OPTIMAL ANALYSIS OF FINITE ELEMENT METHODS FOR THE STOCHASTIC STOKES EQUATIONS

BUYANG LI, SHU MA, AND WEIWEI SUN

ABSTRACT. Numerical analysis for the stochastic Stokes equations is still challenging even though it has been well done for the corresponding deterministic equations. In particular, the pre-existing error estimates of finite element methods for the stochastic Stokes equations in the $L^\infty(0,T;L^2(\Omega;L^2))$ norm all suffer from the order reduction with respect to the spatial discretizations. The best convergence result obtained for these fully discrete schemes is only half-order in time and first-order in space, which is not optimal in space in the traditional sense. The objective of this article is to establish strong convergence of $O(\tau^{1/2}+h^2)$ in the $L^\infty(0,T;L^2(\Omega;L^2))$ norm for approximating the velocity, and strong convergence of $O(\tau^{1/2}+h)$ in the $L^\infty(0,T;L^2(\Omega;L^2))$ norm for approximating the time integral of pressure, where τ and h denote the temporal step size and spatial mesh size, respectively. The error estimates are of optimal order for the spatial discretization considered in this article (with MINI element), and consistent with the numerical experiments. The analysis is based on the fully discrete Stokes semigroup technique and the corresponding new estimates.

1. Introduction

We consider the time-dependent stochastic Stokes equations in a domain $D\subset\mathbb{R}^d$, $d\in\{2,3\}$, under the stress boundary condition, i.e.,

$$\begin{cases} \mathrm{d}u=[\nabla\cdot\mathbb{T}(u,p)+f]\,\mathrm{d}t+B(u)\,\mathrm{d}W(t) & \text{in } D\times(0,T],\\ \nabla\cdot u=0 & \text{in } D\times(0,T],\\ \mathbb{T}(u,p)\mathrm{n}=0 & \text{on } \partial D\times(0,T],\\ u=u^0 & \text{at } D\times\{0\}, \end{cases} \tag{1.1}$$

where u and p denote the velocity and the pressure of the fluid, respectively, f is a given source field and n denotes the outward unit normal vector on the boundary ∂D. Moreover, the stress tensor $\mathbb{T}(u,p)$ is defined by

$$\mathbb{T}(u,p)=2\mathbb{D}(u)-p\mathbb{I} \qquad \text{and} \qquad \mathbb{D}(u)=\frac{1}{2}\big(\nabla u+(\nabla u)^T\big), \tag{1.2}$$

where $\mathbb{I}$ denotes the identity tensor. The stochastic noise is determined by an $L^2(D)^d$-valued Q-Wiener process $\{W(t);t\geq 0\}$ on a filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq0})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t\geq0}$, and a linear operator $B(u):L^2(D)^d\to L^2(D)^d$ which depends on the solution nonlinearly.

The numerical approximations of deterministic Navier–Stokes (NS) equations have been well-understood nowadays; see [21, 22, 27, 31, 33–35]. For the stochastic NS equations driven by multiplicative non-solenoidal noises, Brzeźniak, Carelli & Prohl [9] proposed practical time-stepping schemes based on the finite element methods (FEMs) and established the convergence for velocity approximation (as a function sequence) to weak martingale solutions in 3D and to strong solutions in 2D, using the compactness argument. To obtain convergence rates for space-time discretizations of the stochastic NS equations, a main tool is the localization of the nonlinear term over a probability space of large probability, leading to a convergence rate in probability, as discussed in [3, 7, 8, 12].

- For the 2D stochastic NS equations with non-solenoidal noises under the periodic boundary condition, Carelli & Prohl [12] investigated implicit and semi-implicit time discretizations with FEMs, demonstrating a convergence in probability in the $L^\infty(0,T;L^2)$ norm with rate (almost) 1/4 in time and linear convergence in space for the velocity.

Key words and phrases. stochastic Stokes equation, multiplicative noise, Wiener process, semi-implicit Euler scheme, mixed FEM, analytic semigroup, error estimate.

- For the 2D stochastic NS equations with non-solenoidal noises under the periodic boundary condition, Bessaih, Brzeźniak & Millet [3] studied the convergence of a time-splitting method based on the Lie-Trotter formula. They proved that the speed of the convergence in probability is almost 1/2 for the velocity approximations, which is shown by means of an $L^2(\Omega, \mathbb{P})$ convergence localized on a set of arbitrarily large probability.
- For the 2D stochastic NS equations with non-solenoidal noises under the periodic boundary condition, Breit & Dodgson [7] recently established convergence in probability for the fully discrete implicit FEMs based on a stochastic pressure decomposition technique. They obtained a convergence in probability with rate (almost) 1/2 in time and linear convergence in space, measured in the norm of $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$. This improves the earlier results in [12], where the convergence rate in time was only (almost) 1/4.
- For the 2D stochastic NS equations with solenoidal noises under the Dirichlet boundary condition, Breit & Prohl [8] established convergence rates for the fully discrete semi-implicit FEMs using an approach based on discrete stopping times. They showed the convergence of velocity approximations in the $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ norm with respect to convergence in probability, achieving the rate (almost) 1/2 in time and linear convergence in space.

In addition to previously discussed convergence in probability, the strong rates of convergence (i.e., rates in $L^2(\Omega)$) for the stochastic NS equations have also been explored, see [4–6].

- The first study on the strong convergence for the 2D stochastic NS equations was conducted by Bessaih & Millet [4] under periodic boundary conditions. They focused on the splitting scheme from Bessaih et al. [3] and the implicit Euler schemes used in Carelli & Prohl [12].
- Further exploration of strong convergence for the fully discrete schemes of the 2D stochastic NS equations was carried out in [5]. They focused on the implicit Euler scheme coupled with FEMs for non-solenoidal noises under periodic boundary conditions. This research refines previous results in the stochastic NS equations, which had only established the convergence in probability of these fully discrete numerical approximations.

The stochastic Stokes system (1.1) is a simplified version of the stochastic NS equations with non-solenoidal noises. Most of the numerical analyses discussed above can be applied to the 2D stochastic Stokes equation. However, the convergence of pressure was not provided for the stochastic NS equations. Studies on the convergence of velocity approximations in 3D and pressure approximations for the stochastic Stokes equations have emerged only recently.

- For the 2D and 3D stochastic Stokes equations with non-solenoidal noises under the periodic boundary condition, Feng & Qiu [16] developed the fully discrete semi-implicit mixed FEMs and established strong convergence with rates for both velocity approximation in the norm of $L^\infty(0,T;L^2(\Omega;L^2))$ and pressure approximation in a time-averaged norm. The error estimates provided in [16] were derived based on a τ-dependent stability of the pressure approximations, as discussed in [16, Lemma 2], leading to a sub-optimal error estimate of order $O(\tau^{\frac{1}{2}} + h\tau^{-\frac{1}{2}})$. This τ-dependent stability of pressure approximations can be avoided in the case of solenoidal noises (i.e., $B(u)$ maps $L^2(D)^d$ into its divergence-free subspace, as considered in [11]) or pointwise divergence-free FEMs, as discussed in [12]. The convergence order was improved to $O(\tau^{\frac{1}{2}} + h + h^2\tau^{-\frac{1}{2}})$ in Feng & Vo [17] based on the Chorin-type projection methods. However, all spatial error constants presented in [16,17] include a growth factor $\tau^{-\frac{1}{2}}$.
- Recently, for 2D and 3D non-solenoidal noises under the periodic boundary condition, Feng, Prohl & Vo [15] proposed new fully discrete mixed FEMs for the stochastic Stokes equations by utilizing the Helmholtz decomposition to the noises. By removing a gradient part from the non-solenoidal noise, the modified noise becomes divergence-free. This modification results in an improved stability estimate of the new pressure approximations, which is not dependent on the temporal stepsize τ. Consequently, the inf-sup stable mixed FEMs for the stochastic Stokes equation proposed in [15] achieve strong convergence with

rate $O(\tau^{\frac{1}{2}}+h)$ for velocity in the $L^\infty(0,T;L^2(\Omega;L^2))$ norm and pressure in a time-averaged norm. This improvement addresses the sub-optimal estimates in [16].

The numerical analysis in [15–17] is based on the certain conditions of noise, which can be viewed as the following Lipschitz continuity and growth conditions in the case that B is an $\mathcal{L}_2^0$-valued function:

$$\|B(v)-B(w)\|_{\mathcal{L}_2^0(\mathbb{L}^2,\mathbb{L}^2)} \le C\|v-w\|_{L^2} \quad \text{and} \quad \|B(v)\|_{\mathcal{L}_2^0(\mathbb{L}^2,\mathbb{L}^2)} \le C\big(1+\|v\|_{L^2}\big) \tag{1.3}$$

for $v,w \in L^2(D)^d$, where $\mathcal{L}_2^0(\mathbb{L}^2,\mathbb{L}^2)$ is the space of Hilbert–Schmidt operators on $\mathbb{L}^2 = L^2(D)^d$. In the presence of non-solenoidal noises, the half-order temporal convergence shown in the above-mentioned analyses is optimal and consistent with the numerical experiments. However, the first-order spatial convergence in the $L^\infty(0,T;L^2(\Omega;L^2))$ norm is not optimal and inconsistent with the numerical experiments.

As far as we know, the numerical analysis of the stochastic heat equation has been studied extensively [10,32,39,40] and the second-order convergence in space has been proved. However, for the stochastic Stokes equations, the existing approach applies only to a simple case with a solenoidal noise and a pointwise divergence-free finite element space, for which the error analysis actually reduces to the analysis for an abstract parabolic equation. The second-order convergence in space in the norm of $L^\infty(0,T;L^2(\Omega;L^2))$ for (1.1), under the common setting involving general non-solenoidal noises and frequently used inf-sup stable mixed FEMs, has not been proved. The objective of this article is to address this question under the following noise condition for some $\beta > \frac{d}{2}$:

$$\begin{cases} \|B(v)-B(w)\|_{\mathcal{L}_2^0(\mathbb{L}^2,\mathbb{H}^{-1/2})} \le C\|v-w\|_{L^2}, \\ \|B(v)-B(w)\|_{\mathcal{L}_2^0(\mathbb{L}^2,\mathbb{H}^1)} \le C\|v-w\|_{H^\beta} \text{ and } \|B(v)\|_{\mathcal{L}_2^0(\mathbb{L}^2,\mathbb{H}^1)} \le C\big(1+\|v\|_{H^\beta}\big), \end{cases} \tag{1.4}$$

which also covers many noises that were considered in the literature (e.g., [15, 17]); see the examples in Remark 2.2 and the numerical experiments in Section 6. The error analyses for the stochastic Stokes equations in the previous articles, based on energy approach, do not yield second-order convergence in space in the $L^\infty(0,T;L^2(\Omega;L^2))$ norm under noise condition (1.4), for the same difficulty caused by the low regularity of pressure; see the discussions in [15–17]. Therefore, a new approach of error analysis needs to be developed to address this difficulty.

In this article, we establish optimal convergence of fully discrete mixed methods with standard inf-sup stable finite element pairs for the stochastic Stokes equations driven by a non-solenoidal multiplicative noise under condition (1.4) and the regularity of the mild solution (see Proposition 3.1). In particular, the strong convergence of $O(\tau^{1/2}+h^2)$ in the $L^\infty(0,T;L^2(\Omega;L^2))$ norm is proved for approximating the velocity, and the strong convergence of $O(\tau^{1/2}+h)$ in the $L^\infty(0,T;L^2(\Omega;L^2))$ norm is proved for approximating the time integral of pressure (see Theorem 2.4), where τ and h denote the temporal stepsize and spatial mesh size, respectively. The error estimates are of optimal order for MINI element in the traditional sense and consistent with the numerical experiments.

The analysis presented in this article is based on the fully discrete Stokes semigroup technique and the corresponding new estimates (refer to Lemma 4.1 and Remark 4.1) for general non-solenoidal functions. The regularity of the pressure solution to the stochastic Stokes problem is generally low due to the influence of the non-solenoidal noise. This low regularity of the pressure is a main obstacle in proving second-order convergence in space, as discussed in [15, 16]. We overcome this difficulty by using the fully discrete Stokes semigroup technique to avoid using the τ-dependent estimates of the pressure approximations. Specifically, we have developed technical estimates in Lemma 4.1, which do not require v to be divergence-free (where v represents the noise term in the error estimates, as indicated by T_3 in the error equation (5.10)). This was achieved by proving and utilizing the H^1-stability of the orthogonal projection onto the discrete divergence-free finite element subspace (see Subsection 4.1) and based on the error estimates of the fully-discrete FEMs for the deterministic Stokes problem. In the previous papers, the semigroup estimates provided in Lemma 4.1 were only shown and utilized for the abstract parabolic equation, which requires v to be divergence-free and requires the finite element space to be pointwise divergence-free when the results are applied to the

Stokes equations. This distinction is crucial for our error analysis of the stochastic Stokes equations, and makes it possible to prove a better convergence rate for non-solenoidal noises using non-divergence-free finite elements.

The rest of this article is organized as follows. In Section 2, we collect the assumptions and describe a fully discrete method with a standard inf-sup stable FE pair for the stochastic Stokes and then, present our main theorem. In Section 3, we present the abstract formulation of the stochastic Stokes equations under the stress boundary condition and define the mild solution based on the abstract formulation. In Section 4, we present some technical estimates for the discrete semigroup associated to the Stokes operator. The results are used in the error analysis of the fully discrete FEMs for the stochastic problem in Section 5. In Section 6, we present numerical experiments to support our theoretical analysis by illustrating the convergence orders of the velocity and pressure approximations. The current paper is a reduced version of [29], which contains complete proofs of some technical intermediate results.

2. Main results

In this section, we present some notations and assumptions to be used in this article, as well as the numerical scheme for the stochastic Stokes equations. Then we present the main theoretical result on the convergence of the numerical scheme.

2.1. Basic notations

We assume that the domain D exhibits elliptic H^2 regularity when considering the deterministic Stokes equations with the stress boundary condition. This assumption implies that solutions (v, q) of the linear Stokes equations

$$\begin{cases} v - \nabla \cdot \mathbb{T}(v, q) = g & \text{in } D, \\ \nabla \cdot v = 0 & \text{in } D, \\ \mathbb{T}(v, q)\mathrm{n} = 0 & \text{on } \partial D, \end{cases} \tag{2.1}$$

satisfy the following estimates:

$$\|v\|_{H^2} + \|q\|_{H^1} \leq C\|g\|_{L^2}. \tag{2.2}$$

This H^2 elliptic regularity estimate holds for Stokes equations in two-dimensional convex polygons under both Dirichlet boundary condition [23] and Neumann/Stress boundary condition [28,30], and three-dimensional convex polyhedron under the Dirichlet boundary condition (see [13, Eq. (1.8)]). If the domain is smooth then the H^2 elliptic regularity estimate holds for both Dirichlet [13, Eq. (1.5)] and Neumann/Stress boundary conditions [36, Theorem 1.1]. In this paper, we focus on domains on which the H^2 elliptic regularity estimate in (2.2) holds.

Let $H^s(D)$, $s \geq 0$, denote the conventional Sobolev space of functions defined on D, with $L^2(D) = H^0(D)$, spaces with blackboard letters (e.g., $\mathbb{H}^s(D) = H^s(D)^d$) represent the spaces of vector valued functions. The dual space of $H^s(D)$ is denoted by $H^{-s}(D)$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ denote a filtered probability space with the probability measure $\mathbb{P}$, the σ-algebra $\mathcal{F}$ and the continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The expectation of a random variable v defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is denoted by $\mathbb{E}v$.

For a Hilbert space $\mathcal{K}$, let $W(t)$ be a $\mathcal{K}$-valued Q-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, with expression

$$W(t) = \sum_{\ell} \sqrt{\mu_\ell}\phi_\ell W_\ell(t) \qquad \forall\, t \in [0, T], \tag{2.3}$$

where $\{W_\ell(t)\}_{j\geq 1}$ is a family of independent real-valued Wiener processes and the trace operator $Q : \mathcal{K} \to \mathcal{K}$ is bounded, self-adjoint, positive semi-definite, with eigenvalues $\{\mu_\ell\}_{\ell\geq 1}$ and eigenfunctions $\{\phi_\ell\}_{\ell\geq 1}$.

Let $\mathcal{L}_2(\mathcal{K}, \mathcal{H})$ and $\mathcal{L}_2^0(\mathcal{K}, \mathcal{H})$ be the spaces of Hilbert-Schmidt operators from $\mathcal{K}$ to $\mathcal{H}$ and from $Q^{1/2}(\mathcal{K})$ to $\mathcal{H}$, respectively, satisfying

$$\|\Phi\|_{\mathcal{L}_2^0(\mathcal{K},\mathcal{H})} := \Big(\sum_{\ell} \mu_\ell \|\Phi\phi_\ell\|_{\mathcal{H}}^2\Big)^{\frac{1}{2}} = \|\Phi Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{K},\mathcal{H})}.$$

For a progressively measurable process $\Phi : [0,T] \to \mathcal{L}_2^0(\mathcal{K},\mathcal{H})$ with $\int_0^T \|\Phi(s)\|^2_{\mathcal{L}_2^0(\mathcal{K},\mathcal{H})}\,\mathrm{d}s < \infty$ $\mathbb{P}$-a.s., the stochastic integral $\int_0^t \Phi(s)\,\mathrm{d}W(s)$ is well defined and Itô's isometry holds, i.e.,

$$\mathbb{E}\left\| \int_0^t \Phi(s)\,\mathrm{d}W(s) \right\|^2_{\mathcal{H}} = \mathbb{E}\int_0^t \|\Phi(s)\|^2_{\mathcal{L}_2^0(\mathcal{K},\mathcal{H})}\,\mathrm{d}s. \tag{2.4}$$

For the simplicity of notations, we denote by $\mathcal{L}_2^0 = \mathcal{L}_2^0(\mathbb{L}^2, \mathbb{L}^2)$ and denote by $x \lesssim y$ (or $y \gtrsim x$) the statement "$x \le Cy$ (or $x \ge Cy$) for some positive constant C which is independent of the stepsize τ and the mesh size h in the numerical approximation".

2.2. Assumptions on the noise and nonlinearity

For the existence and uniqueness of mild solutions to problem (1.1), as well as the numerical approximation to the mild solutions, we work with the following assumptions on the noise and nonlinearity.

Assumption 2.1. (Stochastic noise) We assume that the Q-Wiener process $W(t)$ has the following property:

$$\|(-\Delta)^{\frac{1}{2}}\|_{\mathcal{L}_0^2} \lesssim 1, \tag{2.5}$$

where $\Delta : H_N^2(D) \to L^2(D)$ denotes the Neumann Laplacian operator with the domain

$$H_N^2(D) = \{v \in H^2(D) : \partial_n v = 0 \text{ on } \partial D\},$$

and $(-\Delta)^{\frac{1}{2}} : H^1(D) \to L^2(D)$ denotes the fractional power of $-\Delta$.

Remark 2.1. In the case that Q and $-\Delta$ have the same eigenfunctions, the condition (2.5) is equivalent to

$$\sum_{\ell} \mu_\ell \lambda_\ell \lesssim 1,$$

where λ_ℓ is an eigenvalue of $-\Delta$.

Assumption 2.2. (Nonlinearity and source term) We assume that $B(v) : L^2(D)^d \to L^2(D)^d$ is a bounded nonlinear operator for any $v \in L^2(D)^d$, satisfying the following Lipschitz continuity and growth conditions for some $\beta \in (\frac{d}{2}, 2)$:

$$\|B(v) - B(w)\|_{\mathcal{L}_2^0(\mathbb{L}^2,\,\mathbb{H}^{-1/2})} \lesssim \|v - w\|_{L^2}, \tag{2.6}$$

$$\|B(v) - B(w)\|_{\mathcal{L}_2^0(\mathbb{L}^2,\,\mathbb{H}^1)} \lesssim \|v - w\|_{H^\beta} \quad \text{and} \quad \|B(v)\|_{\mathcal{L}_2^0(\mathbb{L}^2,\,\mathbb{H}^1)} \lesssim 1 + \|v\|_{H^\beta}. \tag{2.7}$$

Moreover, we assume that the function $f : [0,T] \times L^2(D)^d \to L^2(D)^d$ satisfies

$$\|f(t)\|_{L^2} \lesssim 1 \qquad \forall\, 0 \le t \le T, \tag{2.8}$$

$$\|f(t_1) - f(t_2)\|_{L^2} \lesssim (t_1 - t_2)^{\frac{1}{2}} \qquad \forall\, 0 \le t_1 \le t_2 \le T. \tag{2.9}$$

Remark 2.2. The conditions in (2.6)-(2.7) are satisfied if $B(v)$ satisfies the following estimates:

$$\|B(v)\phi_\ell\|_{H^1} \lesssim (1 + \|v\|_{H^\beta})\|\phi_\ell\|_{H^1} \qquad \forall\, v \in H^\beta(D)^d, \tag{2.10}$$

$$\|(B(v_1) - B(v_2))\phi_\ell\|_{H^{-\frac{1}{2}}} \lesssim \|v_1 - v_2\|_{L^2}\|\phi_\ell\|_{H^1} \qquad \forall\, v_1, v_2 \in L^2(D)^d, \tag{2.11}$$

$$\|(B(v_1) - B(v_2))\phi_\ell\|_{H^1} \lesssim \|v_1 - v_2\|_{H^\beta}\|\phi_\ell\|_{H^1} \qquad \forall\, v_1, v_2 \in H^\beta(D)^d. \tag{2.12}$$

for some $\beta \in (\frac{d}{2}, 2)$. The proofs in this paper exclusively rely on the use of (2.6)–(2.7). A 2D example of a suitable operator $B(u)$ and noise $W(t)$, satisfying (2.10)–(2.12), is given by

$$\begin{aligned} B(u) &= \begin{pmatrix} \sqrt{u_1^2+1} & \sqrt{u_1^2+1} \\ \sqrt{u_2^2+1} & \sqrt{u_2^2+1} \end{pmatrix} \quad \text{for} \quad u = (u_1, u_2), \\ W(t,\mathbf{x}) &= \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} \sqrt{\mu_{\ell_1\ell_2}} \begin{pmatrix} \phi_{\ell_1\ell_2}(\mathbf{x}) \\ \phi_{\ell_1\ell_2}(\mathbf{x}) \end{pmatrix} w_{\ell_1\ell_2}(t) \qquad \forall\, t \in [0,T], \end{aligned} \tag{2.13}$$

with

$$\mu_{\ell_1\ell_2} = \begin{cases} 0 & \text{for } (\ell_1,\ell_2)=(0,0), \\ (\ell_1^2+\ell_2^2)^{-(2+\varepsilon)} & \text{for } (\ell_1,\ell_2)\in\mathbb{Z}^2/\{(0,0)\},\ \varepsilon=0.1, \end{cases}$$

$$\phi_{\ell_1\ell_2}(\mathbf{x}) = \sin(\ell_1\pi x_1)\sin(\ell_2\pi x_2), \quad \ell_1,\ell_2=0,1,2,\dots,$$

which forms an orthonormal basis of $L^2(D)$. The noise term $B(u)\mathrm{d}W$ determined by (2.13) and series $W(t)$ can be written as

$$B(u)\mathrm{d}W(t) = \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sqrt{\mu_{\ell_1\ell_2}}\begin{pmatrix}\sqrt{u_1^2+1}\\ \sqrt{u_2^2+1}\end{pmatrix}\phi_{\ell_1\ell_2}(\mathbf{x})\,w_{\ell_1\ell_2}(t),$$

which is non-solenoidal and was used in [15,17] to measure the effectiveness of numerical methods for the stochastic Stokes/NS equation. This example of noise satisfies both Assumption 2.1 and conditions (2.6)–(2.7) in Assumption 2.2.

Remark 2.3. In the case that $(B(v)\phi_\ell)(x) = b_\ell(v(x))\phi_\ell(x)$ for some functions $b_\ell:\mathbb{R}\to\mathbb{R}$, $\ell=1,2,\dots$, the conditions (2.10)–(2.12) are satisfied if the functions b_ℓ are uniformly Lipschitz continuous with respect to ℓ, i.e.,

$$|b_\ell(\sigma)| \lesssim 1+|\sigma| \qquad \forall\,\sigma\in\mathbb{R},$$
$$|b_\ell(\sigma_1)-b_\ell(\sigma_2)| \lesssim |\sigma_1-\sigma_2| \quad \forall\,\sigma_1,\sigma_2\in\mathbb{R}.$$

Assumption 2.3. (Initial value) We assume that the initial value $u_0:\Omega\to L^2(D)^d$ is an $\mathcal{F}_0/\mathcal{B}(L^2(D)^d)$-measurable function with $u_0\in L^2(\Omega,\mathcal{D}(A))$.

2.3. The numerical method and its convergence

Let $V_h\times Q_h\subset H^1(D)^d\times L^2(D)$ be a pair of finite element spaces subject to a quasi-uniform triangulation of D with the mesh size $h>0$, satisfying the following properties (see [19, Chapter II]):

(1) There exists a projection operator $\Pi_h: H^1(D)^d\to V_h$, called the Fortin projection, satisfying

$$\big(\nabla\cdot(v-\Pi_h v), q_h\big) = 0 \qquad \forall\, v\in H^1(D)^d \text{ and } q_h\in Q_h, \tag{2.14}$$

$$\|v-\Pi_h v\|_{H^m} \lesssim h^{1-m}\|v\|_{H^1} \qquad \forall\, v\in H^1(D)^d \text{ and } m=0,1, \tag{2.15}$$

$$\|\Pi_h v\|_{H^1} \lesssim \|v\|_{H^1} \qquad \forall\, v\in H^1(D)^d, \tag{2.16}$$

(2) The following approximation properties hold:

$$\inf_{q_h\in Q_h}\|q-q_h\|_{L^2} \lesssim h^m\|q\|_{H^m} \quad \forall\, q\in H^m(D) \text{ and } m=1,2, \tag{2.17}$$

$$\inf_{q_h\in Q_h\cap H_0^1(D)}\|q-q_h\|_{L^2} \lesssim h^m\|q\|_{H^m} \quad \forall\, q\in H^m(D)\cap H_0^1(D) \text{ and } m=1,2. \tag{2.18}$$

Therefore, the L^2 projection $P_{Q_h}: L^2(D)\to Q_h$ satisfies the following estimates:

$$\|q-P_{Q_h}\|_{L^2} \lesssim h^m\|q\|_{H^m} \quad \forall\, q\in H^m(D) \text{ and } m=1,2, \tag{2.19}$$

$$\begin{aligned}\|q-P_{Q_h}q\|_{H^{-1}} &= \sup_{\substack{\eta\in H^1(D)\\ \eta\neq 0}}\frac{(q-P_{Q_h}q,\eta)}{\|\eta\|_{H^1}} = \sup_{\eta\in H^1(D)}\frac{(q-P_{Q_h}q,\eta-P_{Q_h}\eta)}{\|\eta\|_{H^1}}\\ &\lesssim h^2\|q\|_{H^1}.\end{aligned} \tag{2.20}$$

(3) The following inverse inequality holds

$$\|v_h\|_{H^1} \lesssim h^{-1}\|v_h\|_{L^2} \quad \forall\, v_h\in V_h. \tag{2.21}$$

(4) The inf-sup condition holds:

$$\|q_h\|_{L^2} \lesssim \sup_{\substack{v_h\in V_h\\ v_h\neq 0}}\frac{(q_h,\nabla\cdot v_h)}{\|v_h\|_{H^1}} \quad \forall\, q_h\in Q_h. \tag{2.22}$$

Several finite element spaces $V_h \times Q_h$ are known to satisfy the properties above, such as the standard inf-sup finite element spaces including the mini element space in [2] and the Taylor–Hood finite element space in [37]. We assume that the triangulation may contain curved triangles/tetrahedra which fits the boundary exactly in order to avoid making the problem more complicated with additional errors in approximating the boundary.

The natural function spaces associated to incompressible flow is the divergence-free subspaces of $L^2(D)^d$ and $H^1(D)^d$, defined by

$$X = \{v \in L^2(D)^d : \nabla \cdot v = 0\} \quad \text{and} \quad V = X \cap H^1(D)^d. \tag{2.23}$$

Let $X_h := \big\{v_h \in V_h : (\nabla \cdot v_h, q_h) = 0, \ \forall\, q_h \in Q_h\big\}$ be a discrete divergence–free subspace of V_h, and denote by $P_{X_h} : L^2(D)^d \to X_h$ the L^2-orthogonal projection onto X_h. On a uniform partition $t_n = n\tau$, $n = 0, 1, \ldots, N$, of the time interval $[0, T]$ with the stepsize $\tau = T/N$, we consider the following fully discrete semi-implicit Euler method for problem (1.1): For the given initial value $u_h^0 = P_{X_h} u^0$, find a pair of processes $(u_h^n, p_h^n) \in V_h \times Q_h$, $n = 1, \ldots, N$, such that the weak formulation

$$\begin{cases} (u_h^n, v_h) + 2\tau\big(\mathbb{D}(u_h^n), \mathbb{D}(v_h)\big) = (u_h^{n-1}, v_h) + \tau(p_h^n, \nabla \cdot v_h) + \tau(f(t_n), v_h) & \forall\, v_h \in V_h \\ \qquad\qquad\qquad\qquad\qquad\qquad + (B(u_h^{n-1})\Delta W_n, v_h), & \\ \qquad\qquad (\nabla \cdot u_h^n, q_h) = 0 & \forall\, q_h \in Q_h \end{cases} \tag{2.24}$$

holds $\mathbb{P}$-a.s. for all test functions $(v_h, q_h) \in V_h \times Q_h$, where $\Delta W_n := W(t_n) - W(t_{n-1})$ is a random variable with $N(0, \tau Q)$ distribution.

By choosing $v_h \in X_h$ in (2.24), the fully discrete method in (2.24) can be equivalently written as finding a X_h-valued process u_h^n, $n = 1, \ldots, N$, such that $\mathbb{P}$-a.s.

$$\begin{cases} (u_h^n, v_h) + 2\tau\big(\mathbb{D}(u_h^n), \mathbb{D}(v_h)\big) = (u_h^{n-1}, v_h) + \tau(f(t_n), v_h) + (B(u_h^{n-1})\Delta W_n, v_h) \quad \forall\, v_h \in X_h, \\ \qquad\qquad\qquad\qquad u_h^0 = P_{X_h} u^0. \end{cases} \tag{2.25}$$

If we denote by $A_h : X_h \to X_h$ the discrete Stokes operator defined by

$$(A_h v_h, w_h) = 2\big(\mathbb{D}(v_h), \mathbb{D}(w_h)\big) \qquad \forall\ v_h,\, w_h \in X_h. \tag{2.26}$$

Then the fully discrete method in (2.25) is equivalent to finding a X_h-valued process u_h^n, $n = 1, \ldots, N$ such that $\mathbb{P}$-a.s.

$$\begin{cases} u_h^n = \bar{E}_{h,\tau} u_h^{n-1} + \tau \bar{E}_{h,\tau} P_{X_h} f(t_n) + \bar{E}_{h,\tau} P_{X_h} [B(u_h^{n-1})\Delta W_n], \\ u_h^0 = P_{X_h} u^0, \end{cases} \tag{2.27}$$

where $\bar{E}_{h,\tau}$ denotes the discrete semigroup in the full discretization defined by

$$\bar{E}_{h,\tau} v_h = (I + \tau A_h)^{-1} v_h. \tag{2.28}$$

The main result of this article is the following theorem, which provides the convergence of the numerical solution to the mild solution of the stochastic Stokes equations.

Theorem 2.4. *Let $\{W(t); t \geq 0\}$ be a Q-Wiener process on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. Let Assumptions 2.1–2.3 be fulfilled and assume that the finite element space $V_h \times Q_h$ has Properties (1)–(4). Then the numerical solution (u_h^n, p_h^n), $n = 1, \ldots, N$, determined by* (2.24) *has the following error bounds:*

$$\max_{1\leq n\leq N} \Big(\mathbb{E}\|u(t_n) - u_h^n\|_{L^2}^2\Big)^{\frac{1}{2}} \lesssim \tau^{\frac{1}{2}} + h^2, \tag{2.29}$$

$$\max_{1\leq m\leq N} \bigg(\mathbb{E}\Big\| \int_0^{t_m} p(s)\,\mathrm{d}s - \tau \sum_{n=1}^{m} p_h^n \Big\|_{L^2}^2\bigg)^{\frac{1}{2}} \lesssim \tau^{\frac{1}{2}} + h\,. \tag{2.30}$$

Remark 2.4. Since the numerical scheme in (2.24) is linearly implicit, the existence and uniqueness of numerical solutions are standard. The convergence rates presented in the Theorem 2.4 is optimal in space for the inf-sup stable MINI element space in [2]. The half-order convergence in time is the same as the previous results and consistent with the numerical experiments.

Remark 2.5. The numerical scheme and analysis presented in this article can be extended to noises of the type $B(s,u)$, provided that certain Hölder continuity conditions of $B(s,u)$ with respect to s are assumed, as stated in Assumption 2.2.

The proof of Theorem 2.4 will be presented in the next three sections based on the techniques of continuous and discrete analytic semigroups.

3. The abstract formulation under the stress boundary condition

In this section, we present the abstract formulation and functional setting of the stochastic Stokes equations under the stress boundary condition, and define the mild solution of the stochastic Stokes equations to be approximated by the numerical solutions.

The divergence-free subspace X of $L^2(D)^d$ is endowed with the $L^2(D)^d$ norm. It is known that the following orthogonal decomposition holds:

$$L^2(D)^d = X \oplus U \quad \text{with} \quad U = \{\nabla q : q \in H_0^1(D)\}.$$

In particular, any $v \in L^2(D)^d$ can be decomposed as $v = P_X v + \nabla\eta$, where

$$P_X v = v - \nabla\eta$$

denotes the L^2-orthogonal projection from $L^2(D)^d$ onto X, with η being the solution of the equation

$$\begin{cases} \Delta\eta = \nabla\cdot v & \text{in } D, \\ \eta = 0 & \text{on } \partial D. \end{cases} \tag{3.1}$$

Since the solution η of the Poisson equation satisfies

$$\|\eta\|_{H^{s+1}} \le C\|v\|_{H^s} \quad \text{for } s\in[0,2],$$

it follows that the L^2 projection $P_X v = v - \nabla\eta$ satisfies

$$\|P_X v\|_{H^s} \lesssim \|v\|_{H^s} \quad \text{for } s\in[0,2]. \tag{3.2}$$

Since the L^2 projection operator $P_X : L^2(D)^d \to X$ is self-adjoint, by a duality argument it holds that

$$\|P_X v\|_{H^{-s}} \lesssim \|v\|_{H^{-s}} \quad \text{for } s\in[0,2]. \tag{3.3}$$

Let $E(t) : X \to X$ be the semigroup of bounded linear operators defined by $E(t)v^0 := v(t)$ as the solution of the linear Stokes equations

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nabla\cdot\mathbb{T}(v,q) = 0 & \text{in } D\times\mathbb{R}_+ \\ \nabla\cdot v = 0 & \text{in } D\times\mathbb{R}_+ \\ \mathbb{T}(v,q)n = 0 & \text{on } \partial D\times\mathbb{R}_+ \\ v(0) = v^0 & \text{in } D. \end{cases} \tag{3.4}$$

Let $-A$ be the generator of the semigroup $E(t)$ with the domain

$$D(A) = \Big\{ v^0 \in X : \lim_{t\to0}\frac{E(t)v^0 - v^0}{t} \text{ exists in } X \Big\},$$

which is a dense subspace of X. Then $v^0 \in D(A)$ if and only if $\partial_t v \in C([0,T];X)$, which is equivalent to

$$\nabla\cdot\mathbb{T}(v,q) \in C([0,T];X) \text{ and } \nabla\cdot v = 0.$$

The above condition holds if and only if

$$\begin{aligned} &-\Delta v + \nabla q \in C([0,T];X) \text{ and } \nabla\cdot v = 0, \\ &2\mathbb{D}(v)\mathrm{n} - q\mathrm{n} = 0 \text{ on } \partial D, \\ &q \text{ is a harmonic function with boundary condition } q = 2\mathbb{D}(v)\mathrm{n}\cdot\mathrm{n}, \end{aligned} \tag{3.5}$$

which is equivalent to

$$v \in C([0,T]; H^2(D)^d), \ \ \nabla \cdot v = 0 \ \text{ and } \ \mathbb{D}(v)\mathrm{n} \times \mathrm{n} = 0 \ \text{ on } \ \partial D, \tag{3.6}$$

where the last equivalence relation follows from the H^2 regularity of the stationary Stokes equations. Since $v^0 \in D(A)$ if and only if $Av \in C([0,T]; X)$, it follows from (3.6) that

$$D(A) = \dot{H}^2(D)^d := \big\{ v \in H^2(D)^d : \nabla \cdot v = 0 \ \text{ and } \ \mathbb{D}(v)\mathrm{n} \times \mathrm{n} = 0 \text{ on } \partial D \big\}.$$

Moreover, from (3.5) we see that the operator $A : D(A) \to X$ can be written as

$$Av = -\nabla \cdot \mathbb{T}(v, q_v) = -\Delta v + \nabla q_v, \tag{3.7}$$

where q_v is determined by v through the following equation:

$$\begin{cases} \Delta q_v = 0 & \text{in } D, \\ \ \ q_v = 2\mathbb{D}(v)\mathrm{n} \cdot \mathrm{n} & \text{on } \partial D. \end{cases} \tag{3.8}$$

For $v \in D(A)$, testing (3.7) with w and using integration by parts, we obtain by utilizing the boundary condition $q_v = 2\mathbb{D}(v)\mathrm{n} \cdot \mathrm{n}$ and $\mathbb{D}(v)\mathrm{n} \times \mathrm{n} = 0$ (which imply $2\mathbb{D}(v)\mathrm{n} - q_v\mathrm{n} = 0$ on ∂D)

$$\big(Av, w\big) = \big(2\mathbb{D}(v), \mathbb{D}(w)\big) \quad \forall\, w \in V. \tag{3.9}$$

The Stokes operator $A : D(A) \to X$ has an extension $A : V \to V'$ as defined above.

The boundary condition $\mathbb{T}(u,p)\mathrm{n} = 0$ in (1.1) implies that $\mathbb{D}(u)\mathrm{n} \times \mathrm{n} = 0$ and $p = 2\mathbb{D}(u)\mathrm{n} \cdot \mathrm{n}$, and therefore

$$-P_X \mathbb{T}(u,p) = -\Delta u + P_X \nabla p = -\Delta u + \nabla(p - \eta),$$

where η is the solution of

$$\begin{cases} \Delta \eta = \Delta p & \text{in } D \\ \ \ \eta = 0 & \text{on } \partial D. \end{cases} \tag{3.10}$$

This implies that

$$-P_X \mathbb{T}(u,p) = -\Delta u + \nabla q_u = Au, \tag{3.11}$$

where $q_u = p - \eta$ is the harmonic function satisfying $q_u = 2\mathbb{D}(u)\mathrm{n} \cdot \mathrm{n}$ on ∂D.

As a result, applying P_X to (1.1) yields the following abstract formulation of the stochastic Stokes problem in (1.1):

$$\begin{cases} \mathrm{d}u = [-Au + P_X f]\,\mathrm{d}t + P_X[B(u)\,\mathrm{d}W] & \text{in } D \times (0,T], \\ \ \ u = u^0 & \text{at } D \times \{0\}. \end{cases} \tag{3.12}$$

A predictable process $u \in C([0,T]; L^2(\Omega; X))$ is called a mild solution of problem (3.12) if

$$u(t) = E(t)u^0 + \int_0^t E(t-s)P_X f(s)\mathrm{d}s + \int_0^t E(t-s)P_X B(u(s))\,\mathrm{d}W(s) \quad \mathbb{P}\text{-a.s.} \tag{3.13}$$

The proof of the following proposition can be found in [29, Appendix A], where [29] refers to the arXiv version of this paper, where certain technical intermediate results are proved.

Proposition 3.1. (Well-posedness and regularity) *Under Assumptions 2.1–2.3, problem* (3.12) *has a unique mild solution in the sense of* (3.13). *Moreover, the mild solution satisfies the following regularity estimates:*

$$\sup_{t\in[0,T]} \mathbb{E}\|u(t)\|_{H^2}^2 \lesssim \big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big), \tag{3.14}$$

$$\mathbb{E}\|u(t) - u(s)\|_{H^1}^2 \lesssim \big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big)(t-s) \qquad \forall\, 0 \le s \le t \le T \tag{3.15}$$

$$\mathbb{E}\|u(t) - u(s)\|_{H^\beta}^2 \lesssim \big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big)(t-s)^{2-\beta} \qquad \forall\, 0 \le s \le t \le T, \ \ \forall\, \beta \in (\tfrac{d}{2}, 2). \tag{3.16}$$

The error estimates for the numerical approximations will be proved with the regularity results in Proposition 3.1. Since the mild solution has the regularity $u \in C([0,T]; L^2(\Omega, H^1(D)^d))$, the inf-sup condition [18, Theorem 4.1] implies that there exits $\int_0^t p(s)\mathrm{d}s \in C([0,T]; L^2(\Omega, L^2(D)))$

satisfying following relation:

$$\Big(\int_0^t p(s)\,\mathrm{d}s, \nabla\cdot v\Big) = (u(t)-u^0, v) + 2\int_0^t \big(\mathbb{D}(u(s)), \mathbb{D}(v)\big)\,\mathrm{d}s$$
$$-\int_0^t (f(s), v)\,\mathrm{d}s - \Big(\int_0^t B(u(s))\,\mathrm{d}W(s), v\Big) \quad \forall\, v \in H^1(D)^d. \tag{3.17}$$

4. Estimates for the discrete semigroups

In this section, we establish some technical estimates of the discrete analytic semigroup associated to the Stokes operator. The main results are the following three types of error estimates about approximating the semigroup $E(t)P_X$ by the discrete semigroup $\bar{E}^n_{h,\tau}P_{X_h}$, $n = 1, \ldots, N$, defined in (2.28), which play a key role in our theoretical analysis.

Lemma 4.1. *For any $v \in H^1(D)^d$, there holds*

$$\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|[E(s)P_X - \bar{E}^j_{h,\tau}P_{X_h}]\,v\|^2_{L^2}\,\mathrm{d}s \le C(\tau + h^4)\|v\|^2_{H^1}. \tag{4.1}$$

In addition, for all $v \in L^2(D)^d$, there holds

$$\Big\|\sum_{j=1}^n \int_{t_{j-1}}^{t_j} [E(s)P_X - \bar{E}^j_{h,\tau}P_{X_h}]\,v\,\mathrm{d}s\Big\|^2_{L^2} \le C(\tau + h^4)\|v\|^2_{L^2}, \tag{4.2}$$

and

$$\Big\|\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \nabla[E(s)P_X - \bar{E}^j_{h,\tau}P_{X_h}]\,v\,\mathrm{d}s\Big\|^2_{L^2} \le C(\tau + h^2)\|v\|^2_{L^2}. \tag{4.3}$$

Remark 4.1. Note that the estimates above were analyzed in [38] for the abstract parabolic equation, including the Stokes equations with divergence-free initial value $v \in X$ using pointwise divergence-free finite element spaces. Since we allow $B(u) : L^2(D)^d \to L^2(D)^d$ rather than $B(u) : L^2(D)^d \to X$, the estimates under such a weaker assumption are not straightforward and also the key to establish the second-order convergence in space for the numerical solution of the stochastic Stokes equations.

In order to prove Lemma 4.1, we need to extend the H^1-stability of the L^2 projection P_{X_h} from $X \cap H^1(D)^d$ to $H^1(D)^d$. This is obtained by characeterizing the orthogonal complement of X_h in V_h, as discussed in subsection 4.1. The proof of Lemma 4.1 is presented at the end of this section after introducing the orthogonal decomposition and some properties of the discrete semigroup.

4.1. Orthogonal complement of X_h in V_h

Let $X_h^\perp$ be the orthogonal complement of X_h in V_h, i.e., $V_h = X_h \oplus X_h^\perp$, namely, any $v_h \in V_h$ has an orthogonal decomposition

$$v_h = w_h + z_h \quad \text{with} \quad w_h \in X_h \text{ and } z_h \in X_h^\perp \text{ satisfying } (w_h, z_h) = 0. \tag{4.4}$$

This decomposition (see [7] for a slightly different presentation) is stable in the H^1 norm, as shown in the following lemma.

Lemma 4.2. *The orthogonal decomposition in* (4.4) *is stable in the H^1 norm, i.e.,*

$$\|w_h\|_{H^1} + \|z_h\|_{H^1} \lesssim \|v_h\|_{H^1} \quad \forall\, v_h \in V_h. \tag{4.5}$$

Proof. For any given $v_h \in V_h$, the inf-sup condition (2.22) implies that there exists a unique solution $(w_h, q_h) \in V_h \times Q_h$ of the following equations:

$$\begin{aligned} (w_h, a_h) - (q_h, \nabla\cdot a_h) &= (v_h, a_h) && \forall\, a_h \in V_h,\\ (\nabla\cdot w_h, \eta_h) &= 0 && \forall\, \eta_h \in Q_h \end{aligned}$$

which shows that $w_h \in X_h$ and

$$(w_h, a_h) = (v_h, a_h) \qquad \forall\, a_h \in X_h.$$

Let $z_h = v_h - w_h$ where w_h and z_h are the functions in the orthogonal decomposition (4.4). Before estimating (w_h, q_h) directly, we first introduce $(w, q) \in H^1(D)^d \times H^1(D)$ to be the solution of the continuous problem

$$\begin{aligned} (w, a) - (q, \nabla \cdot a) &= (v_h, a) && \forall\, a \in H^1(D)^d, \\ (\nabla \cdot w, \eta) &= 0 && \forall\, \eta \in L^2(D). \end{aligned}$$

Via integration by parts in the first equation above, one can obtain that $w = v_h - \nabla q$, with $q \in H^1(D)$ being the weak solution of

$$\begin{cases} \Delta q = \nabla \cdot v_h & \text{in } D, \\ \quad q = 0 & \text{on } \partial D. \end{cases} \tag{4.6}$$

The equation above has the standard H^2 elliptic regularity, i.e.,

$$\|q\|_{H^2} \lesssim \|\nabla \cdot v_h\|_{L^2} \lesssim \|v_h\|_{H^1}, \tag{4.7}$$

which implies that

$$\|w\|_{H^1} = \|v_h - \nabla q\|_{H^1} \lesssim \|v_h\|_{H^1}\,. \tag{4.8}$$

Denote $\theta_h = w_h - \Pi_h w$ and $\phi_h = q_h - P_{Q_h} q$. They satisfy

$$\begin{aligned} (\theta_h, a_h) - (\phi_h, \nabla \cdot a_h) &= (w - \Pi_h w, a_h) + (P_{Q_h} q - q, \nabla \cdot a_h) && \forall\, a_h \in V_h, \\ (\nabla \cdot \theta_h, \eta_h) &= 0 && \forall\, \eta_h \in Q_h. \end{aligned}$$

Substituting $a_h = \theta_h$ and $\eta_h = \phi_h$ into the equations above and using the inverse inequality (2.21) we obtain

$$\begin{aligned} \|\theta_h\|_{L^2}^2 &= (w - \Pi_h w, \theta_h) + (P_{Q_h} q - q, \nabla \cdot \theta_h) \\ &\le \|w - \Pi_h w\|_{L^2} \|\theta_h\|_{L^2} + \|P_{Q_h} q - q\|_{L^2} \|\nabla \cdot \theta_h\|_{L^2} \\ &\lesssim h \|w\|_{H^1} \|\theta_h\|_{L^2} + h^2 \|q\|_{H^2} \|\theta_h\|_{H^1} \\ &\lesssim h\big(\|w\|_{H^1} + \|q\|_{H^2}\big) \|\theta_h\|_{L^2}, \end{aligned}$$

which together with (4.7) and (4.8) implies

$$\|\theta_h\|_{H^1} \lesssim h^{-1} \|\theta_h\|_{L^2} \lesssim \|w\|_{H^1} + \|q\|_{H^2} \le \|v_h\|_{H^1}.$$

Therefore, the H^1-stability (2.16) of the Fortin projection operator gives

$$\|w_h\|_{H^1} = \|\theta_h + \Pi_h w\|_{H^1} \lesssim \|v_h\|_{H^1},$$

which yields

$$\|z_h\|_{H^1} = \|v_h - w_h\|_{H^1} \lesssim \|v_h\|_{H^1}.$$

This proves the desired H^1-stability result in (4.5). ∎

Remark 4.2. The H^1 stability in Lemma 4.2 implies the following properties:

$$\|P_{X_h} v\|_{H^1} \le C \|v\|_{H^1} \qquad \text{for } v \in H^1(D)^d, \tag{4.9}$$

$$\|v - P_{X_h} v\|_{H^1} \lesssim \inf_{v_h \in X_h} \|v - v_h\|_{H^1} \le Ch \|v\|_{H^2} \qquad \text{for } v \in \mathcal{D}(A), \tag{4.10}$$

$$\|v - P_{X_h} v\|_{L^2} \lesssim \inf_{v_h \in X_h} \|v - v_h\|_{L^2} \le Ch^2 \|v\|_{H^2} \qquad \text{for } v \in \mathcal{D}(A). \tag{4.11}$$

For any $v \in \mathcal{D}(A)$, we denote by q_v the solution of (3.8), and denote by $R_{X_h} : V \to X_h$ the Stokes–Ritz projection defined by

$$\big(v - R_{X_h} v, w_h\big) + \big(A_h (v - R_{X_h} v), w_h\big) - (q_v, \nabla \cdot w_h) = 0 \qquad \forall\, w_h \in X_h, \tag{4.12}$$

which is equivalent to finding $(R_{X_h} v, q_{v,h}) \in V_h \times Q_h$ satisfying

$$\begin{aligned} \big(v - R_{X_h} v, w_h\big) + 2\big(\mathbb{D}(v - R_{X_h} v), \mathbb{D}(w_h)\big) - (q_v - q_{v,h}, \nabla \cdot w_h) &= 0 && \forall\, w_h \in V_h, \\ \big(\nabla \cdot (v - R_{X_h} v), q_h\big) &= 0 && \forall\, q_h \in Q_h. \end{aligned} \tag{4.13}$$

The Stokes–Ritz projection has the following error bound (cf. [14, Lemma 2.44 and Lemma 2.45] and [14, Proposition 4.18]):

$$\|v - R_{X_h} v\|_{L^2} + h\|v - R_{X_h} v\|_{H^1} \lesssim h^2\big(\|v\|_{H^2} + \|q_v\|_{H^1}\big) \leq Ch^2\|v\|_{H^2} \quad \forall\, v \in \mathcal{D}(A). \quad (4.14)$$

4.2. Fractional powers of $I + A$

Since the Stokes operator A defined in (3.7) is self-adjoint and positive semi-definite, it follows that the operator $z + A$ is invertible on X for $z \in \mathbb{C}\backslash(-\infty, 0]$, and $-A$ generates a bounded analytic semigroup $E(t) = e^{-tA}$ on X; see [1, Example 3.7.5]. The fractional powers of the positive definite operator $I + A$ (with compact inverse) can be defined by means of the spectral decomposition (see [25, Appendix B.2]). We present some estimates in the following two lemmas. The proof is based on complex interpolation and the details can be found in [29, Appendix B].

Lemma 4.3. *The following equivalence relations hold for* $s \in [0, 2]$:

$$\|v\|_{H^s} \lesssim \|(I + A)^{\frac{s}{2}} v\|_{L^2} \lesssim \|v\|_{H^s} \qquad \forall\, v \in \mathcal{D}(A^{\frac{s}{2}}), \quad (4.15)$$

$$\|v\|_{H^{-s}} \lesssim \|(I + A)^{-\frac{s}{2}} v\|_{L^2} \lesssim \|v\|_{H^{-s}} \qquad \forall\, v \in \mathcal{D}(A^{\frac{s}{2}})', \quad (4.16)$$

where $\mathcal{D}(A^{\frac{s}{2}}) = (X, \mathcal{D}(A))_{[s/2]} = X \cap H^s(D)^d$ *for* $s \in [0, 2]$ *denotes the complex interpolation spaces between* X *and* $\mathcal{D}(A)$.

Lemma 4.4. *For any* $v \in X$, *the following results hold for* $t \in (0, T]$:

$$\|(I + A)^{\gamma} E(t) v\|_{L^2} \lesssim t^{-\gamma}\|v\|_{L^2} \qquad \forall\, \gamma \geq 0, \quad (4.17)$$

$$\|(I + A)^{-\mu}(I - E(t)) v\|_{L^2} \lesssim t^{\mu}\|v\|_{L^2} \qquad \forall\, \mu \in [0, 1]. \quad (4.18)$$

The following results hold for $0 \leq t_1 \leq t_2 \leq T$:

$$\int_{t_1}^{t_2} \|(I + A)^{\frac{\rho}{2}} E(t_2 - s) v\|_{L^2}^2 \, \mathrm{d}s \lesssim (t_2 - t_1)^{1-\rho}\|v\|_{L^2}^2 \qquad \forall\, \rho \in [0, 1], \quad (4.19)$$

$$\Big\|(I + A)^{\rho} \int_{t_1}^{t_2} E(t_2 - s) v \, \mathrm{d}s\Big\|_{L^2} \lesssim (t_2 - t_1)^{1-\rho}\|v\|_{L^2} \qquad \forall\, \rho \in [0, 1]. \quad (4.20)$$

4.3. The discrete semigroup from spatial discretization

Let $0 = \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,M_h}$ be the eigenvalues of the discrete Stokes operator $A_h : X_h \to X_h$ with corresponding orthonormal eigenvectors $\{\varphi_{h,j}\}_{j=1}^{M_h} \subset X_h$ and $\dim(X_h) = M_h$. The operator $-A_h$ generates a bounded analytic semigroup on X_h defined by

$$E_h(t) = e^{-tA_h}, \quad (4.21)$$

which can be expressed as $E_h(t)v_h = \sum_{j=1}^{M_h} e^{-t\lambda_{h,j}}(v_h, \varphi_{h,j})\varphi_{h,j}$ for all $v_h \in X_h$. Hence, following the proof [38, Lemma 3.9] and [26, Lemma 3.2 (iii)], we have

$$\|(I + A_h)^{\frac{\gamma}{2}} E_h(t) v_h\|_{L^2} \lesssim t^{-\frac{\gamma}{2}}\|v_h\|_{L^2} \qquad \text{for } v_h \in X_h,\ \gamma \in [0, 1], \quad (4.22)$$

$$\int_0^t \|(I + A_h)^{\frac{1}{2}} E_h(s) v_h\|_{L^2}^2 \mathrm{d}s \lesssim \|v_h\|_{L^2}^2, \qquad \text{for } v_h \in X_h. \quad (4.23)$$

Let

$$\Phi_h(t) := E(t)P_X - E_h(t)P_{X_h} \qquad \text{for} \quad t \in [0, T], \quad (4.24)$$

denote the error in approximating the continuous semigroup. The main results of this subsection are the estimates of $\Phi_h(t)$ presented in the following lemma.

Lemma 4.5. *For* $v \in H^1(D)^d$, *the following estimates hold*:

$$\big\|\Phi_h(t) v\big\|_{L^2} \leq C t^{-\frac{1}{2}} h^2 \|v\|_{H^1}, \quad (4.25)$$

and

$$\Big(\int_0^t \big\|\Phi_h(s) v\big\|_{L^2}^2 \, \mathrm{d}s\Big)^{\frac{1}{2}} \leq Ch^2\|v\|_{H^1}. \quad (4.26)$$

For $v \in L^2(D)^d$, the following estimates hold:

$$\big\|\Phi_h(t)v\big\|_{L^2} \le Ct^{-1}h^2\|v\|_{L^2}, \tag{4.27}$$

and

$$\Big\| \int_0^t \Phi_h(s)v\,\mathrm{d}s\Big\|_{L^2} + h\Big\| \int_0^t \nabla\Phi_h(s)v\,\mathrm{d}s\Big\|_{L^2} \le Ch^2\|v\|_{L^2}. \tag{4.28}$$

Moreover for $v \in \mathcal{D}(A^{\frac{1+\rho}{2}})$, it holds that

$$\big\|\Phi_h(t)v\big\|_{L^2} + h\big\|\nabla\Phi_h(t)v\big\|_{L^2} \le Ct^{-\frac{1-\rho}{2}}h^2\|v\|_{H^{1+\rho}} \quad \textit{for } \rho \in [0,1]. \tag{4.29}$$

Remark 4.3. Lemma 4.5 is proved later based on an orthogonal decomposition of $v = P_X v + \nabla q$ for $v \in L^2(D)^d$ at the end of this subsection, and it does not require $\nabla\cdot v = 0$ in the cases $v \in L^2(D)^d$ and $v \in H^1(D)^d$.

The proof of Lemma 4.5 is based on error estimates for the semi-discrete FEM for the deterministic linear Stokes problem

$$\begin{cases} \partial_t u - \nabla\cdot\mathbb{T}(u,p) = 0 & \text{in } D\times[0,T],\\ \nabla\cdot u = 0 & \text{in } D\times[0,T],\\ \mathbb{T}(u,p)\mathrm{n} = 0 & \text{on } \partial D\times[0,T],\\ u(0) = u^0 & \text{in } D. \end{cases} \tag{4.30}$$

which can be rewritten into the abstract formulation (by applying P_X to the first relation)

$$\partial_t u + Au = 0 \qquad \text{with} \qquad u(0) = u^0 \in X. \tag{4.31}$$

The solution of (4.31) can be represented by $u(t) = E(t)P_X u^0$ in terms of the semigroup generated by the Stokes operator. The FEM for (3.4) reads: for given $u_h(0) = P_{X_h}u^0$, find $(u_h(t), p_h(t)) \in V_h \times Q_h$ such that

$$\begin{cases} \big(\partial_t u_h(t), v_h\big) + 2\big(\mathbb{D}(u_h(t)), \mathbb{D}(v_h)\big) - (p_h(t), \nabla\cdot v_h) = 0 & \forall\, v_h \in V_h,\\ (\nabla\cdot u_h(t), q_h) = 0 & \forall\, q_h \in Q_h \end{cases} \tag{4.32}$$

which can be expressed in the following abstract form by choosing $v_h \in X_h$:

$$\frac{\mathrm{d}}{\mathrm{d}t}u_h(t) + A_h u_h(t) = 0 \quad \text{with} \quad u_h(0) = P_{X_h}u^0 \tag{4.33}$$

or equivalently $u_h(t) = E_h(t)P_{X_h}u^0$.

In the following lemma, we present some estimate for the difference between the continuous and discrete resolvent operators, i.e., $(z+A)^{-1}$ and $(z+A_h)^{-1}$. The proof can be found in [29, Appendix B].

Lemma 4.6. *For any $f \in L^2(D)^d$, let w and w_h be the solution and finite element solution of the Stokes equations*

$$\begin{cases} zw - \nabla\cdot\mathbb{T}(w,p) = f & \text{in } D,\\ \nabla\cdot w = 0 & \text{in } D,\\ \mathbb{T}(w,p)\mathrm{n} = 0 & \text{on } \partial D, \end{cases} \tag{4.34}$$

and

$$\begin{cases} (zw_h, v_h) + 2\big(\mathbb{D}(w_h), \mathbb{D}(v_h)\big) - (p_h, \nabla\cdot v_h) = (f, v_h) & \forall\, v_h \in V_h,\\ (\nabla\cdot w_h, q_h) = 0 & \forall\, q_h \in Q_h, \end{cases} \tag{4.35}$$

where $z \in \Sigma_\phi := \{1 + z' \in \mathbb{C} : |\arg(z')| < \phi\}$ for some $\phi \in (0,\pi)$. Then the following results hold:

$$\|w - w_h\|_{L^2} + h\|\nabla(w - w_h)\|_{L^2} \le Ch^2(\|w\|_{H^2} + \|p\|_{H^1}), \tag{4.36}$$

$$\|w\|_{H^2} + \|p\|_{H^1} \le C\|f\|_{L^2}, \tag{4.37}$$

where the constant C is independent of $z \in \Sigma_\phi$ (but may depend on ϕ).

Remark 4.4. The first equation in (4.34) can be written as

$$\nabla p = f - (z - \Delta)w.$$

By applying P_X to the first equation in (4.34) and using relation $-P_X\mathbb{T}(w,p) = -\Delta w + \nabla q_w$, as shown in (3.11), we obtain

$$\nabla q_w = P_X f - (z - \Delta)w.$$

Combining the two equations above yields

$$\nabla p = f - P_X f + \nabla q_w.$$

In the case $f \in X$, we obtain $p = q_w$ and therefore (4.36) reduces to

$$\|w - w_h\|_{L^2} + h\|\nabla w - \nabla w_h\|_{L^2} \le Ch^2\|w\|_{H^2}, \tag{4.38}$$

where we have used the inequality $\|q_w\|_{H^1} \lesssim \|w\|_{H^2}$.

In the following lemma, we study the case with $v \in X \cap H^1(D)^d$.

Lemma 4.7. *It holds that*

$$\int_0^t \|\Phi_h(s)g\|_{L^2}^2 \,\mathrm{d}s \le Ch^4\|g\|_{H^1}^2 \quad \forall\, g \in X \cap H^1(D)^d. \tag{4.39}$$

Proof. Let (u,p) and (u_h,p_h) be the solution of (3.4) and (4.32) with $u(0) = g \in X \cap H^1(D)^d$ and $u_h(0) = P_{X_h}g$, respectively. By noting (4.24),

$$\Phi_h(s)g = u(s) - u_h(s). \tag{4.40}$$

Subtracting (4.32) from (3.4) yields

$$\begin{cases} \big(\partial_t(u - u_h), v_h\big) + 2\big(\mathbb{D}(u - u_h), \mathbb{D}(v_h)\big) - (p - p_h, \nabla\cdot v_h) = 0 & \forall\, v_h \in V_h, \\ (\nabla\cdot(u - u_h), \eta_h) = 0 & \forall\, \eta_h \in Q_h, \end{cases}$$

and choosing $v_h \in X_h$ and using the Ritz projection R_{X_h} defined in (4.12), we have

$$\begin{aligned} &\big(\partial_t(u - u_h), v_h\big) + \big((I + A_h)(R_{X_h}u - u_h), v_h\big) \\ &= \big(u - u_h, v_h\big) + (q - P_{Q_h}q, \nabla\cdot v_h) - (q_u - P_{Q_h}q_u, \nabla\cdot v_h), \end{aligned} \tag{4.41}$$

where q_u is defined in (3.8) (with v replaced by u there). By denoting $\widetilde{e}_h = P_{X_h}u - u_h \in X_h$ and choosing $v_h = (I + A_h)^{-1}\widetilde{e}_h \in X_h$ in (4.41), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(I + A_h)^{-\frac12}\widetilde{e}_h\|_{L^2}^2 + \|\widetilde{e}_h\|_{L^2}^2 \qquad (4.42)\\ &= \|(I + A_h)^{-\frac12}\widetilde{e}_h\|_{L^2}^2 + \big(P_{X_h}u - R_{X_h}u, \widetilde{e}_h\big) + R_q(\widetilde{e}_h) + R_{q_u}(\widetilde{e}_h) \\ &\le \|(I + A_h)^{-1}\widetilde{e}_h\|_{L^2}^2 + C\|R_{X_h}u - P_{X_h}u\|_{L^2}^2 + \frac14\|\widetilde{e}_h\|_{L^2}^2 + |R_q(\widetilde{e}_h)| + |R_{q_u}(\widetilde{e}_h)| \\ &\le \|(I + A_h)^{-1}\widetilde{e}_h\|_{L^2}^2 + Ch^4\|u\|_{H^2}^2 + \frac14\|\widetilde{e}_h\|_{L^2}^2 + |R_q(\widetilde{e}_h)| + |R_{q_u}(\widetilde{e}_h)|, \end{aligned}$$

where

$$R_\xi(\widetilde{e}_h) = \big(\xi - P_{Q_h}\xi, \nabla\cdot[(I + A_h)^{-1}\widetilde{e}_h]\big), \qquad \xi = q, q_u$$

It is easy to see that $w = (I + A)^{-1}P_X\widetilde{e}_h$ and $w_h = (I + A_h)^{-1}P_{X_h}\widetilde{e}_h$ are the solutions of (4.34) and (4.35), respectively, with $z = 1$ and $f = \widetilde{e}_h$. Then applying Lemma 4.6 yields

$$\big\|\big[(I + A)^{-1}P_X - (I + A_h)^{-1}P_{X_h}\big]\widetilde{e}_h\big\|_{H^1} \lesssim h\|\widetilde{e}_h\|_{L^2},$$

which implies that

$$\begin{aligned} |R_q(\widetilde{e}_h)| &= \big|\big(q - P_{Q_h}q, \nabla\cdot\big[(I + A)^{-1}P_X\widetilde{e}_h\big]\big)\big| \\ &\quad + \big|\big(q - P_{Q_h}q, \nabla\cdot\big[(I + A)^{-1}P_X - (I + A_h)^{-1}P_{X_h}\big]\widetilde{e}_h\big)\big| \\ &\lesssim \|q - P_{Q_h}q\|_{H^{-1}}\|\nabla\cdot\big[(I + A)^{-1}P_X\widetilde{e}_h\big]\|_{H^1} \\ &\quad + \|q - P_{Q_h}q\|_{L^2}\big\|\big[(I + A)^{-1}P_X - (I + A_h)^{-1}P_{X_h}\big]\widetilde{e}_h\big\|_{H^1} \\ &\lesssim h^2\|q\|_{H^1}\|(I + A)^{-1}P_X\widetilde{e}_h\|_{H^2} + h\|q\|_{H^1}h\|\widetilde{e}_h\|_{L^2} \\ &\lesssim h^2\|q\|_{H^1}\|\widetilde{e}_h\|_{L^2} \end{aligned}$$

$$\leq Ch^4\|q\|_{H^1}^2 + \frac{1}{8}\|\widetilde{e}_h\|_{L^2}^2, \tag{4.43}$$

where we have used the inequality $\|q - P_{Q_h}q\|_{H^{-1}} \lesssim h^2\|q\|_{H^1}$ and Young's inequality; see Property (2) in Section 2.3. Similarly, since $\|q_u\|_{H^1}^2 \lesssim \|u\|_{H^2}$, it follows that

$$|R_{q_u}(\widetilde{e}_h)| \leq Ch^4\|q_u\|_{H^1}^2 + \frac{1}{8}\|\widetilde{e}_h\|_{L^2}^2 \leq Ch^4\|u\|_{H^2}^2 + \frac{1}{8}\|\widetilde{e}_h\|_{L^2}^2. \tag{4.44}$$

Substituting (4.43)–(4.44) into (4.42), and using Gronwall's inequality with $\widetilde{e}_h(0) = 0$, we see that

$$\begin{aligned}\|(I + A_h)^{-\frac{1}{2}}\widetilde{e}_h(t)\|_{L^2}^2 + \int_0^t \|\widetilde{e}_h(s)\|_{L^2}^2 \,\mathrm{d}s \lesssim h^4 \int_0^t \left(\|u\|_{H^2}^2 + \|q\|_{H^1}^2\right) \mathrm{d}s \lesssim h^4\|u(0)\|_{H^1}^2 \\ = h^4\|g\|_{H^1}^2,\end{aligned} \tag{4.45}$$

where the basic energy inequality for the Stokes equations has been used. Since $u(s) - u_h(s) = u(s) - P_{X_h}u(s) + \widetilde{e}_h$, by using the inequality above we get

$$\int_0^t \|u(s) - u_h(s)\|_{L^2}^2 \,\mathrm{d}s \lesssim h^4\|g\|_{H^1}^2.$$

This proves the result of Lemma 4.7 in view of (4.40). ∎

Now we turn back to the proof of Lemma 4.5.

Proof of Lemma 4.5. For $v \in L^2(D)^d$ we denote

$$w = (z + A)^{-1}P_X v \quad \text{ and } \quad w_h = (z + A_h)^{-1}P_{X_h}v,$$

which are the solutions of (4.34) and (4.35) with $f = v$.

Since $-A$ generates a bounded analytic semigroup on X, there exists an angle $\phi \in (\frac{\pi}{2}, \pi)$ such that the operator $(z + A)^{-1}$ is analytic with respect to z in the sector $\Sigma_\phi = \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \phi\}$. Moreover, the semigroup $E(t) = e^{-tA}$ can be expressed in terms of the resolvent operator $(z + A)^{-1}$ through a contour integral on the complex plane, i.e.,

$$E(t)P_X v = \frac{1}{2\pi i}\int_\Gamma (z + A)^{-1}P_X v e^{zt}\,\mathrm{d}z, \quad \text{with } \Gamma = \{1 + z : |\arg(z)| = \phi\} \subset \Sigma_\phi. \tag{4.46}$$

Similarly,

$$E_h(t)P_{X_h}v = \frac{1}{2\pi i}\int_\Gamma (z + A_h)^{-1}P_{X_h}v e^{zt}\,\mathrm{d}z. \tag{4.47}$$

Therefore,

$$\begin{aligned}\|\Phi_h(t)v\|_{L^2} &= \|E(t)P_X v - E_h(t)P_{X_h}v\|_{L^2} \\ &= \left\|\frac{1}{2\pi i}\int_\Gamma \left[(z + A)^{-1}P_X v - (z + A_h)^{-1}P_{X_h}v\right]e^{zt}\,\mathrm{d}z\right\|_{L^2} \\ &\lesssim \int_\Gamma \|w - w_h\|_{L^2}e^{\mathrm{Re}(z)t}\,|\mathrm{d}z| \\ &\lesssim \int_\Gamma h^2\|v\|_{L^2}e^{\mathrm{Re}(z)t}\,|\mathrm{d}z| \quad \text{(This follows from Lemma 4.6 with } f = v\text{)} \\ &\lesssim t^{-1}h^2\|v\|_{L^2} \quad \text{for } t \in (0, T].\end{aligned} \tag{4.48}$$

This proves (4.27).

For $v \in \mathcal{D}(A^{\frac{1+\rho}{2}})$, by (4.38) we have

$$\begin{aligned}\|w - w_h\|_{L^2} + h\|\nabla(w - w_h)\|_{L^2} &\lesssim h^2\|w\|_{H^2} \\ &= h^2\|(z + A)^{-1}v\|_{H^2} \\ &\lesssim h^2\|(I + A)(z + A)^{-1}v\|_{L^2} \\ &= h^2\big\|(I + A)^{\frac{1-\rho}{2}}(z + A)^{-1}(I + A)^{\frac{1+\rho}{2}}v\big\|_{L^2}\end{aligned}$$

$$\lesssim h^2|z|^{-\frac{1+\rho}{2}}\big\|(I+A)^{\frac{1+\rho}{2}}v\big\|_{L^2} \quad \forall\,\rho\in[0,1], \tag{4.49}$$

where we have used the interpolation inequality to get

$$\|(I+A)^\theta(z+A)^{-1}v\|_{L^2}\le C_\theta|z|^{-(1-\theta)}\|v\|_{L^2} \quad \forall\,\theta\in[0,1],\ v\in X. \tag{4.50}$$

Substituting (4.49) into (4.48) and using (4.15) give

$$\begin{aligned}\|\Phi_h(t)v\|_{L^2}+h\|\nabla\Phi_h(t)v\|_{L^2} &\lesssim h^2\big\|(I+A)^{\frac{1+\rho}{2}}v\big\|_{L^2}\int_\Gamma |z|^{-\frac{1+\rho}{2}}e^{\mathrm{Re}(z)t}\,|\mathrm{d}z|.\\ &\lesssim t^{-\frac{1-\rho}{2}}h^2\|v\|_{H^{1+\rho}} \qquad \forall\,\rho\in[0,1],\ v\in\mathcal{D}(A^{\frac{1+\rho}{2}}).\end{aligned} \tag{4.51}$$

This proves (4.29).

In order to prove (4.25)–(4.26), we consider the orthogonal decomposition $v=P_Xv+\nabla q$ for a function $v\in H^1(D)^d$, where $q\in H^1(D)$ is the weak solution of

$$\begin{cases}\Delta q=\nabla\cdot v & \text{in } D\\ \ \ q=0 & \text{on } \partial D,\end{cases}$$

with the regularity

$$\|q\|_{H^2}\lesssim\|\nabla\cdot v\|_{L^2}\lesssim\|v\|_{H^1}. \tag{4.52}$$

Since $\Phi_h(t)\nabla q=E(t)P_X\nabla q-E_h(t)P_{X_h}\nabla q=-E_h(t)P_{X_h}\nabla q\in X_h$, by using the self-adjointness of $E_h(t)$ we have

$$\begin{aligned}(\Phi_h(t)\nabla q,a_h)&=-(P_{X_h}\nabla q,E_h(t)a_h)\\ &=(q,\nabla\cdot[E_h(t)a_h]) \quad \text{(since } E_h(t)a_h\in X_h \text{ and } q=0 \text{ on } \partial D)\\ &=(q-q_h,\nabla\cdot[E_h(t)a_h]) \quad \text{(for any } q_h\in H^1_0(D)\cap Q_h)\\ &=-(E_h(t)P_{X_h}\nabla(q-q_h),a_h)\\ &=-((I+A_h)^{\frac12}E_h(t)(I+A_h)^{-\frac12}P_{X_h}\nabla(q-q_h),a_h),\end{aligned}$$

which implies that

$$\Phi_h(t)\nabla q=-(I+A_h)^{\frac12}E_h(t)(I+A_h)^{-\frac12}P_{X_h}\nabla(q-q_h) \quad \forall\, q_h\in Q_h\cap H^1_0(D). \tag{4.53}$$

By using (4.22)–(4.23) and (2.18), we can obtain the following estimates:

$$\|\Phi_h(t)\nabla q\|_{L^2}\lesssim \inf_{q_h\in Q_h\cap H^1_0(D)} t^{-\frac12}\|q-q_h\|_{L^2}\lesssim t^{-\frac12}h^2\|q\|_{H^2},$$

$$\left(\int_0^T\|\Phi_h(t)\nabla q\|_{L^2}^2\mathrm{d}t\right)^{\frac12}\lesssim \inf_{q_h\in Q_h\cap H^1_0(D)}\|q-q_h\|_{L^2}\lesssim h^2\|q\|_{H^2}.$$

Substituting (4.52) into the two inequalities above gives

$$\|\Phi_h(t)\nabla q\|_{L^2}\lesssim t^{-\frac12}h^2\|v\|_{H^1} \quad\text{and}\quad \left(\int_0^t\|\Phi_h(s)\nabla q\|_{L^2}^2\,\mathrm{d}s\right)^{\frac12}\lesssim h^2\|v\|_{H^1}. \tag{4.54}$$

By using inequality (4.51) with $\rho=0$ (with v replaced by P_Xv therein) and Lemma 4.7 with $g=P_Xv$, we see that

$$\|\Phi_h(t)P_Xv\|_{L^2}\lesssim t^{-\frac12}h^2\|v\|_{H^1} \quad\text{and}\quad \left(\int_0^t\|\Phi_h(s)P_Xv\|_{L^2}^2\,\mathrm{d}s\right)^{\frac12}\lesssim h^2\|v\|_{H^1}. \tag{4.55}$$

By noting the decomposition $v=P_Xv+\nabla q$, (4.25)-(4.26) follow from the last two inequalities immediately.

To this end, we combine (4.46) and (4.47) to get

$$\begin{aligned}\int_0^t\Phi_h(s)v\,\mathrm{d}s&=\int_0^t\frac{1}{2\pi i}\int_\Gamma[(z+A)^{-1}P_Xv-(z+A_h)^{-1}P_{X_h}v]e^{zs}\,\mathrm{d}z\mathrm{d}s\\ &=\frac{1}{2\pi i}\int_\Gamma z^{-1}[(z+A)^{-1}P_Xv-(z+A_h)^{-1}P_{X_h}v]e^{zs}\,\mathrm{d}z\end{aligned}$$

$$= \frac{1}{2\pi i}\int_\Gamma z^{-1}[w-w_h]e^{zt}\,\mathrm{d}z.$$

By applying Lemma 4.6 with $f=v$, we have

$$\begin{aligned}
&\Big\|\int_0^t \Phi_h(s)v\,\mathrm{d}s\Big\|_{L^2} + h\Big\|\nabla\int_0^t \Phi_h(s)v\,\mathrm{d}s\Big\|_{L^2}\\
&\lesssim \int_\Gamma |z|^{-1}\big(\|w-w_h\|_{L^2} + h\|\nabla w-\nabla w_h\|_{L^2}\big)e^{\mathrm{Re}(z)t}\,|\mathrm{d}z|\\
&\lesssim h^2\|v\|_{L^2}\int_\Gamma |z|^{-1}e^{\mathrm{Re}(z)t}\,|\mathrm{d}z| \lesssim h^2\|v\|_{L^2}. \qquad (4.56)
\end{aligned}$$

This proves (4.28). ∎

4.4. The discrete semigroup in the full discretization

Let $\lambda_j^*\geq 0$ and ϕ_j^*, $j=1,2,\dots$, be the eigenvalues and eigenfunctions of the operator $A:D(A)\to X$. Similarly, let $\lambda_{h,j}^*\geq 0$ and $\phi_{h,j}^*$, $j=1,\dots,M_h$, be the eigenvalues and eigenfunctions of the operator $A_h:X_h\to X_h$.

We denote by $R(\tau A_h):X_h\to X_h$ the linear operator defined by

$$R(\tau A_h)v_h = \sum_{j=1}^{M_h} R(\tau\lambda_{h,j}^*)(v_h,\phi_{h,j}^*)\phi_{h,j}^* \quad\text{with}\ \ R(z)=\frac{1}{1+z}\ \ \text{for}\ \ z\in\mathbb{R},\ z\neq -1 \qquad (4.57)$$

with which we can rewrite the discrete semigroup $\bar E_{h,\tau}$ defined in (2.28) by

$$\bar E_{h,\tau}v_h = R(\tau A_h)v_h \quad\text{for}\quad v_h\in X_h.$$

As a time discrete version of (4.22) and (4.23), the following estimates hold for any $v_h\in X_h$,

$$\|(I+A_h)^{\frac{\gamma}{2}}\bar E_{h,\tau}^n v_h\|_{L^2} \lesssim t_n^{-\frac{\gamma}{2}}\|v_h\|_{L^2} \qquad \text{for } 1\leq n\leq N \ \text{ and } \gamma\in[0,1], \qquad (4.58)$$

$$\tau\sum_{j=1}^n \|(I+A_h)^{\frac12}\bar E_{h,\tau}^j v_h\|_{L^2}^2 \lesssim C\|v_h\|_{L^2}^2 \qquad \text{for } 1\leq n\leq N. \qquad (4.59)$$

Remark 4.5. Inequality (4.58) is equivalent to

$$|(1+\lambda_{h,j}^*)^{\frac{\gamma}{2}}R(\tau\lambda_{h,j}^*)^n| \lesssim t_n^{-\frac{\gamma}{2}} \quad\text{for }\ \lambda_{h,j}^*\geq 0\ \text{ and }\ t_n=n\tau\in[0,T]. \qquad (4.60)$$

The proof of this inequality can be found in [38, Lemma 7.3]). Since the function $w_h^j=\bar E_{h,\tau}^j v_h$, $n=1,2,\dots$, are solutions of the equation

$$\frac{w_h^j-w_h^{j-1}}{\tau} + A_h w_h^j = 0.$$

Testing the equation with w_h^j and summing up the results for $j=1,\dots,n$, yield the basic energy inequality

$$\max_{1\leq j\leq n}\frac12\|w_h^j\|_{L^2}^2 + \tau\sum_{j=1}^n \|A_h^{\frac12}w_h^j\|_{L^2}^2 \leq \frac12\|w_h^0\|_{L^2}^2 = \frac12\|v_h\|_{L^2}^2,$$

which implies (4.59).

The error of $\bar E_{h,\tau}^n P_{X_h}$ in approximating $E_h(t_n)P_{X_h}$ is analyzed in the following lemma. The proof can be found in [29, Lemma 4.8].

Lemma 4.8. *Let $\bar\Phi_{h,\tau}^n v := E_h(t_n)P_{X_h}v - \bar E_{h,\tau}^n P_{X_h}v$ for $v\in L^2(D)^d$. Then the following estimates hold:*

$$\big\|\bar\Phi_{h,\tau}^n v\big\|_{L^2} \lesssim \tau^{\frac12}\|v\|_{H^1} \qquad \forall\, v\in H^1(D)^d, \qquad (4.61)$$

$$\big\|\bar\Phi_{h,\tau}^n v\big\|_{L^2} \lesssim t_n^{-\frac12}\tau^{\frac12}\|v\|_{L^2} \qquad \forall\, v\in L^2(D)^d, \qquad (4.62)$$

$$\Big(\tau\sum_{j=1}^{n}\|\bar\Phi^j_{h,\tau}v\|^2_{L^2}\Big)^{\frac12}+\Big\|\tau\sum_{j=1}^{n}\nabla\bar\Phi^j_{h,\tau}v\Big\|_{L^2}\lesssim\tau^{\frac12}\|v\|_{L^2}\qquad\forall\, v\in L^2(D)^d. \tag{4.63}$$

Proof of Lemma 4.1. By using the expressions

$$\Phi_h(t):=E(t)P_X-E_h(t)P_{X_h}\quad\text{and}\quad\bar\Phi^n_{h,\tau}v:=E_h(t_n)P_{X_h}v-\bar E^n_{h,\tau}P_{X_h}v,$$

and the triangle inequality, we have

$$\begin{aligned}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[E(s)P_X-\bar E^j_{h,\tau}P_{X_h}]\,v\|^2_{L^2}\,\mathrm{d}s&\lesssim\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[E_h(s)-E_h(t_j)]P_{X_h}v\|^2_{L^2}\,\mathrm{d}s\\&\quad+\int_0^{t_n}\|\Phi_h(s)v\|^2_{L^2}\,\mathrm{d}s+\tau\sum_{j=1}^{n}\|\bar\Phi^j_{h,\tau}v\|^2_{L^2}.\end{aligned}\tag{4.64}$$

The first term on the right-hand side of (4.64) can be estimated by using (4.23), i.e.,

$$\begin{aligned}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[E_h(s)-E_h(t_j)]P_{X_h}v\|^2_{L^2}\mathrm{d}s&=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|(1-e^{-(t_j-s)A_h})e^{-sA_h}P_{X_h}v\|^2_{L^2}\mathrm{d}s\\&\lesssim\tau\int_0^{t_n}\|A_h^{\frac12}e^{-sA_h}P_{X_h}v\|^2_{L^2}\mathrm{d}s\\&\lesssim\tau\|v\|^2_{L^2}\quad\forall\ v\in L^2(D)^d.\end{aligned}\tag{4.65}$$

The last two terms on the right-hand side of (4.64) have been estimated in (4.26) and (4.63), respectively, which imply (4.1).

Similarly, (4.2) can be proved by using (4.28), (4.63) and (4.65).

To prove (4.3), we rewrite it into

$$\begin{aligned}\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\nabla[E(s)P_X-\bar E^j_{h,\tau}P_{X_h}]\,v\,\mathrm{d}s\Big\|^2_{L^2}&\lesssim\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\nabla[E_h(s)-E_h(t_j)]P_{X_h}\,v\,\mathrm{d}s\Big\|^2_{L^2}\\&\quad+\Big\|\int_0^{t_n}\nabla\Phi_h(s)\,v\,\mathrm{d}s\Big\|^2_{L^2}+\Big\|\tau\sum_{j=1}^{n}\nabla\bar\Phi^j_{h,\tau}v\Big\|^2_{L^2}\\&\lesssim(\tau+h^2)\|v\|^2_{L^2}.\end{aligned}\tag{4.66}$$

The first term on the right-hand side of the last equation was estimated in [24, p. 236, with $\rho=1$ and P_hx replaced by $A_h^{\frac12}P_hx$], i.e.,

$$\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E_h(s)-E_h(t_j)](I+A_h)^{\frac12}P_{X_h}v\mathrm{d}s\Big\|^2_{L^2}\lesssim\tau\|P_{X_h}v\|^2_{L^2}\lesssim\tau\|v\|^2_{L^2}.$$

The estimates for the last two terms follow from (4.28) and (4.63), respectively. The proof of Lemma 4.1 is completed. ∎

5. Error analysis for the stochastic problem

In this section, we first present some estimates of the noise term in the Hilbert–Schmidt norm in Subsection 5.1 and then, the proof of our main theorem in Section 5.2.

5.1. Estimates in the Hilbert–Schmidt norm

Lemma 5.1. *Under Assumptions 2.1–2.3, the operator $B(v):L^2(D)^d\to L^2(D)^d$ satisfies*

$$\|E(t)P_X[B(u)-B(v)]\|^2_{\mathcal{L}_2^0}\lesssim t^{-\frac12}\,\|u-v\|^2_{L^2}\quad\forall\,u,v\in L^2(D)^d,\tag{5.1}$$

and

$$\|B(u)\|^2_{\mathcal{L}_2^0}\lesssim1+\|u\|^2_{H^\beta}\qquad\forall\ u\in H^\beta(D)^d,\ \ \beta\in(\tfrac d2,2),\tag{5.2}$$

$$\|(I+A)^{\frac{1}{2}}P_XB(u)\|^2_{\mathcal{L}_2^0} \lesssim 1+\|u\|^2_{H^\beta} \qquad \forall\, u\in H^\beta(D)^d,\;\; \beta\in(\tfrac{d}{2},2). \tag{5.3}$$

Proof. By using (4.16) with $s=\frac{1}{2}$, (4.17) with $\gamma=\frac{1}{4}$, (3.3) and (2.6), we have

$$\begin{aligned}
&\|E(t)P_X(B(u)-B(v))\|^2_{\mathcal{L}_2^0}\\
&=\sum_\ell \mu_\ell \|E(t)P_X[(B(u)-B(v))\phi_\ell]\|^2_{L^2}\\
&\lesssim \sum_\ell \mu_\ell \|(I+A)^{\frac{1}{4}}E(t)(I+A)^{-\frac{1}{4}}P_X[(B(u)-B(v))\phi_\ell]\|^2_{L^2}\\
&\lesssim t^{-\frac{1}{2}}\sum_\ell \mu_\ell\,\|(B(u)-B(v))\phi_\ell\|^2_{H^{-\frac{1}{2}}}\\
&\lesssim t^{-\frac{1}{2}}\|(B(u)-B(v))\|_{\mathcal{L}_2^0(\mathbb{L}^2,\,\mathbb{H}^{-1/2})}\\
&\lesssim t^{-\frac{1}{2}}\,\|u-v\|^2_{L^2},
\end{aligned} \tag{5.4}$$

and by using (2.7), we have

$$\|B(u)\|^2_{\mathcal{L}_2^0}\le \|B(u)\|^2_{\mathcal{L}_2^0(\mathbb{L}^2,\,\mathbb{H}^1)} \lesssim 1+\|u\|^2_{H^\beta}, \tag{5.5}$$

which imply (5.1)–(5.2). Similarly,

$$\begin{aligned}
\|(I+A)^{\frac{1}{2}}P_XB(u)\|^2_{\mathcal{L}_2^0} &= \sum_\ell \mu_\ell\|(I+A)^{\frac{1}{2}}P_XB(u)\phi_\ell\|^2_{L^2} \qquad (5.6)\\
&\le \|B(u)\|^2_{\mathcal{L}_2^0(\mathbb{L}^2,\,\mathbb{H}^1)} \lesssim 1+\|u\|^2_{H^\beta},
\end{aligned}$$

This proves (5.3). ∎

Remark 5.1. As a result of the estimates in Lemma 5.1, the regularity results in Proposition 3.1 imply that

$$\sup_{t\in[0,T]}\mathbb{E}\|B(u(t))\|^2_{\mathcal{L}_2^0}+\sup_{t\in[0,T]}\mathbb{E}\|(I+A)^{\frac{1}{2}}P_XB(u(t))\|^2_{\mathcal{L}_2^0}\lesssim 1. \tag{5.7}$$

The following stability estimates for the numerical solution can be proved by using Lemma 5.1. The proof can be found in [29, Proof of Lemma 5.2].

Lemma 5.2. *Under Assumptions 2.1–2.3, the numerical solution of the fully discrete method* (2.27) *satisfies the following energy inequality:*

$$\max_{1\le n\le N}\mathbb{E}\|u_h^n\|^2_{H^{\frac{1}{2}}}+\sum_{n=1}^N\mathbb{E}\|u_h^n-u_h^{n-1}\|^2_{L^2}+\tau\sum_{n=1}^N\mathbb{E}\|u_h^n\|^2_{H^1}\lesssim 1. \tag{5.8}$$

5.2. The proof of Theorem 2.4

By iterating (2.27) with respect to n, the full discrete method can be rewritten as

$$u_h^n=\bar{E}^n_{h,\tau}P_{X_h}u_0+\tau\sum_{i=0}^{n-1}\bar{E}^{n-i}_{h,\tau}P_{X_h}f(t_{i+1})+\sum_{i=0}^{n-1}\bar{E}^{n-i}_{h,\tau}P_{X_h}[B(u_h^i)\Delta W_{i+1}]. \tag{5.9}$$

Then, after subtracting (5.9) from (3.13), we obtain the following error equation:

$$\begin{aligned}
u(t_n)-u_h^n =&\,(E(t_n)P_X-\bar{E}^n_{h,\tau}P_{X_h})u_0\\
&+\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[E(t_n-s)P_Xf(s)-\bar{E}^{n-i}_{h,\tau}P_{X_h}f(t_{i+1})]\mathrm{d}s\\
&+\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[E(t_n-s)P_XB(u(s))-\bar{E}^{n-i}_{h,\tau}P_{X_h}B(u_h^i)]\mathrm{d}W(s)\\
=:&\,T_1+T_2+T_3,
\end{aligned} \tag{5.10}$$

which implies that

$$\mathbb{E}\|u(t_n)-u_h^n\|_{L^2}^2 \lesssim \sum_{j=1}^{3}\mathbb{E}\|T_j\|_{L^2}^2. \tag{5.11}$$

The three terms are estimated below one by one.

Since $T_1=\Phi_h(t_n)u^0+\bar{\Phi}_{h,\tau}^n u^0$, by applying (4.29) and (4.61) with $\rho=1$ and $v=u_0\in\mathcal{D}(A)$, we obtain

$$\mathbb{E}\|T_1\|_{L^2}^2 \lesssim \mathbb{E}\|\Phi_h(t_n)u^0\|_{L^2}^2+\mathbb{E}\|\bar{\Phi}_{h,\tau}^n u^0\|_{L^2}^2 \lesssim (\tau+h^4)\mathbb{E}\|u^0\|_{H^2}^2. \tag{5.12}$$

To estimate $\mathbb{E}\|T_2\|_{L^2}$, we rewrite it into

$$\begin{aligned}\mathbb{E}\|T_2\|_{L^2}^2 \lesssim &\mathbb{E}\Big\|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}E(t_n-s)P_X[f(s)-f(t_{i+1})]\mathrm{d}s\Big\|_{L^2}^2 \\ &+\mathbb{E}\Big\|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[E(t_n-s)P_X-\bar{E}_{h,\tau}^{n-i}P_{X_h}]f(t_{i+1})\mathrm{d}s\Big\|_{L^2}^2 \\ &=:T_{21}+T_{22}.\end{aligned} \tag{5.13}$$

By using the Hölder continuity of $f(t)$ in (2.9), we have

$$T_{21}\lesssim \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\|f(s)-f(t_{i+1})\|_{L^2}^2\mathrm{d}s \lesssim \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)\mathrm{d}s\lesssim\tau. \tag{5.14}$$

Through a change of variables $\sigma=t_n-s$ and $j=n-i$, we further split T_{22} into

$$\begin{aligned}T_{22}=&\mathbb{E}\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E(\sigma)P_X-\bar{E}_{h,\tau}^{j}P_{X_h}]f(t_{n-j+1})\mathrm{d}\sigma\Big\|_{L^2}^2 \\ \lesssim&\mathbb{E}\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E(\sigma)-E(t_j)]P_X[f(t_{n-j+1})-f(t_n)]\mathrm{d}\sigma\Big\|_{L^2}^2 \\ &+\mathbb{E}\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E(t_j)P_X-\bar{E}_{h,\tau}^{j}P_{X_h}][f(t_{n-j+1})-f(t_n)]\mathrm{d}\sigma\Big\|_{L^2}^2 \\ &+\mathbb{E}\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E(\sigma)P_X-\bar{E}_{h,\tau}^{j}P_{X_h}]f(t_n)\mathrm{d}\sigma\Big\|_{L^2}^2 \\ =:&T_{22}^a+T_{22}^b+T_{22}^c.\end{aligned} \tag{5.15}$$

Using (4.17) with $\gamma=\frac{1}{2}$, (4.18) with $\mu=\frac{1}{2}$, and the Hölder continuity of f in (2.9), we have the following bound for T_{22}^a

$$\begin{aligned}T_{22}^a\lesssim&\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|(I+A)^{\frac{1}{2}}E(\sigma)(I+A)^{-\frac{1}{2}}[I-E(t_j-\sigma)]P_X[f(t_{n-j+1})-f(t_n)]\|_{L^2}^2\mathrm{d}\sigma \\ \lesssim&\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\sigma^{-1}(t_j-\sigma)\|f(t_{n-j+1})-f(t_n)\|_{L^2}^2\mathrm{d}\sigma \\ \lesssim&\tau\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\sigma^{-1}(t_n-t_{n-j+1})\mathrm{d}\sigma\lesssim\tau\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\sigma^{-1}\sigma\mathrm{d}\sigma\lesssim\tau,\end{aligned} \tag{5.16}$$

where the conditions $t_j-\sigma\le\tau$ and $t_n-t_{n-j+1}=t_{j-1}\le\sigma$ are satisfied for $\sigma\in[t_{j-1},t_j]$, ensuring the integrability in the last line.

By using (4.27), (4.62) and the Assumption 2.9, we have the bound

$$T_{22}^b\lesssim\mathbb{E}\Big(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[\Phi_h(t_j)+\bar{\Phi}_{h,\tau}^j][f(t_{n-j+1})-f(t_n)]\|_{L^2}\mathrm{d}\sigma\Big)^2 \tag{5.17}$$

$$\begin{aligned}
&\lesssim \mathbb{E}\Big(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(t_j^{-1}h^2+t_j^{-\frac{1}{2}}\tau^{\frac{1}{2}})\|f(t_{n-j+1})-f(t_n)\|_{L^2}\mathrm{d}\sigma\Big)^2\\
&\lesssim (\tau+h^4)\Big(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}t_j^{-1}(t_n-t_{n-j+1})^{\frac{1}{2}}\mathrm{d}\sigma\Big)^2\\
&\lesssim (\tau+h^4)\Big(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\sigma^{-1}\sigma^{\frac{1}{2}}\mathrm{d}\sigma\Big)^2\lesssim \tau+h^4.
\end{aligned}$$

for T_{22}^b. Moreover, by (4.2) it is easy to see

$$T_{22}^c\lesssim(\tau+h^4)\|f(t_n)\|_{L^2}^2\lesssim\tau+h^4. \tag{5.18}$$

Substituting (5.14)–(5.18) into (5.13) yields

$$\mathbb{E}\|T_2\|_{L^2}^2\lesssim\tau+h^4. \tag{5.19}$$

It remains to estimate the term $\mathbb{E}\|T_3\|_{L^2}$ in (5.10). Taking the same approach as above, by using (2.4) and a change of variables $\sigma=t_n-s$ and $j=n-i$, we first rewrite the bound of $\mathbb{E}\|T_3\|_{L^2}$ into three parts as follows:

$$\begin{aligned}
\mathbb{E}\|T_3\|_{L^2}^2=&\,\mathbb{E}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\|E(t_n-s)P_XB(u(s))-\bar{E}_{h,\tau}^{n-i}P_{X_h}B(u_h^i)\|_{\mathcal{L}_2^0}^2\mathrm{d}s\\
=&\,\mathbb{E}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[E(\sigma)P_XB(u(t_n-\sigma))-\bar{E}_{h,\tau}^{j}P_{X_h}B(u_h^{n-j})\|_{\mathcal{L}_2^0}^2\mathrm{d}\sigma\\
\lesssim&\,\mathbb{E}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|E(\sigma)P_X[B(u(t_n-\sigma))-B(u(t_{n-j}))]\|_{\mathcal{L}_2^0}^2\mathrm{d}\sigma\\
&+\mathbb{E}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[E(\sigma)P_X-\bar{E}_{h,\tau}^{j}P_{X_h}]B(u(t_{n-j}))\|_{\mathcal{L}_2^0}^2\mathrm{d}\sigma\\
&+\tau\mathbb{E}\sum_{j=1}^{n}\|\bar{E}_{h,\tau}^{j}P_{X_h}[B(u(t_{n-j}))-B(u_h^{n-j})]\|_{\mathcal{L}_2^0}^2\\
=:&\,T_{31}+T_{32}+T_{33}.
\end{aligned} \tag{5.20}$$

By (5.1) and Hölder continuity (3.15), T_{31} is bounded by

$$\begin{aligned}
T_{31}\le\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\sigma^{-\frac{1}{2}}\mathbb{E}\|u(t_n-\sigma)-u(t_{n-j})\|_{L^2}^2\mathrm{d}\sigma&\le C\big(1+\mathbb{E}\|u^0\|_{H^2}^2\big)\mathbb{E}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\sigma^{-\frac{1}{2}}(t_j-\sigma)\mathrm{d}\sigma\\
&\le C\tau\big(1+\mathbb{E}\|u^0\|_{H^2}^2\big).
\end{aligned} \tag{5.21}$$

To estimate T_{32}, we make a decomposition again

$$\begin{aligned}
T_{32}\lesssim&\,\mathbb{E}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[E(\sigma)-E(t_j)]P_X[B(u(t_{n-j}))-B(u(t_{n-1}))]\|_{\mathcal{L}_2^0}^2\mathrm{d}\sigma \qquad (5.22)\\
&+\mathbb{E}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[E(t_j)P_X-\bar{E}_{h,\tau}^{j}P_{X_h}][B(u(t_{n-j}))-B(u(t_{n-1}))]\|_{\mathcal{L}_2^0}^2\mathrm{d}\sigma\\
&+\mathbb{E}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|[E(\sigma)P_X-\bar{E}_{h,\tau}^{j}P_{X_h}]B(u(t_{n-1}))\|_{\mathcal{L}_2^0}^2\mathrm{d}\sigma\\
=:&\,T_{32}^a+T_{32}^b+T_{32}^c.
\end{aligned}$$

By using (2.6), (4.17) with $\gamma = \frac{3}{4}$, (4.18) with $\mu = \frac{1}{2}$ and Hölder continuity (3.15), we have the following bound for T_{32}^a

$$
\begin{aligned}
T_{32}^a &\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbb{E}\|E(\sigma)[I - E(t_j - \sigma)]P_X[B(u(t_{n-j})) - B(u(t_{n-1}))]\|_{\mathcal{L}_2^0}^2 \mathrm{d}\sigma \\
&\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbb{E}\sum_{\ell} \mu_\ell \|(I+A)^{\frac{3}{4}} E(\sigma)(I+A)^{-\frac{3}{4}}[I - E(t_j - \sigma)]P_X[B(u(t_{n-j})) - B(u(t_{n-1}))]\phi_\ell\|_{L^2}^2 \mathrm{d}\sigma \\
&\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{3}{2}} \mathbb{E}\sum_{\ell} \mu_\ell \|(I+A)^{-\frac{1}{2}}[I - E(t_j - \sigma)](I+A)^{-\frac{1}{4}} P_X[B(u(t_{n-j})) - B(u(t_{n-1}))]\phi_\ell\|_{L^2}^2 \mathrm{d}\sigma \\
&\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{3}{2}}(t_j - \sigma) \mathbb{E}\sum_{\ell} \mu_\ell \|B(u(t_{n-j})) - B(u(t_{n-1}))\phi_\ell\|_{H^{-\frac{1}{2}}}^2 \mathrm{d}\sigma \\
&\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{3}{2}}(t_j - \sigma) \mathbb{E}\|u(t_{n-j}) - u(t_{n-1})\|_{L^2}^2 \mathrm{d}\sigma \\
&\lesssim \tau\big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{3}{2}}(t_{n-1} - t_{n-j}) \mathrm{d}\sigma \\
&\lesssim \tau\big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{3}{2}} \sigma \mathrm{d}\sigma \\
&\lesssim \tau\big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big).
\end{aligned} \tag{5.23}
$$

where (2.6) is used in the derivation of the fourth to last inequality, inequalities $t_j - \sigma \leq \tau$ and (3.15) are used in deriving the third to last inequality, and estimate $t_{n-1} - t_{n-j} = t_{j-1} \leq \sigma$ for $\sigma \in [t_{j-1}, t_j]$ is used in deriving the second to last inequality.

By noting $E(t_j)P_X - \bar{E}_{h,\tau}^j P_{X_h} = \Phi_h(t_j) + \bar{\Phi}_{h,\tau}^j$, T_{32}^b is bounded by

$$
\begin{aligned}
T_{32}^b &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbb{E}\|[\Phi_h(t_j) + \bar{\Phi}_{h,\tau}^j][B(u(t_{n-j})) - B(u(t_{n-1}))]\|_{\mathcal{L}_2^0}^2 \mathrm{d}\sigma \qquad (5.24)\\
&\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} t_j^{-1} h^4\, \mathbb{E}\|(I+A)^{\frac{1}{2}} P_X[B(u(t_{n-j})) - B(u(t_{n-1}))]\|_{\mathcal{L}_2^0}^2 \mathrm{d}\sigma \\
&\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} t_j^{-1} \tau\, \mathbb{E}\|B(u(t_{n-j})) - B(u(t_{n-1}))\|_{\mathcal{L}_2^0}^2 \mathrm{d}\sigma \\
&\lesssim (\tau + h^4) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} t_j^{-1} \mathbb{E}\|u(t_{n-j}) - u(t_{n-1})\|_{H^\beta}^2 \mathrm{d}\sigma \\
&\lesssim (\tau + h^4)\big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} t_j^{-1} (t_{n-1} - t_{n-j})^{2-\beta} \mathrm{d}\sigma \\
&\lesssim (\tau + h^4)\big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big) \int_0^{t_n} \sigma^{1-\beta} \mathrm{d}\sigma \\
&\lesssim (\tau + h^4)\big(1 + \mathbb{E}\|u^0\|_{H^2}^2\big),
\end{aligned}
$$

where (4.25) and (4.62) are used in the derivation of the fifth to last inequality, (2.7) is used in the fourth to last inequality, (3.16) is used in the third to last inequality, and $t_{n-1} - t_{n-j} = t_{j-1} \leq \sigma$ for $\sigma \in [t_{j-1}, t_j]$ is used in the second to last inequality. Directly applying (4.1) and (5.7) shows that

$$
T_{32}^c \lesssim (\tau + h^4) \sup_{t \in [0,T]} \mathbb{E}\|(I+A)^{\frac{1}{2}} P_X B(u(t))\|_{\mathcal{L}_2^0}^2 \lesssim \tau + h^4. \tag{5.25}
$$

Moreover, by (4.58) with $\gamma = \frac{1}{2}$ and (2.6), we further have

$$\begin{aligned}
\mathbb{E}\|T_{33}\|_{L^2}^2 &\lesssim \tau \sum_{j=1}^{n} \mathbb{E}\|(I+A_h)^{\frac{1}{4}} \bar{E}_{h,\tau}^{j} (I+A_h)^{-\frac{1}{4}} P_{X_h}[B(u(t_{n-j})) - B(u_h^{n-j})]\|_{\mathcal{L}_2^0}^2 \\
&\lesssim \tau \sum_{j=1}^{n} t_j^{-\frac{1}{2}} \mathbb{E} \sum_{\ell} \mu_\ell \, \|P_{X_h}[(B(u(t_{n-j})) - B(u_h^{n-j}))\phi_\ell]\|_{H^{-\frac{1}{2}}}^2 \\
&\lesssim \tau \sum_{j=1}^{n} t_j^{-\frac{1}{2}} \mathbb{E}\|B(u(t_{n-j})) - B(u_h^{n-j})\|_{\mathcal{L}_2^0(\mathbb{L}^2, \mathbb{H}^{-1/2})}^2 \\
&\lesssim \tau \sum_{j=1}^{n} t_j^{-\frac{1}{2}} \mathbb{E}\|u(t_{n-j}) - u_h^{n-j}\|_{L^2}^2 \\
&\lesssim \tau \sum_{i=0}^{n-1} (t_n - t_i)^{-\frac{1}{2}} \mathbb{E}\|u(t_i) - u_h^i\|_{L^2}^2. \qquad (5.26)
\end{aligned}$$

In view of (5.20)–(5.26), we see that

$$\mathbb{E}\|T_3\|_{L^2}^2 \lesssim \tau + h^4 + \tau \sum_{i=0}^{n-1} (t_n - t_i)^{-\frac{1}{2}} \mathbb{E}\|u(t_i) - u_h^i\|_{L^2}^2. \qquad (5.27)$$

Finally, substituting (5.12), (5.19) and (5.27) into (5.11) yields

$$\mathbb{E}\|u(t_n) - u_h^n\|_{L^2}^2 \lesssim \tau + h^4 + \tau \sum_{i=0}^{n-1} (t_n - t_i)^{-\frac{1}{2}} \mathbb{E}\|u(t_i) - u_h^i\|_{L^2}^2. \qquad (5.28)$$

By using the discrete version of generalized Gronwall's inequality in [25, Lemma A.4], we obtain the error estimate (2.29) for the velocity.

The error estimates for the pressure can be obtained by following classic approach and utilizing the result above. We omit the details and refer to [29, Section 4.3].

6. Numerical experiments

In this section, we present numerical tests to support the theoretical analysis in Theorem 2.4 by illustrating the convergence of the fully discrete finite element solutions. For a stable discretization in space, we use the prototypical MINI element; cf. [2, 20] for details. All the computations are performed using the software package NGSolve (`https://ngsolve.org/`).

We solve the stochastic Stokes equations (1.1) in the two-dimensional square $D = [0,1] \times [0,1]$ under the stress boundary condition by the proposed scheme (2.24) up to time T, with initial value $u_0 = (0,0)$ and source term $f = (1,1)^\top$. The noise term $B(u)\mathrm{d}W$ determined by

$$B(u) = \frac{1}{2} \begin{pmatrix} \sqrt{u_1^2+1} & \sqrt{u_1^2+1} \\ \sqrt{u_2^2+1} & \sqrt{u_2^2+1} \end{pmatrix}, \qquad (6.1)$$

$$W(t,\mathbf{x}) = \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=1}^{\infty} \sqrt{\mu_{\ell_1\ell_2}} \begin{pmatrix} \phi_{\ell_1\ell_2}(\mathbf{x}) \\ \phi_{\ell_1\ell_2}(\mathbf{x}) \end{pmatrix} w_{\ell_1\ell_2}(t) \qquad \forall\, t \in [0,T], \qquad (6.2)$$

for $\mathbf{x} = (x_1, x_2) \in D$, where

$$\{w_{\ell_1\ell_2}(t) : \ell_1, \ell_2 = 0, 1, 2, \dots\}$$

is a set of independent $\mathbb{R}$-valued Wiener processes,

$$\{\phi_{\ell_1\ell_2}(\mathbf{x}) = \cos(\ell_1 \pi x_1)\cos(\ell_2 \pi x_2) : \ell_1, \ell_2 = 0, 1, 2, \dots\} \qquad (6.3)$$

is an orthonormal basis of $L^2(D)$, and

$$\mu_{\ell_1\ell_2} = \begin{cases} 0 & \text{for } (\ell_1,\ell_2) = (0,0), \\ (\ell_1^2 + \ell_2^2)^{-(r+\varepsilon)} & \text{for } (\ell_1,\ell_2) \in \mathbb{Z}^2/\{(0,0)\}, \end{cases} \qquad (6.4)$$

with $\varepsilon = 0.1$ and $r \in (0, 2]$ determining the regularity of the noise.

The noise term $B(u)\mathrm{d}W$ determined by (6.1)–(6.2) can be written as

$$B(u)\mathrm{d}W(t) = \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} \sqrt{\mu_{\ell_1\ell_2}} \begin{pmatrix} \sqrt{u_1^2+1} \\ \sqrt{u_2^2+1} \end{pmatrix} \phi_{\ell_1\ell_2}(\mathbf{x})\, w_{\ell_1\ell_2}(t), \tag{6.5}$$

which is non-solenoidal and was used to measure the effectiveness of numerical methods for the stochastic Stokes/NS equation (with $\phi_{\ell_1\ell_2}(\mathbf{x}) = 2\sin(\ell_1\pi x_1)\sin(\ell_2\pi x_2)$ therein); see [15, 17]. Based on the coefficients $\mu_{\ell_1\ell_2}$ given in (6.4), the regularity of the series $W(t)$ presented in (6.2) can be characterized as follows:

$$\left\|(-\Delta)^{\frac{r-1}{2}}\right\|_{\mathcal{L}_2^0} = \left\|(-\Delta)^{\frac{r-1}{2}} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_2(\mathbb{L}^2,\mathbb{L}^2)} = \sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty} \lambda_{\ell_1\ell_2}^{r-1}\mu_{\ell_1\ell_2} \lesssim 1 \quad \text{for} \quad r \in (0, 2], \tag{6.6}$$

where $\lambda_{\ell_1,\ell_2} = \pi^2(\ell_1^2 + \ell_2^2)$ represents eigenvalues of $-\Delta$. In the case $r = 2$, the series given in (6.2) constitutes a Q-Wiener process which satisfies Assumption 2.1, and the noise in (6.5) fulfills the conditions (2.6)–(2.7) in Assumption 2.2. In particular, the Wiener processes in (6.2) is in $L^2(\Omega, H^1(D))$ but not in $L^2(\Omega, H^{1+\varepsilon}(D))$.

We consider three cases in the numerical experiments:

Case I : $r = 2$. In this case, the Assumption 2.1 is satisfied.
Case II : $r = 1$. The noise is trace-class, but Assumption 2.1 is not satisfied.
Case III : $r = 0.5$. The noise is not trace-class.

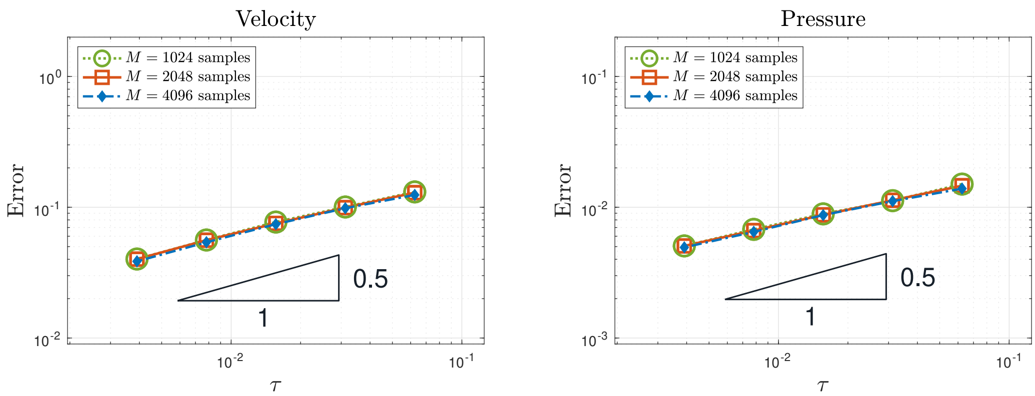

FIGURE 1. Time discretization errors at $T = 1$ for Case I with $h = 2^{-6}$.

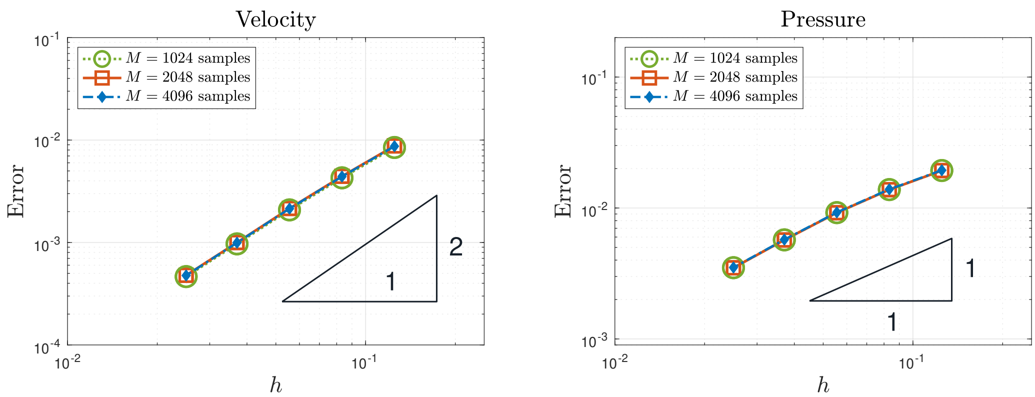

FIGURE 2. Spatial discretization errors at $T = 1$ for Case I with $\tau = 2^{-8}$.

The errors of the numerical solutions in Case I are presented in Figures 1 and 2. The expectations of the errors are computed as averages over M samples, where $M = 1024, 2048, 4096$, respectively. The numerical results in Figures 1 and 2 indicate that the numerical solutions have half-order convergence in time for both velocity and pressure, second-order convergence in space for the velocity, and first-order convergence in space for the pressure. Therefore, the convergence orders observed in the numerical experiments are consistent with the theoretical results proved in Theorem 2.4.

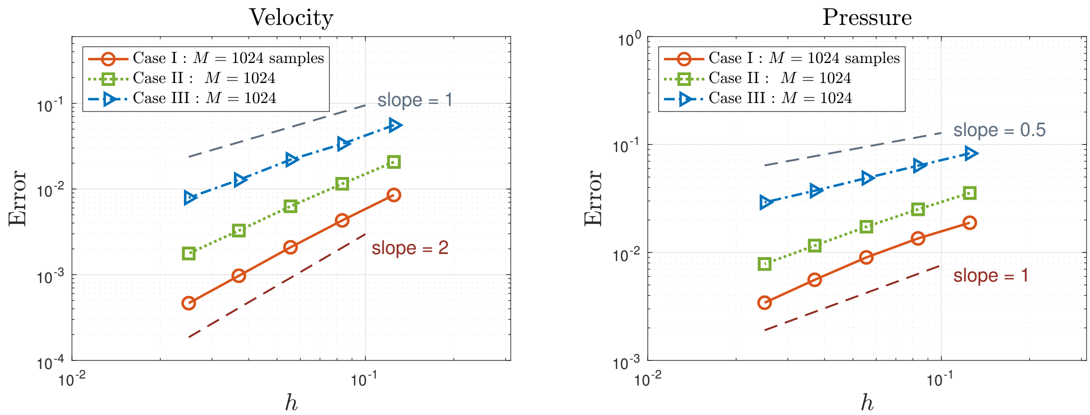

FIGURE 3. Spatial discretization errors at $T = 1$ for Cases I, II and III.

The spatial discretization errors for noises in Cases I, II, and III are presented in Figure 3, where the expectations are approximated by computing averages over $M = 1024$ samples. Notably, Figures 1 and 2 demonstrate that the number of samples, $M = 1024$, is already sufficiently large to capture the influence of the noise on the convergence rate. The numerical results in Figure 3 indicate that order reduction may occur if Assumption 2.1 is not satisfied.

Acknowledgement

The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work is supported in part by the NSFC key programs (project no. 12231003), NSFC general program (project no. 12071020), Guangdong Provincial Key Laboratory IRADS (2022B1212010006, UIC-R0400001-22) and Guangdong Higher Education Upgrading Plan (UIC-R0400024-21), the Research Grants Council of Hong Kong (GRF project no. PolyU15301321), and an internal grant of The Hong Kong Polytechnic University (Project ID: P0038843).

References

[1] W. Arendt, C. J. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*. Birkhäuser, second edition, 2011.
[2] D. N. Arnold, F. Brezzi, and M. Fortin. A stable finite element for the Stokes equations. *Calcolo*, 21(4):337–344, 1984.
[3] H. Bessaih, Z. Brzeźniak, and A. Millet. Splitting up method for the 2D stochastic Navier–Stokes equations. *Stoch. Partial Differ. Equ.-Anal. Comput.*, 2(4):433–470, 2014.
[4] H. Bessaih and A. Millet. Strong L^2 convergence of time numerical schemes for the stochastic two-dimensional Navier–Stokes equations. *IMA J. Numer. Anal.*, 39(4):2135–2167, 2019.
[5] H. Bessaih and A. Millet. Space-time Euler discretization schemes for the stochastic 2D Navier–Stokes equations. *Stoch. Partial Differ. Equ.-Anal. Comput.*, pages 1–44, 2021.
[6] H. Bessaih and A. Millet. Strong rates of convergence of space-time discretization schemes for the 2D Navier–Stokes equations with additive noise. *Stochastics and Dynamics*, page 2240005, 2022.
[7] D. Breit and A. Dodgson. Convergence rates for the numerical approximation of the 2D stochastic Navier–Stokes equations. *Numer. Math.*, 147(3):553–578, 2021.
[8] D. Breit and A. Prohl. Error analysis for 2D stochastic Navier–Stokes equations in bounded domains with Dirichlet data. *Found. Comput. Math.*, pages 1–30, 2023.
[9] Z. Brzeźniak, E. Carelli, and A. Prohl. Finite-element-based discretizations of the incompressible Navier–Stokes equations with multiplicative random forcing. *IMA J. Numer. Anal.*, 33(3):771–824, 2013.
[10] Y. Cao, J. Hong, and Z. Liu. Approximating Stochastic evolution equations with additive white and rough noises. *SIAM J. Numer. Anal.*, 55(4):1958–1981, 2017.
[11] E. Carelli, E. Hausenblas, and A. Prohl. Time-splitting methods to solve the stochastic incompressible Stokes equation. *SIAM J. Numer. Anal.*, 50(6):2917–2939, 2012.
[12] E. Carelli and A. Prohl. Rates of convergence for discretizations of the stochastic incompressible Navier–Stokes equations. *SIAM J. Numer. Anal.*, 50(5):2467–2496, 2012.

[13] M. Dauge. Stationary Stokes and Navier–Stokes systems on two-or three-dimensional domains with corners. part i. linearized equations. *SIAM J. Math. Anal.*, 20(1):74–97, 1989.
[14] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159. Springer, 2004.
[15] X. Feng, A. Prohl, and L. Vo. Optimally convergent mixed finite element methods for the stochastic Stokes equations. *IMA J. Numer. Anal.*, 41(3):2280–2310, 2021.
[16] X. Feng and H. Qiu. Analysis of fully discrete mixed finite element methods for time-dependent stochastic Stokes equations with multiplicative noise. *J. Sci. Comput.*, 88(2):1–25, 2021.
[17] X. Feng and L. Vo. Analysis of chorin-type projection methods for the stochastic Stokes equations with general multiplicative noise. *Stoch. Partial Differ. Equ.-Anal. Comput.*, pages 1–38, 2022.
[18] V. Girault and P.-A. Raviart. *Finite element approximation of the Navier-Stokes equations*, volume 749. Springer Berlin, 1979.
[19] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*, volume 5. Springer Science & Business Media, 1986.
[20] V. Girault and L. R. Scott. A quasi-local interpolation operator preserving the discrete divergence. *Calcolo*, 40(1):1–19, 2003.
[21] J. G. Heywood and R. Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem IV: Error analysis for second-order time discretization. *SIAM J. Numer. Anal.*, 27:353–384, 1990.
[22] R. Ingram. A new linearly extrapolated Crank-Nicolson time-stepping scheme for the Navier-Stokes equations. *Math. Comp.*, 82(284):1953–1973, 2013.
[23] R. B. Kellogg and J. E. Osborn. A regularity for the Stokes problem in a convex polygon. *J. Funct. Anal.*, 21:397–431, 1976.
[24] R. Kruse. Optimal error estimates of Galerkin finite element methods for stochastic partial differential equations with multiplicative noise. *IMA J. Numer. Anal.*, 34(1):217–251, 2014.
[25] R. Kruse. *Strong and weak approximation of semilinear stochastic evolution equations*. Springer, 2014.
[26] R. Kruse and S. Larsson. Optimal regularity for semilinear stochastic partial differential equations with multiplicative noise. *Electron. J. Probab.*, 17:1–19, 2012.
[27] W. Layton, N. Mays, M. Neda, and C. Trenchea. Numerical analysis of modular regularization methods for the BDF2 time discretization of the Navier-Stokes equations. *ESAIM: M2AN*, 48(3):765–793, 2014.
[28] B. Li. An explicit formula for corner singularity expansion of the solutions to the Stokes equations in a polygon. *Int. J. Numer. Anal. Model.*, 17(6):900–928, 2020.
[29] B. Li, S. Ma, and W. Sun. Higher-order convergence of finite element methods for the stochastic Stokes equations. arXiv preprint, arXiv:2206.13879.
[30] M. Orlt and A. M. Sändig. *Regularity of viscous Navier–Stokes flows in nonsmooth domains*, pages 185–201. Marcel Dekker, Inc., 1995.
[31] M. Marion and R. Temam. Navier-Stokes equations: Theory and approximation. *Handbook of numerical analysis*, 6:503–689, 1998.
[32] J. D. Mukam and A. Tambue. Strong convergence of the linear implicit Euler method for the finite element discretization of semilinear non-autonomous SPDEs driven by multiplicative or additive noise. *Appl. Numer. Math.*, 147:222–253, 2020.
[33] R. H. Nochetto and J.-H. Pyo. Error estimates for semi-discrete gauge methods for the Navier–Stokes equations. *Math. Comp.*, 74:521–542, 2016.
[34] J. Shen. On error estimates of projection methods for Navier–Stokes equations: first-order schemes. *SIAM J. Numer. Anal.*, 29(1):57–77, 1992.
[35] J. Shen. On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations. *Numer. Math.*, 62(1):49–73, 1992.
[36] Y. Shibata and S. Shimizu. On a resolvent estimate for the Stokes system with Neumann boundary condition. *Differential and Integral Equations*, 16(4):385–426, 2003.
[37] C. Taylor and P. Hood. A numerical solution of the Navier-Stokes equations using the finite element technique. *Comput. Fluids*, 1(1):73–100, 1973.
[38] V. Thomée. *Galerkin finite element methods for parabolic problems*, volume 25. Springer Science & Business Media, 2007.
[39] X. Wang. Strong convergence rates of the linear implicit Euler method for the finite element discretization of SPDEs with additive noise. *IMA. J. Numer. Anal.*, 37:965–984, 2017.
[40] Y. Yan. Galerkin finite element methods for stochastic parabolic partial differential equations. *SIAM J. Numer. Anal.*, 43:1363–1384, 2005.

Buyang Li: Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong. *E-mail address*: `buyang.li@polyu.edu.hk`

Shu Ma: Department of Mathematics, National University of Singapore, Singapore 119076. *E-mail address*: `shuma@nus.edu.sg`

Weiwei Sun: Research Center for Mathematics, Beijing Normal University, Zhuhai, P. R. China; Division of Science and Technology, BNU-HKBU United International College, Zhuhai, P. R. China. *E-mail address*: `maweiw@uic.edu.cn`