \text{ has compact support}) \\
&= \|\widehat{\psi}_R\widehat{\widetilde{w}}\|^2_{L^2(\mathbb{R})} \\
&\lesssim \|\widetilde{w}\|^2_{H^{-1}(\mathbb{R})} \quad (\text{since } \widehat{\psi}_R \text{ is a Schwartz function}) \\
&\lesssim \|\widetilde{w}\|^2_{H^{-1/2}(\mathbb{R})}.
\end{aligned} \tag{18}
$$

It follows from (16), (17), (18) and (13) that, for $w \in \widetilde{H}^{-1/2}(\Gamma)$,

$$
\begin{aligned}
\int_{-\infty}^{\infty} |m(\xi)||\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi &= \int\limits_{|\xi|\geq\xi_\rho} |m(\xi)||\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi + \int\limits_{|\xi|<\xi_\rho} |m(\xi)||\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi \\
&\lesssim \|\widetilde{w}\|^2_{\widetilde{H}^{-1/2}(\mathbb{R})} = \|w\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.
\end{aligned}
$$

Therefore, for $w, \varphi \in \widetilde{H}^{-1/2}(\Gamma)$,

$$
\begin{aligned}
\left| \int_\Gamma \mathbf{G}_\rho w(x)\overline{\varphi}(x)\mathrm{d}x \right| &= \left| \int_{-\infty}^{\infty} m(\xi)\widehat{\widetilde{w}}(\xi)\overline{\widehat{\widetilde{\varphi}}}(\xi)\mathrm{d}\xi \right| \\
&\leq \left(\int_{-\infty}^{\infty} |m(\xi)||\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi \right)^{1/2} \left(\int_{-\infty}^{\infty} |m(\xi)||\widehat{\widetilde{\varphi}}(\xi)|^2 \mathrm{d}\xi \right)^{1/2} \\
&\leq C\|w\|_{\widetilde{H}^{-1/2}(\Gamma)}\|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)}.
\end{aligned}
$$

This proves the first inequality of Lemma 1.

Since $\mathrm{Re}(m(\xi)) \geq \alpha(1 + |\xi|^2)^{-1/2}$ for $|\xi| \geq \xi_\rho$, where α denotes some positive constant, it follow that

$$
\begin{aligned}
\mathrm{Re} \int_\Gamma \mathbf{G}_\rho w(x)\overline{w}(x)\mathrm{d}x &= \int\limits_{|\xi|\geq\xi_\rho} \mathrm{Re}(m(\xi))|\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi + \int\limits_{|\xi|<\xi_\rho} \mathrm{Re}(m(\xi))|\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi \\
&\geq \alpha \int_{-\infty}^{\infty} \frac{|\widehat{\widetilde{w}}(\xi)|^2}{(1 + |\xi|^2)^{1/2}} \mathrm{d}\xi - \int\limits_{|\xi|<\xi_\rho} (\alpha + |m(\xi)|)|\widehat{\widetilde{w}}(\xi)|^2 \mathrm{d}\xi \\
&\geq \alpha \int_{-\infty}^{\infty} \frac{|\widehat{\widetilde{w}}(\xi)|^2}{(1 + |\xi|^2)^{1/2}} \mathrm{d}\xi - \beta \int_{-\infty}^{\infty} \frac{|\widehat{\widetilde{w}}(\xi)|^2}{1 + |\xi|^2} \mathrm{d}\xi \\
&= \alpha\|w\|^2_{\widetilde{H}^{-1/2}(\Gamma)} - \beta\|\widetilde{w}\|^2_{H^{-1}(\mathbb{R})},
\end{aligned}
$$

where β is some positive constant. This proves the second inequality of Lemma 1. □

Now we are ready to prove Theorem 1. The proof consists of two steps.

Part 1: Uniqueness

It suffices to show that the homogeneous equation

$$B(u, w; v, \varphi) = 0 \ \text{ for any } v \in H^1(\Omega) \text{ and } \varphi \in \widetilde{H}^{-1/2}(\Gamma) \tag{19}$$

admits only the zero solution $(u, w) = (0, 0)$.

Substituting $v = 0$ into the equation (19), we obtain $u = -\mathbf{G}_\rho \phi$ on Γ where $\phi = w + \rho u$. Substituting $v = \overline{u}$ and $\varphi = -\overline{\phi}$ into the equation (19), we have

$$\begin{aligned}
&\int\limits_{\Omega} \left(\frac{1}{a} \nabla u \cdot \overline{\nabla u} - k_0^2 b |u|^2 \right) \mathrm{d}x\mathrm{d}y \\
&= \int\limits_{S} \rho |u|^2 \mathrm{d}s + 2\mathrm{Re} \int\limits_{\Gamma} w\overline{u}\mathrm{d}x + \int\limits_{\Gamma} (\mathbf{G}_\rho \phi)\overline{\phi}\mathrm{d}x + \int\limits_{\Gamma} \overline{\rho} |\mathbf{G}_\rho \phi|^2 \mathrm{d}x
\end{aligned} \tag{20}$$

Considering the imaginary part of (20), we get

$$\begin{aligned}
&\int\limits_{\Omega} \left(-\frac{\mathrm{Im}(a)}{|a|^2} \nabla u \cdot \overline{\nabla u} - k_0^2 \mathrm{Im}(b) |u|^2 \right) \mathrm{d}x\mathrm{d}y \\
&= \int\limits_{S} \mathrm{Im}(\rho) |u|^2 \mathrm{d}s + \mathrm{Im} \left(\int\limits_{\Gamma} (\mathbf{G}_\rho \phi)\overline{\phi}\mathrm{d}x + \overline{\rho} \int\limits_{\mathbb{R}} |\mathbf{G}_\rho \phi|^2 \mathrm{d}x \right) + \int\limits_{\mathbb{R}\backslash\Gamma} \mathrm{Im}(\rho) |\mathbf{G}_\rho \phi|^2 \mathrm{d}x,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\int\limits_{\Omega} \left(-\frac{\mathrm{Im}(a)}{|a|^2} \nabla u \cdot \overline{\nabla u} - k_0^2 \mathrm{Im}(b) |u|^2 \right) \mathrm{d}x\mathrm{d}y \\
&= \mathrm{Im}(\rho) \int\limits_{S} |u|^2 \mathrm{d}s + \int\limits_{|\xi|<k_0} \sqrt{k_0^2 - \xi^2} |m(\xi)|^2 |\widehat{\widetilde{\phi}}(\xi)|^2 \mathrm{d}\xi + \mathrm{Im}(\rho) \int\limits_{\mathbb{R}\backslash\Gamma} |\mathbf{G}_\rho \phi|^2 \mathrm{d}x.
\end{aligned}$$

With $\mathrm{Im}(a) \geq 0$, $\mathrm{Im}(b) \geq 0$ and $\mathrm{Im}(\rho) \geq 0$, the above equation implies that $\widehat{\widetilde{\phi}}(\xi) = 0$ for $|\xi| < k_0$. Since $\widetilde{\phi}$ has compact support, its Fourier transform $\widehat{\widetilde{\phi}}$ is an entire analytic function, which means that $\widehat{\widetilde{\phi}}(\xi) = 0$ for all $\xi \in \mathbb{R}$. This proves that $\widetilde{\phi} = 0$, which also implies that $u = -\mathbf{G}_\rho \phi = 0$ and $w = \phi - \rho u = 0$ on Γ. Now (19) reduces to

$$\int\limits_{\Omega} \left(\frac{1}{a} \nabla u \cdot \nabla v - k_0^2 b u v \right) \mathrm{d}x\mathrm{d}y - \int\limits_{S} \rho u v \mathrm{d}s = 0, \quad \text{for any } \ v \in H^1(\Omega),$$

which implies that

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{a} \nabla u \right) + k_0^2 b u = 0, \ \text{in} \ \ \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial y} = 0, \qquad \quad \text{on } \ \Gamma. \end{cases}$$

Then the strong unique continuation theorem [53] implies $u = 0$ in Ω.

Part 2: Existence

It is easy to show that, for any $(u, w), (v, \varphi) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$,

$$|B(u, w; \overline{v}, \overline{\varphi})| \leq C \left(\|u\|_{H^1(\Omega)} + \|w\|_{\widetilde{H}^{-1/2}(\Gamma)} \right) \left(\|v\|_{H^1(\Omega)} + \|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)} \right).$$

It is obvious that $l(\cdot,\cdot)$ is a continuous linear functional on $H^1(\Omega)\times\widetilde{H}^{-1/2}(\Gamma)$. By Lemma 1, we have

$$
\begin{aligned}
&\mathrm{Re}(B(u,w;\overline{u},\overline{w}))\\
=&\mathrm{Re}\int\limits_{\Omega}\left(\frac{1}{a}\nabla u\cdot\overline{\nabla u}-k_0^2b|u|^2\right)\mathrm{d}x\mathrm{d}y-\mathrm{Re}\int\limits_{S}\rho|u|^2\mathrm{d}s+\mathrm{Re}\int\limits_{\Gamma}(\mathbf{G}_\rho(w+\rho u))\overline{w}\mathrm{d}x\\
\geq& a_0\|a\|^{-2}_{L^\infty(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}-k_0^2\|b\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)}-|\rho|\|u\|^2_{L^2(S)}\\
&+\alpha\|w\|^2_{\widetilde{H}^{-1/2}(\Gamma)}-\beta\|\widetilde{w}\|^2_{H^{-1}(\mathbb{R})}-C\|\rho u\|_{\widetilde{H}^{-1/2}(\Gamma)}\|w\|_{\widetilde{H}^{-1/2}(\Gamma)}\\
\geq& a_0\|a\|^{-2}_{L^\infty(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}-k_0^2\|b\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)}-|\rho|\|u\|^2_{L^2(S)}\\
&+\frac{\alpha}{2}\|w\|^2_{\widetilde{H}^{-1/2}(\Gamma)}-\beta\|\widetilde{w}\|^2_{H^{-1}(\mathbb{R})}-\frac{C^2}{2\alpha}\|\rho u\|^2_{\widetilde{H}^{-1/2}(\Gamma)}\\
\geq&\alpha_0\left(\|u\|^2_{H^1(\Omega)}+\|w\|^2_{\widetilde{H}^{-1/2}(\Gamma)}\right)-\beta_0\left(\|u\|^2_{L^2(\Omega)}+\|\widetilde{w}\|^2_{H^{-1}(\mathbb{R})}\right),
\end{aligned}
\tag{21}
$$

where α_0 and β_0 are some positive constants.

Let $X=H^1(\Omega)\times\widetilde{H}^{-1/2}(\Gamma)$ and let $\mathbf{A}:X\to X'$ be defined by

$$
\langle\mathbf{A}(u,w),(v,\varphi)\rangle=B(u,w;v,\varphi)+\beta_0\int\limits_{\Omega}uv\mathrm{d}x\mathrm{d}y+\beta_0\int\limits_{\Gamma}(\mathbf{B}w)\varphi\mathrm{d}x,
\tag{22}
$$

where β_0 is given in (21), and $\mathbf{B}$ is a compact operator from $\widetilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ defined by

$$
\mathbf{B}w(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{\widehat{\widetilde{w}}(\xi)}{1+|\xi|^2}e^{\mathrm{i}x\xi}\mathrm{d}\xi.
$$

From (21) we see that the bilinear form on the right-hand side of (22) is strongly elliptic, which implies that the operator $\mathbf{A}:X\to X'$ is well defined and invertible. Define the compact operator $\mathbf{C}:X\to X'$ by

$$
\mathbf{C}(u,w):=(u,\mathbf{B}w).
$$

Consider the operator $\mathbf{L}=\mathbf{A}-\beta_0\mathbf{C}$, which maps X into X'. Clearly,

$$
\langle\mathbf{L}(u,w),(v,\varphi)\rangle=B(u,w;v,\varphi),\quad\text{for any }(u,w),(v,\varphi)\in X.
$$

By the uniqueness of solution for the problem (19), we know that the null space of $\mathbf{L}$ only consists of zero. Since $\mathbf{I}-\beta_0\mathbf{A}^{-1}\mathbf{C}=\mathbf{A}^{-1}\mathbf{L}$, then the null space of the Fredholm operator $\mathbf{I}-\beta_0\mathbf{A}^{-1}\mathbf{C}$ only consists of zero, which means $\mathbf{I}-\beta_0\mathbf{A}^{-1}\mathbf{C}$ is invertible. Therefore, the operator $\mathbf{L}$ is invertible and the existence of the solution for the problem (14) follows.

The proof of Theorem 1 is complete.

5 | NUMERICAL SIMULATION

The physical parameter of interest is the RCS defined by

$$
\sigma(\vartheta)=\lim_{r\to\infty}2\pi r\frac{|u^s(r\cos\vartheta,r\sin\vartheta)|^2}{|u^i|^2}
$$

where ϑ is the observation angle with respect to the positive x-axis. When the incident and observation directions are the same $(\theta=\vartheta)$, we have the backscatter RCS

$$
\text{Backscatter RCS}(\vartheta)=10\log_{10}\sigma(\vartheta)\text{ dB}.
$$

By (11), the impedance boundary condition, the field continuity conditions, and the far field behavior of the impedance Green' function G_ρ, we can evaluate $\sigma(\vartheta)$ as

$$
\sigma(\vartheta)=\frac{4}{k_0}|P(\vartheta)|^2,
$$

where $P(\vartheta)$ is the far-field coefficient given by

$$
P(\vartheta)=\frac{1}{2}\frac{\mathrm{i}k_0\sin\vartheta}{\rho+\mathrm{i}k_0\sin\vartheta}\int\limits_{\Gamma}\left(\frac{1}{a}\frac{\partial u}{\partial y}+\rho u\right)\mathrm{e}^{\mathrm{i}k_0x\cos\vartheta}\mathrm{d}x.
$$

In the following we present finite element simulations for calculating the RCS from a rectangular cavity $\Omega = (0, L) \times (-D, 0)$ based on the weak formulation (14).

Let Ω be partitioned into regular triangles τ_j, $j = 1, \ldots, \mathcal{N}$, which also yields a uniform partition of Γ into intervals I_j, $j = 1, \ldots, \mathcal{M}$. Let

$$H_h^1(\Omega) = \{v \in C(\Omega) : v|\tau_j \in P_1(\tau_j),\ j = 1, \ldots, \mathcal{N}\} \subseteq H^1(\Omega),$$
$$\widetilde{H}_h^{-1/2}(\Gamma) = \{w \in C(\Gamma) : v|\tau_j \in P_1(I_j),\ j = 1, \ldots, \mathcal{M}\} \subseteq \widetilde{H}^{-1/2}(\Gamma)$$

be the spaces of piecewise linear finite element basis functions on Ω and Γ respectively. We look for a pair of finite element functions $(u_h, w_h) \in H_h^1(\Omega) \times \widetilde{H}_h^{-1/2}(\Gamma)$ to approximate the solution $(u, w) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$, satisfying the weak form:

$$B(u_h, w_h; v_h, \varphi_h) = l(v_h, \varphi_h), \quad \forall\, (v_h, \varphi_h) \in H_h^1(\Omega) \times \widetilde{H}_h^{-1/2}(\Gamma). \tag{23}$$

Specifically, let $\psi_i(x, y)$ and $\phi_i(x)$ be the basis functions of $H_h^1(\Omega)$ and $\widetilde{H}_h^{-1/2}(\Gamma)$, respectively, and expressed the finite element solution by

$$u_h(x, y) = \sum_{i=1}^{\mathcal{N}} u_i \psi_i(x, y), \quad w_h(x) = \sum_{m=1}^{\mathcal{M}} w_m \phi_m(x),$$

where u_i and w_m are the nodal values of the functions u_h and w_h. These nodal values are stored in the vectors $\mathbf{u}$ and $\mathbf{w}$, respectively.

The mass and stiffness matrices and forcing vectors are assembled as

$$\mathbf{S}^\Omega = \left[S_{ij}^\Omega\right]_{\mathcal{N}\times\mathcal{N}}, \qquad S_{ij}^\Omega = \int_\Omega \frac{1}{a} \nabla\psi_j \cdot \nabla\psi_i \mathrm{d}x\mathrm{d}y$$

$$\mathbf{M}^\Omega = \left[M_{ij}^\Omega\right]_{\mathcal{N}\times\mathcal{N}}, \qquad M_{ij}^\Omega = \int_\Omega b\psi_j\psi_i \mathrm{d}x\mathrm{d}y$$

$$\mathbf{K}^S = \left[K_{ij}^S\right]_{\mathcal{N}\times\mathcal{N}}, \qquad K_{ij}^S = \int_S b\psi_j\psi_i \mathrm{d}s$$

$$\mathbf{K}^\Gamma = \left[K_{im}^\Gamma\right]_{\mathcal{N}\times\mathcal{M}}, \qquad K_{im}^\Gamma = \int_\Gamma b\phi_m\psi_i \mathrm{d}s$$

$$\mathbf{G}^\Gamma = \left[G_{mm'}^\Gamma\right]_{\mathcal{M}\times\mathcal{M}}, \qquad G_{mm'}^\Gamma = \int_\Gamma \int_\Gamma G_\rho(x, x_0)\phi_{m'}(x_0)\phi_m(x)\mathrm{d}x_0\mathrm{d}x$$

$$\mathbf{g}^\Gamma = \left[g_m^\Gamma\right]_{\mathcal{M}\times 1}, \qquad g_m^\Gamma = \int_\Gamma g\phi_m \mathrm{d}s.$$

The matrices $\mathbf{S}^\Omega$, $\mathbf{M}^\Omega$, $\mathbf{K}^S$, $\mathbf{K}^\Gamma$ and the vector $\mathbf{g}^\Gamma$ can be evaluated in the standard way, while the evaluation of the matrix

$$G_{mm'}^\Gamma = -\int_\Gamma \int_\Gamma \left(\frac{1}{\pi} \int_0^\infty \frac{\cos((x - x_0)\xi)}{\rho - \sqrt{\xi^2 - k_0^2}} d\xi \right) \phi_m(x_0)\phi_{m'}(x)\, \mathrm{d}x_0\mathrm{d}x,$$

requires more technical treatment of the improper integral with respect to ξ. Since the kernel $\frac{\cos((x-x_0)\xi)}{\rho-\sqrt{\xi^2-k_0^2}}$ is not absolutely integrable with respect to ξ on $(0, \infty)$, standard quadrature for the interior integral would fail to converge. To overcome this difficulty, we shall write $G_{mm'}^\Gamma$ into an alternative form:

$$G_{mm'}^\Gamma = -\frac{1}{\pi} \int_0^{k_0} \frac{g_{mm'}(\xi)}{\rho + i\sqrt{k_0^2 - \xi^2}} \mathrm{d}\xi - \frac{1}{\pi} \int_{k_0}^\infty \frac{g_{mm'}(\xi)}{\rho - \sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi,$$

with a matrix $g_{mm'}(\xi) = O(\xi^{-4})$ decaying sufficiently fast as $\xi \to \infty$. Then both integrals in the expression above are evaluated numerically. The expression of $g_{mm'}(\xi)$ can be found in appendix.

The global discretized algebraic system is assembled as

$$\mathcal{A}\boldsymbol{v} = \boldsymbol{f}, \tag{24}$$

where

$$\mathcal{A}=\begin{bmatrix}\mathbf{S}^{\Omega}-k_0^2\mathbf{M}^{\Omega}-\rho\mathbf{K}^{S} & -\mathbf{K}^{\Gamma}\\ \left(\mathbf{K}^{\Gamma}\right)^{\mathrm{T}}+\rho\mathbf{G}^{\Gamma}\mathbf{R}^{\Gamma} & \mathbf{G}^{\Gamma}\end{bmatrix},\quad \boldsymbol{v}=\begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix},\quad \boldsymbol{f}=\begin{bmatrix}\mathbf{0}\\ \mathbf{g}^{\Gamma}\end{bmatrix}.$$

Here the restriction matrix $\mathbf{R}^{\Gamma}$ projects $\mathbf{u}$ onto boundary Γ. After solving (24), we can compute $P(\vartheta)$ by

$$P(\vartheta)\approx\frac{1}{4}\frac{\mathrm{i}k_0h_x\sin\vartheta}{\rho+\mathrm{i}k_0\sin\vartheta}\sum_{l=1}^{M-1}\Big(\big(w_l+\rho u(x_l,0)\big)\,\mathrm{e}^{\mathrm{i}k_0x_l\cos\vartheta}+\big(w_{l+1}+\rho u(x_{l+1},0)\big)\,\mathrm{e}^{\mathrm{i}k_0x_{l+1}\cos\vartheta}\Big).$$

We report computational results for a rectangular cavity with 1 meter wide and 0.25 meter deep ($L=1.0$ and $D=0.25$). Our focus is on the efficiency of the proposed model and the finite element method for RCS calculation. Two different cases (see Figure 2) are considered.

FIGURE 2 The empty cavity (left) and the filled cavity (right).

(i) The cavity is empty, i.e., $a(x,y)=b(x,y)=1$ in Ω. This is a standard test problem [19, 47]. The magnitudes of the total fields at normal incidence ($\theta=\pi/2$) on the aperture and backscatter RCS of the empty cavity for $k_0=2\pi$ and difference values of the impedance parameter ρ are given in Figure 3 . Numerical results are obtained by using our finite element method with $M=129,\mathcal{N}=16641$ and the method given in [19]. From the computational results, we observe that as ρ approaches to 0, the total field approaches to the Neumann case. The magnitude of the total field at normal incidence on the aperture of the empty cavity and the backscatter RCS for $k_0=4\pi$ and $\rho=\mathrm{i}k_0$ are given in Figure 4 . The total field at normal incidence in the empty cavity can also be visualized in Figure 5 .

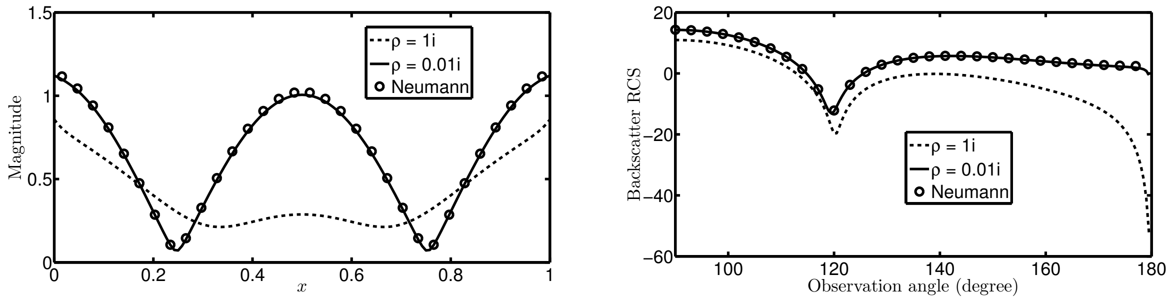

FIGURE 3 Aperture field (left) at normal incidence $\theta=\pi/2$ and backscatter RCS (right) for the empty cavity with $k_0=2\pi$. For $\rho=1\mathrm{i},0.01\mathrm{i}$, the results are obtained via the proposed finite element method. For the Neumann cases, the result is obtained via the method in [19].

(ii) There is a target inside the cavity. The parameters $a(x,y)$ and $b(x,y)$ in Ω are defined as follows:

$$a(x,y)=\begin{cases}4+\mathrm{i}, & 0.2<x<0.8,\quad -0.25<y<-0.20,\\ 1, & \text{otherwise},\end{cases}$$
$$b(x,y)=1.$$

The magnitude of the total field at normal incidence on the aperture of the filled cavity and the backscatter RCS for $k_0=4\pi$ and $\rho=\mathrm{i}k_0$ are given in Figure 6 . Numerical results are obtained by using our finite element method with $M=129,\mathcal{N}=16641$.

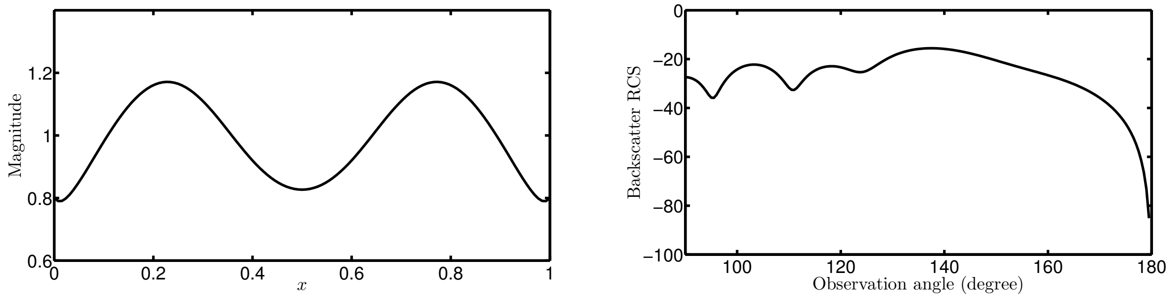

FIGURE 4 Aperture field (left) at normal incidence $\theta = \pi/2$ and backscatter RCS (right) for the empty cavity with $k_0 = 4\pi$ and $\rho = \mathrm{i}k_0$.

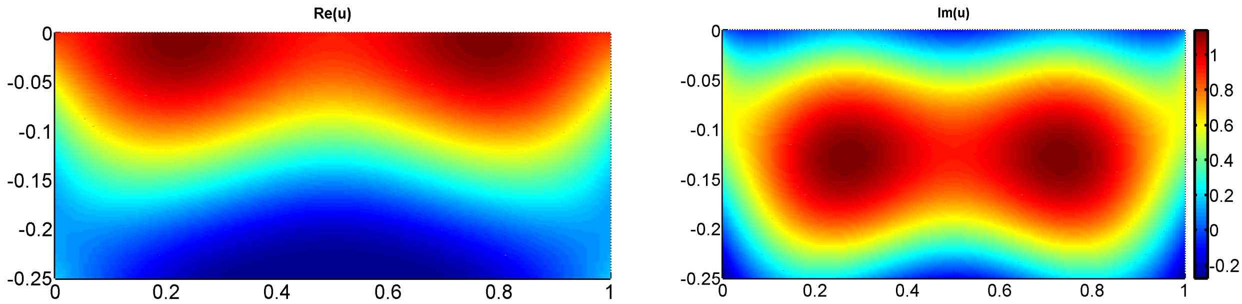

FIGURE 5 Total field (real and imaginary parts) at normal incidence $\theta = \pi/2$ for the empty cavity with $k_0 = 4\pi$ and $\rho = \mathrm{i}k_0$.

Compared with the empty case (Figure 4), both the magnitude of the aperture field and the backscatter RCS are significantly different. The total field at normal incidence in the filled cavity can also be visualized in Figure 7 .

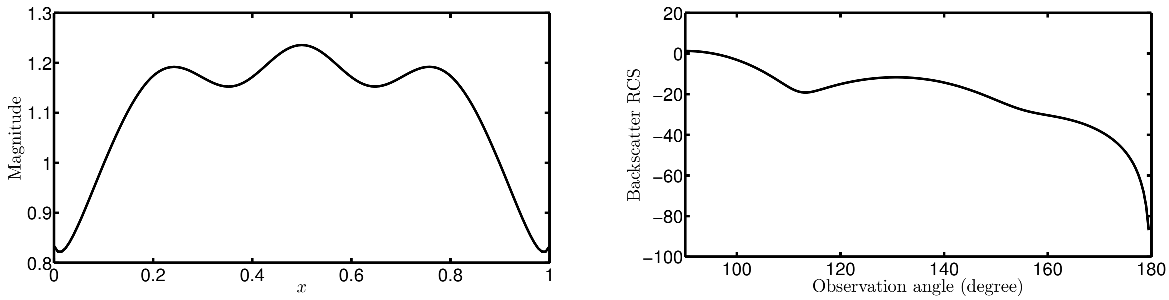

FIGURE 6 Aperture field (left) at normal incidence $\theta = \pi/2$ and backscatter RCS (right) for the filled cavity with $k_0 = 4\pi$ and $\rho = \mathrm{i}k_0$.

6 | CONCLUDING REMARKS

We have proposed a bounded domain model for the scattering from two dimensional cavities embedded in an impedance ground plane. It is shown that a unique weak solution exists. For the calculation of RCS, it is sufficient to solve the problem in the cavity because of the homogeneous medium in the upper half-plane. A finite element method is given to solve the problem in the cavity. Our algorithm has the advantages of being simple in structures and easy to implement.

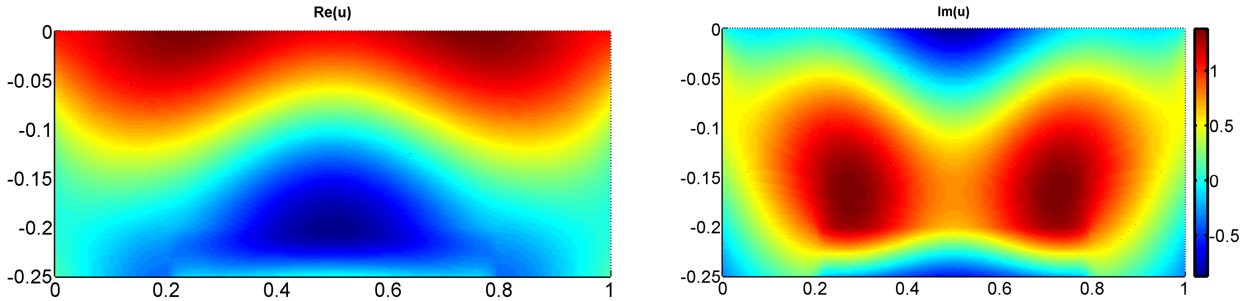

FIGURE 7 Total field (real and imaginary parts) at normal incidence $\theta = \pi/2$ for the filled cavity with $k_0 = 4\pi$ and $\rho = \mathrm{i}k_0$.

The problem with large wave numbers is of significant interest, but the computation is especially challenging [17, 30] because of the highly oscillatory nature of the fields. Low-order methods often require much more mesh points per wavelength due to the pollution effect [54, 55, 56] of the computed solutions, therefore, extremely large scale indefinite linear systems occur. It is well known that high-order methods are more attractive for solving Helmholtz problems with large wave numbers since they can offer relative higher accurate solutions by utilizing fewer mesh points; e.g., see [57, 58, 21, 59, 60, 61, 62, 63]. Efficient high-order methods and the corresponding fast algorithms for large wave number cavity problems with impedance boundary conditions are being considered.

APPENDIX: EVALUATION OF THE MATRIX $G^{\Gamma}_{MM'}$

To evaluate the matrix

$$G^{\Gamma}_{mm'} = -\frac{1}{\pi}\int_0^{\infty} \frac{1}{\rho - \sqrt{\xi^2 - k_0^2}} \int_{\Gamma}\int_{\Gamma} \cos((x - x_0)\xi)\phi_{m'}(x_0)\phi_m(x)\mathrm{d}x_0\mathrm{d}x d\xi,$$

we denote

$$\begin{aligned} K_{m'}(x,\xi) &= \int_{\Gamma} \cos((x-x_0)\xi)\phi_{m'}(x_0)\mathrm{d}x_0 \\ &= \begin{cases} \dfrac{1-\cos(h_x\xi)}{h_x\xi^2}\cos((x-x_{m'})\xi) + \dfrac{h_x\xi - \sin(h_x\xi)}{h_x\xi^2}\sin((x-x_{m'})\xi), & \text{if } m' = 1 \\ \dfrac{1-\cos(h_x\xi)}{h_x\xi^2}\cos((x-x_{m'})\xi) - \dfrac{h_x\xi - \sin(h_x\xi)}{h_x\xi^2}\sin((x-x_{m'})\xi), & \text{if } m' = \mathcal{M} \\ \dfrac{2(1-\cos(h_x\xi))}{h_x\xi^2}\cos((x-x_{m'})\xi), & \text{otherwise.} \end{cases} \end{aligned}$$

where we have used the expressions of the basis functions $\phi_m(x)$, i.e.,

$$\phi_1(x) = \begin{cases} \dfrac{x_2 - x}{h_x}, & x \in (x_1, x_2), \\ 0, & x \in [x_2, L], \end{cases} \qquad \phi_M(x) = \begin{cases} \dfrac{x - x_{M-1}}{h_x}, & x \in (x_{\mathcal{M}-1}, x_{\mathcal{M}}) \\ 0, & x \in [0, x_{\mathcal{M}-1}], \end{cases}$$

$$\phi_m(x) = \begin{cases} \dfrac{x - x_{m-1}}{h_x}, & x \in (x_{m-1}, x_m), \\ \dfrac{x_{m+1} - x}{h_x}, & x \in (x_m, x_{m+1}), \\ 0, & x \in [0, x_{m-1}] \cup [x_{m+1}, L], \end{cases} \qquad \text{for } 2 \le m \le \mathcal{M}.$$

We also define $H_m(x', \xi)$ and evaluate

$$
\begin{aligned}
H_m(x', \xi) &= \int_\Gamma \sin((x - x')\xi)\phi_m(x)\mathrm{d}x \\
&= \begin{cases}
\dfrac{(1-\cos(h_x\xi))\sin((x_m - x')\xi)}{h_x\xi^2} + \dfrac{\cos((x_m - x')\xi)(h_x\xi - \sin(h_x\xi))}{h_x\xi^2}, & \text{if } m = 1 \\
\dfrac{(1-\cos(h_x\xi))\sin((x_m - x')\xi)}{h_x\xi^2} - \dfrac{\cos((x_m - x')\xi)(h_x\xi + \sin(h_x\xi))}{h_x\xi^2}, & \text{if } m = M \\
\dfrac{2(1-\cos(h_x\xi))\sin((x_m - x')\xi)}{h_x\xi^2}, & \text{otherwise.}
\end{cases}
\end{aligned}
$$

As such, we obtain the key quantities

$$
\begin{aligned}
g_{mm'}(\xi) &= \int_\Gamma \int_\Gamma \cos((x - x_0)\xi)\phi_{m'}(x_0)\phi_m(x)\mathrm{d}x_0\mathrm{d}x \\
&= \int_\Gamma K_{m'}(x, \xi)\phi_m(x)dx \\
&= \begin{cases}
\dfrac{1-\cos(h_x\xi)}{h_x\xi^2}K_m(x_{m'}, \xi) + \dfrac{h_x\xi - \sin(h_x\xi)}{h_x\xi^2}H_m(x_{m'}, \xi), & \text{if } m' = 1 \\
\dfrac{1-\cos(h_x\xi)}{h_x\xi^2}K_m(x_{m'}, \xi) - \dfrac{h_x\xi - \sin(h_x\xi)}{h_x\xi^2}H_m(x_{m'}, \xi), & \text{if } m' = \mathcal{M} \\
\dfrac{2(1-\cos(h_x\xi))}{h_x\xi^2}K_m(x_{m'}, \xi), & \text{otherwise.}
\end{cases}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
g_{mm'}(\xi) &= \frac{4(1-\cos(h_x\xi))^2\cos((x_{m'} - x_m)\xi)}{h_x^2\xi^4} \qquad 2 \le m, m' \le M - 2 \\
g_{mm'}(\xi) &= \frac{2(1-\cos(h_x\xi))^2\cos((x_{m'} - x_m)\xi)}{h_x^2\xi^4} \\
&\quad + \frac{2(1-\cos(h_x\xi))(h_x\xi - \sin(h_x\xi))\sin((x_{m'} - x_m)\xi)}{h_x^2\xi^4} \qquad m = 1,\ 2 \le m' \le M - 2 \\
g_{mm'}(\xi) &= \frac{2(1-\cos(h_x\xi))^2\cos((x_{m'} - x_m)\xi)}{h_x^2\xi^4} \\
&\quad - \frac{2(1-\cos(h_x\xi))(h_x\xi - \sin(h_x\xi))\sin((x_{m'} - x_m)\xi)}{h_x^2\xi^4} \qquad m = M,\ 2 \le m' \le M - 2 \\
g_{mm'}(\xi) &= \frac{(1-\cos(h_x\xi))^2\cos((x_{m'} - x_m)\xi)}{h_x^2\xi^4} + \frac{(h_x\xi - \sin(h_x\xi))^2}{h_x^2\xi^4} \qquad m = 1,\ m' = 1 \\
g_{mm'}(\xi) &= \frac{(1-\cos(h_x\xi))^2\cos((x_{m'} - x_m)\xi)}{h_x^2\xi^4} - \frac{(h_x\xi - \sin(h_x\xi))^2\cos((x_{m'} - x_m)\xi)}{h_x^2\xi^4} \\
&\quad - \frac{2(1-\cos(h_x\xi))(h_x\xi - \sin(h_x\xi))\sin((x_{m'} - x_m)\xi)}{h_x^2\xi^4} \qquad m = \mathcal{M},\ m' = 1 \\
g_{MM}(\xi) &= g_{11}(\xi).
\end{aligned}
$$

The entries of G^Γ are eventually computed by

$$
G^\Gamma_{mm'} = -\frac{1}{\pi}\int_0^{k_0} \frac{g_{mm'}(\xi)}{\rho + i\sqrt{k_0^2 - \xi^2}}\mathrm{d}\xi - \frac{1}{\pi}\int_{k_0}^{\infty} \frac{g_{mm'}(\xi)}{\rho - \sqrt{\xi^2 - k_0^2}}\mathrm{d}\xi.
$$

Both integrals above are evaluated numerically. Since $g_{mm'}(\xi) = O(\xi^{-4})$ decays sufficiently fast as $\xi \to \infty$, the second integral above can be evaluated sufficiently accurately.

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