MATHEMATICS OF COMPUTATION

CONVERGENCE OF A STABILIZED PARAMETRIC FINITE ELEMENT METHOD OF THE BARRETT–GARCKE–NÜRNBERG TYPE FOR CURVE SHORTENING FLOW

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ABSTRACT. The parametric finite element methods of the Barrett–Garcke–Nürnberg (BGN) type have been successful in preventing mesh distortion/degeneration in approximating the evolution of surfaces under various geometric flows, including mean curvature flow, Willmore flow, Helfrich flow, surface diffusion, and so on. However, the rigorous justification of convergence of the BGN-type methods and the characeterization of the particle trajectories produced by these methods still remain open since this class of methods was proposed in 2007. The main difficulty lies in the stability of the artificial tangential velocity implicitly determined by the BGN methods. In this paper, we give the first proof of convergence of a stabilized BGN method for curve shortening flow, with optimal-order convergence in L^2 norm for finite elements of degree $k \geq 2$ under the stepsize condition $\tau \leq ch^{k+1}$ (for any fixed constant c). Moreover, we give the first rigorous characterization of the particle trajectories produced by the BGN-type methods for one-dimensional curves, i.e., we prove that the particle trajectories produced by the stabilized BGN methods converge to the particle trajectories determined by a system of geometric partial differential equations which differs from the standard curve shortening flow by a tangential motion. The characterization of the particle trajectories also rigorously explains, for one-dimensional curves, why the BGN-type methods could maintain the quality of the underlying evolving mesh.

1. INTRODUCTION

Parametric finite element methods for approximating surface evolution under geometric flows were firstly proposed by Dziuk in his 1990 paper [19] for mean curvature flow. For a given approximate surface Γ_h^m at time level $t = t_m$, Dziuk proposed to determine the surface Γ_h^{m+1} at time level $t = t_{m+1}$ as the image of a finite element parametrization function $X_h^{m+1} : \Gamma_h^m \to \mathbb{R}^3$, satisfying the following weak formulation:

$$(1.1) \qquad \int_{\Gamma_h^m} \frac{X_h^{m+1} - \mathrm{id}}{\tau} \cdot \chi_h + \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^{m+1} \cdot \nabla_{\Gamma_h^m} \chi_h = 0 \quad \forall \chi_h \in S_h(\Gamma_h^m),$$

Received by the editor October 30, 2023, and, in revised form, August 6, 2024.

2020 *Mathematics Subject Classification.* Primary 65M12, 65M60, 53E10, 53A04, 35R01, 35R35.

Key words and phrases. Mean curvature flow, curve shortening flow, geometric evolution equation, parametric finite element method, tangential motion, stability, convergence, trajectory, mass lumping, distance projection.

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This work was supported in part by the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. PolyU/RFS2324-5S03, PolyU/GRF15303022) and an internal grant of The Hong Kong Polytechnic University (Project ID: P0051154).

where τ is the size of the time step, and $S_h(\Gamma_h^m)$ denotes the space of vector-valued finite element space on the surface Γ_h^m. At every time level, Dziuk's semi-implicit parametric FEM only requires solving a linear elliptic partial differential equation on a given surface. Since Dziuk's paper was published, parametric FEMs have become successful and widely used for approximating the evolution of surfaces and interfaces in various different geometric flows and related problems, including mean curvature flow, Willmore flow, Helfrich flow, surface diffusion, and so on; see [3, 12, 16, 19, 21].

In practical computations, the accuracy of parametric FEMs in approximating an evolving surface can be greatly influenced by the mesh quality of the triangulation which constitutes the approximate surface. One of the main difficulties in approximating surface evolution under geometric flows, which were not addressed by Dziuk's parametric FEMs, is that the mesh which forms the approximate surface often becomes distorted and degenerate as time grows. One popular approach to overcome this difficulty is to artificially redistribute the mesh points more equally when the mesh quality becomes bad (below some threshold), as proposed in [3]. Another popular approach is to introduce an artificial tangential velocity, which could drive the nodes moving tangentially as a surface evolves to maintain good mesh quality; see [6, 7, 9, 27]. For example, the method proposed by Barrett, Garcke & Nürnberg in [9, Eq. (2.25)] (i.e., the BGN method) for mean curvature flow seeks a parametrization $X_h^{m+1} : \Gamma_h^m \to \mathbb{R}^2$ satisfying the following weak formulation:

(1.2)
$$\int_{\Gamma_h^m}^h \frac{X_h^{m+1} - \mathrm{id}}{\tau} \cdot \bar{n}_h^m \, \phi_h \cdot \bar{n}_h^m + \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^{m+1} \cdot \nabla_{\Gamma_h^m} \phi_h = 0 \quad \forall \, \phi_h \in S_h(\Gamma_h^m),$$

where $\bar{n}_h^m$ is a weighted averaging normal vector at the nodes of the piecewise linear curve Γ_h^m (see [9, Eq. (2.7) and Remark 2.1] and [5, Eq. (47)]), and the superscript h in the integral indicates that the mass lumping technique for piecewise linear FEM is used. In this method, only the normal component of the velocity is explicitly specified, while the tangential component of the velocity is implicitly determined to make the map $X_h^{m+1} : \Gamma_h^m \to \mathbb{R}^2$ approximately harmonic. It turns out that the tangential velocity implicitly determined in this way could maintain good mesh quality of the approximate evolving surfaces. The idea of the BGN methods has become popular and widely used for approximating various geometric flows, including mean curvature flow, Willmore flow, Helfrich flow, surface diffusion, and so on; see [4, 8–10]. However, the convergence of such BGN methods has not been proved for any geometric flow.

Convergence of some semidiscrete and fully discrete parametric FEMs for mean curvature flow and Willmore flow of *curves* was proved by Dziuk [20], Deckelnick & Dziuk [14, 15], Bartels [11], Li [37], Ye & Cui [42], etc. For mean curvature flow and Willmore flow of *closed surfaces*, convergence results are available in the literature only in the following several cases:

- Evolving surface FEMs with finite elements of degree $k \geq 2$ based on reformulations of mean curvature flow and Willmore flow in terms of the evolution equations of normal vector and mean curvature; see [26, 31, 33, 34].
- The semidiscrete version of Dziuk's parametric FEM with finite elements of degree $k \geq 6$ based on H^1 parabolicity of the normal components in the

framework of evolving surface FEM [22,23] and matrix-vector techniques [35]; see [1,38].

- Dziuk's semi-implicit parametric FEM with finite elements of degree $k \geq 3$ based on a new approach which recovers the full H^1 parabolicity of Dziuk's method by measuring the error in terms of the distance between the approximate surface and exact surface; see [2].

The error and stability estimates in these articles all rely on corresponding continuous formulations of the tangential velocity or evolution equations of normal vector and mean curvature, which are not available for the BGN type of methods. Therefore, the convergence analyses in these article cannot be applied/extended to the BGN type of methods.

Apart from the BGN methods, there are other approaches to constructing artificial tangential velocities for parametric finite element approximations of geometric flows. One popular approach, originaly proposed by DeTurck in [18] in the context of Ricci flow and firstly brought into the numerics world by Fritz in his dissertation [28] (see also [27]), is to introduce a tangential reparametrization of the geometric flow. It is also important to mention the work of Mikula and Ševčovič [41] where the authors are able to construct a nontrivial tangential smoothing velocity via solving a nonlocal equation. Error estimates of the evolving surface FEMs for curve shortening flow and related problems based on this approach were established in [6,27,40] based on available continuous formulations of the tangential velocity. Another approach, proposed by Hu & Li in [31], is to construct an artificial tangential velocity in the reformations of mean curvature flow and Willmore flow by Kovacs, Li & Lubich [33,34] to minimize the instantaneous rate of deformation caused by the flow map. Error estimates for this type of methods are based on the H^1 parabolicity in the reformulations by Kovacs, Li & Lubich [33,34] as well as the stability estimates of tangential velocity which further rely on the stability estimates of normal vector and mean curvature from their evolution equations. These works provide insights into the numerical importance of working with coupled systems. However, since the continuous formulations of the tangential velocity produced by the BGN type methods are not available yet, and the evolution equations of normal vector and mean curvature are not available in the BGN type of methods, the convergence analyses in these two approaches cannot be applied/extended to the BGN type of methods.

The main difficulty in the analysis of the BGN type of methods is the lack of stability estimates for the artificial tangential velocity. This is partly reflected by the following aspect: The formal limiting equation of (1.2) as $\tau, h \to 0$, i.e.,

$$(\partial_t X \cdot n)n = (\Delta_{\Gamma[X]}\mathrm{id}) \circ X, \tag{1.3}$$

does not have a unique solution (adding an arbitrary tangential motion to the solution does not change the equation). Therefore, the convergence of the BGN type methods to the original geometric flow, such as curve shortening flow, has not been proved rigorously. Moreover, the question of why the BGN methods could maintain good mesh quality of the evolving surfaces/curves has not been addressed rigorously, though this has been demonstrated intuitively in [5,9]. These two open questions are both addressed in the current paper.

In this paper, we construct a high-order and stabilized version of the BGN method for curve shortening flow, with high-order accuracy in space and as good performance as the original BGN method in improving the distribution of mesh points, and provide rigorous analysis for the convergence of the numerical solutions to the exact solution of curve shortening flow. The continuous formulation of the artificial tangential velocity produced by the BGN method for curve shortening flow is also derived rigorously. Correspondingly, the limit of the particle trajectories produced by the BGN method is completely characterized.

Let Γ_h^m be a closed and continuous piecewise polynomial curve which approximates the smooth curve $\Gamma^m := \Gamma(t_m)$ evolving under curve shortening flow. Each polynomial element K of Γ_h^m is the image of an element $K^0 \subset \Gamma_h^0$ under the discrete flow map. We denote by $K_{\rm f}^0$ the unique flat segment which has the same endpoints as K^0, and denote by $F_K : K_{\rm f}^0 \to K$ the parametrization of K, i.e., F_K is the unique polynomial of degree k that maps $K_{\rm f}^0$ onto K. The finite element space on the approximate curve Γ_h^m is defined as

$$S_h(\Gamma_h^m) = \{v_h \in C(\Gamma_h^m) : v_h \circ F_K \in \mathbb{P}^k(K_{\rm f}^0)^2 \text{ for every element } K \subset \Gamma_h^m\},$$

where $\mathbb{P}^k(K_{\rm f}^0)$ denotes the space of polynomials of degree $k \geq 1$ on the flat segment $K_{\rm f}^0$.

Then we introduce the mass lumping integral for high-order finite elements denoted by the superscript h:

$$\int_{\Gamma_h^m}^h u \cdot n_h^m \, v \cdot n_h^m := \sum_{K\subset\Gamma_h^m} \int_{K_{\rm f}^0} I_h^{GL}\big[(u\circ F_K \cdot n_h^m \circ F_K)(v\circ F_K \cdot n_h^m \circ F_K)|\nabla_{K_{\rm f}^0} F_K|\big], \tag{1.4}$$

where the summation extends over all elements of the curve Γ_h^m, and I_h^{GL} denotes the interpolation operator at the Gauss–Lobatto points of the flat element $K_{\rm f}^0$ (cf. [13, Eq. (10.2.3)]). In the special case of piecewise linear FEM (i.e., $k = 1$), the definition in (1.4) coincides with the definition in [9, Eq. (2.2)].

Let $t_m = m\tau$, $m = 0, 1, \dots, \lfloor T/\tau \rfloor$, be a sequence of grid points in time with stepsize $\tau > 0$, where $\lfloor T/\tau \rfloor$ denotes the maximal integer not exceeding T/τ. We propose the following high-order and stabilized BGN method for curve shortening flow: For a given approximate curve Γ_h^m, find a parametrization $X_h^{m+1} : \Gamma_h^m \to \Gamma_h^{m+1}$ such that $X_h^{m+1} \in S_h(\Gamma_h^m)$ and

$$\begin{aligned} &\int_{\Gamma_h^m}^h \frac{X_h^{m+1} - \mathrm{id}}{\tau} \cdot \bar n_h^m \, \phi_h \cdot \bar n_h^m + \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^{m+1} \cdot \nabla_{\Gamma_h^m} \phi_h \\ &= \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} \mathrm{id} \cdot \nabla_{\Gamma_h^m} I_h[\phi_h - (\phi_h \cdot \bar n_h^m)\bar n_h^m] \quad \forall\, \phi_h \in S_h(\Gamma_h^m), \end{aligned} \tag{1.5}$$

where the right-hand side of (1.5) is a (consistent) stabilization term which plays an important role in proving the convergence of the numerical solutions as well as characterizing the tangential motion produced by the method, and $\bar n_h^m \in S_h(\Gamma_h^m)$ is an averaged normal vector defined as the discrete L^2 projection of the piecewise unit normal vector n_h^m onto the finite element space $S_h(\Gamma_h^m)$, i.e.,

$$\int_{\Gamma_h^m}^h \bar n_h^m \cdot \phi_h = \int_{\Gamma_h^m}^h n_h^m \cdot \phi_h \quad \forall\, \phi_h \in S_h(\Gamma_h^m). \tag{1.6}$$

In the case $k = 1$ (using piecewise linear finite elements), the numerical scheme in (1.5) differs from the BGN method in (1.2) by the stabilization term on the right-hand side of (1.5). The motivation of adding this stabilization term is stated in the text between (1.8)–(1.10).

Using the definition in (1.6), expressing the mass lumping integrals in (1.6) as the summation of the quadrature weights times the evaluations at the quadrature points, and comparing the coefficients of each degree of freedom on the both sides of (1.6), the following relations between $\bar{n}_h^m$ and n_h^m can be shown:

$$\begin{aligned}
&\bar{n}_h^m(p) = n_h^m(p) && \text{if } p \text{ is an interior node of an element,}\\
&\bar{n}_h^m(p) = \frac{|w_{K_1}(p)||K_{1\mathrm{f}}^0|\, n_h^m|_{K_1}(p)}{|w_{K_1}(p)||K_{1\mathrm{f}}^0| + |w_{K_2}(p)||K_{2\mathrm{f}}^0|} + \frac{|w_{K_2}(p)||K_{2\mathrm{f}}^0|\, n_h^m|_{K_2}(p)}{|w_{K_1}(p)||K_{1\mathrm{f}}^0| + |w_{K_2}(p)||K_{2\mathrm{f}}^0|} \\
& && \text{if } p = K_1 \cap K_2 \text{ for two elements } K_1 \text{ and } K_2,
\end{aligned} \tag{1.7}$$

where $w_K(p) = \nabla_{K_\mathrm{f}^0} F_K \circ F_K^{-1}(p)$ for $p \in K$ and $n_h^m|_K$ denotes the normal vector on element K. Note that both the mass lumping in (1.4) and the averaged normal vector in (1.7) are intrinsically defined in the sense that they are independent of the choice of flat segment for parametrization.

The proof of convergence of the proposed stabilized BGN method is based on the recently developed new approach in [2] for the analysis of parametric finite element approximations to geometric flows, where the error of concern is the distance projection from the numerically computed curve to the exact smooth curve, rather than the error between particle trajectories of the curves as in [31, 33, 35]. It has been shown in [2] that this approach (i.e., to estimate the error of distance projection) can recover the full H^1 parabolicity of mean curvature flow and therefore leads to better stability estimates.

The novel contributions of this article to the construction and analysis of parametric approximations to geometric flows include the following several aspects.

- Stabilization and averaged normal vector: We stabilize the BGN method in two ways, including the use of an averaged normal vector $\bar{n}_h^m$ defined in (1.6) and the introduction of the stabilization term to the right-hand side of (1.5). Since the proposed stabilization term vanishes in the continuous case, i.e.,

$$\int_\Gamma \nabla_\Gamma \mathrm{id} \cdot \nabla_\Gamma [\phi - (\phi \cdot n) n] = \int_\Gamma -\Delta_\Gamma \mathrm{id} \cdot [\phi - (\phi \cdot n) n] = \int_\Gamma H n \cdot [\phi - (\phi \cdot n) n] = 0,$$

  the stabilization term is expected to vanish approximately at the discrete level. The advantage of adding this stabilization term is that, for test functions ϕ_h in the finite element tangential subspace

$$S_h(\Gamma_h^m)^\top = \{v_h \in S_h(\Gamma_h^m) : v_h \cdot \bar{n}_h^m = 0 \text{ at the finite element nodes of } \Gamma_h^m\},$$

  the weak formulation in (1.5) reduces to the following relation:

$$\int_{\Gamma_h^m} \nabla_{\Gamma_h^m} \frac{X_h^{m+1} - \mathrm{id}}{\tau} \cdot \nabla_{\Gamma_h^m} \phi_h = 0 \quad \forall\, \phi_h \in S_h(\Gamma_h^m)^\top, \tag{1.8}$$

  which will be used to establish estimates for the tangential velocity of the approximate curve in the (stabilized) BGN method. Therefore, the stabilization term on the right-hand side of (1.5) is to stabilize the tangential velocity in the form of (1.8), rather than enforcing some energy stability.

- Characterization of the tangential motion and the particle trajectories: It was formally shown in [31, Section 1] that the velocity of the approximate curve given by the BGN method converges to the velocity governed by the following elliptic system on the exact curve Γ:

$$\begin{aligned} v\cdot n &= -H, \\ -\Delta_\Gamma v &= \kappa n, \end{aligned} \tag{1.9}$$

which is the Euler-Lagrange equation of the following minimization problem:

$$\min_{v\in H^1(\Gamma)} \int_\Gamma |\nabla_\Gamma v|^2 \quad \text{under the pointwise constraint } v\cdot n = -H.$$

In this paper, we present rigorous justification of this convergence for the stabilized BGN method by utilizing (1.8) (the derivation of this relation requires us to add the stabilization term to the BGN method). This completely characterizes the underlying geometric PDEs to which the stabilized BGN method converges, i.e., the particle trajectories of approximate curve converge to the particle trajectories determined by the following geometric PDEs:

$$\begin{aligned} \partial_t X &= v\circ X, \\ v\cdot n &= -H, \\ -\Delta_\Gamma v &= \kappa n, \\ H &= -\Delta_\Gamma \mathrm{id}\cdot n. \end{aligned} \tag{1.10}$$

As we shall see in the error estimation, the velocity v determined by the elliptic system in (1.9) is compared with the velocity $(X_h^{m+1}-X_h^m)/\tau$ of the approximate curve to establish stability estimates for the tangential velocity. This is one of the reasons that we can prove the convergence of numerical solutions for the stabilized BGN method.

Since the velocity v determined by (1.9) minimizes the rate of the change of deformation at every time $t\in[0,T]$, as explained in [31, Section 1], and the tangential velocity in the stabilized BGN method can be proved convergent to the tangential component of v, this explains why the tangential velocity generated by the stabilized BGN method could improve the mesh quality.

- Stability of the tangential velocity: The key stability structure in the tangential direction follows from testing (1.9) by the tangential vector $(I-nn^\top)v$, i.e.,

$$\int_\Gamma \nabla_\Gamma v\cdot \nabla_\Gamma[(I-nn^\top)v] = 0. \tag{1.11}$$

If we denote by $\underline{D}_j v$ the jth component of $\nabla_\Gamma v$ in the ambient geometry, using integration by parts and Young's inequality, we can obtain the following relation:

$$
\begin{aligned}
&\int_\Gamma |\nabla_\Gamma[(I-nn^\top)v]|^2 \\
&= -\int_\Gamma \nabla_\Gamma(nn^\top v)\cdot\nabla_\Gamma[(I-nn^\top)v] \\
&= -\int_\Gamma \underline{D}_j(nn^\top v)\cdot\underline{D}_j[(I-nn^\top)(I-nn^\top)v] \\
&= -\int_\Gamma \underline{D}_j(nn^\top v)\cdot(I-nn^\top)\underline{D}_j[(I-nn^\top)v] \\
&\quad -\int_\Gamma \underline{D}_j(nn^\top v)\cdot[\underline{D}_j(I-nn^\top)(I-nn^\top)v] \quad \text{(product rule)} \\
&= -\int_\Gamma \Big(\underline{D}_j[(I-nn^\top)nn^\top v]-[\underline{D}_j(I-nn^\top)]nn^\top v\Big)\cdot\underline{D}_j[(I-nn^\top)v] \\
&\quad +\int_\Gamma (nn^\top v)\cdot\Big(\underline{D}_j\underline{D}_j(I-nn^\top)(I-nn^\top)v+\underline{D}_j(I-nn^\top)\underline{D}_j[(I-nn^\top)v]\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(integration by parts)} \\
&\le \epsilon\int_\Gamma |\nabla_\Gamma[(I-nn^\top)v]|^2 + C\epsilon^{-1}\int_\Gamma |v\cdot n|^2, \qquad (1.12)
\end{aligned}
$$

with an arbitrary small constant ϵ, where the last inequality uses the identity $(I-nn^\top)nn^\top v = 0$ and the following Poincaré type of inequality for the tangential velocity field $(I-nn^\top)v$:

$$\int_\Gamma |(I-nn^\top)v|^2 \le C\int_\Gamma |\nabla_\Gamma[(I-nn^\top)v]|^2. \tag{1.13}$$

By choosing a sufficiently small constant ϵ and absorbing the first term on the right-hand side of (1.12) by its left-hand side, we obtain

$$\int_\Gamma |\nabla_\Gamma[(I-nn^\top)v]|^2 \le C\int_\Gamma |v\cdot n|^2. \tag{1.14}$$

Therefore, the H^1 norm of the tangential velocity can be bounded by the L^2 norm of the normal velocity (one derivative is removed). With (1.13) replaced by (see Lemma 3.10)

$$\int_{\hat\Gamma^m_{h,*}} |v_h|^2 \lesssim \int_{\hat\Gamma^m_{h,*}} I_h(|v_h\cdot\bar n^m_{h,*}|^2) + \int_{\hat\Gamma^m_{h,*}} |\nabla_{\hat\Gamma^m_{h,*}} v_h|^2,$$

we manage to extend (1.14) to the discrete level in estimating the tangential velocity generated by the stabilized BGN method. This is another reason that we manage to prove the convergence of numerical solutions for the stabilized BGN method.

- Optimal-order convergence of the numerical solution: By combining the following techniques in the analysis, i.e.,
  - (i) the introduction of stabilization to the BGN method,
  - (ii) the underlying PDEs in (1.9) which characterizes the tangential motion,
  - (iii) the stability of the tangential velocity in light of (1.14),

(iv) the mass lumping techniques based on the Gauss–Lobatto quadrature nodes and the averaged normal vector techniques,
(v) the super-approximation estimates in the consistency analysis,
(vi) the high-order a priori estimate for the shape regularity,

we manage to prove optimal-order convergence of the numerical solutions under the stepsize condition $\tau \le ch^{k+1}$ (for any fixed constant c) in the L^2 norm which measures the distance between the approximate curve and the exact curve, for the stabilized BGN method. The stepsize condition is required in part (vi) mentioned above (also see Remark 2.3). The use of the Gauss–Lobatto quadrature is the main obstacle that prevents us to extend our current proof from the case of curves to surfaces with triangular meshes. Nevertheless, such extension is still possible if we use tensorial parametric finite elements (for example, for approximating two-dimensional surfaces of torus type), where the construction of the tensorial Gauss–Lobatto quadrature is straightforward.

The underlying framework and techniques developed in this paper (with the above-mentioned ingredients) may be applied/extended to other geometric flows and parametric finite element approximations which contain artificial tangential motions of the BGN type.

The rest of this paper is organized as follows. The main theoretical results of this paper are presented in Section 2. The notations and underlying framework for proving the main theorems are presented in Section 3. The convergence of numerical solutions given by the stabilized BGN method and the characterization of the particle trajecteries (continuous formulation of the artificial tangential motion) are presented in Sections 4 and 5, respectively. Finally, numerical examples and conclusions are presented in Sections 6 and 7, respectively.

2. Statement of the main theoretical results

Let $\delta > 0$ be a sufficiently small constant such that every point x in the δ-neighborhood of the exact curve $\Gamma^m = \Gamma(t_m)$, denoted by $D_\delta(\Gamma^m) = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \Gamma^m) \le \delta\}$, has a unique smooth projection of distance retraction onto Γ^m, denoted by $a^m(x)$, satisfying the following relation:

$$x - a^m(x) = \pm |x - a^m(x)| n^m(a^m(x)),$$

where n^m is the unit normal vector on Γ^m. It is known that such a constant δ exists and only depends on the curvature of Γ^m (thus δ is independent of m, but possibly dependent on T); see [29, Lemma 14.17] and [36, Theorem 6.40].

We assume that each element $K^0 \subset \Gamma_h^0$ interpolates the smooth initial curve Γ^0 at $k+1$ nodes and that the parametrization $F_{K^0} : K_{\mathrm{f}}^0 \to K^0$ is a polynomial of degree $\le k$ with the following property:

$$\max_{K^0 \subset \Gamma_h^0} \Big(\|F_{K^0}\|_{W^{k,\infty}(K_{\mathrm{f}}^0)} + \|\nabla_{K^0} F_{K^0}^{-1}\|_{L^\infty(K^0)} \Big) \le \kappa_0, \tag{2.1}$$

where κ_0 is some constant that is independent of h. This property holds for standard parametric finite elements which interpolate the smooth curve Γ^0 and guarantees the following optimal-order approximation to Γ^0 by Γ_h^0:

$$\max_{K^0 \subset \Gamma_h^0} \|a^0 \circ F_{K^0} - F_{K^0}\|_{L^\infty(K_{\mathrm{f}}^0)} \le Ch^{k+1}. \tag{2.2}$$

The projection $a^0(x)$ is well defined for points x in a neighborhood of Γ^0 and therefore well defined on Γ_h^0 for sufficiently small mesh size h.

Let x_j^m, $j = 1, \ldots, J$, be the nodes of the approximate curve Γ_h^m at the time t_m given by the stabilized BGN method in (1.5). The interpolated piecewise polynomial curve $\hat{\Gamma}_{h,*}^m$ is determined by the nodes which are obtained by projecting the nodes of Γ_h^m onto Γ^m. We shall prove that the approximate curve Γ_h^m is in a δ-neighborhood of the smooth curve Γ^m so that the projection of the nodes of Γ_h^m onto Γ^m is well defined (thus the interpolated curve $\hat{\Gamma}_{h,*}^m$ is well defined).

In view of the matrix-vector formulation which was firstly proposed in [35, Section 2.5] in the context of numerical geometric flow and the notational conventions introduced in [2, Section 1], we will always identify a finite element function as a vector consisting of its nodal values. Such representation is unique if we have specified the underlying domain. For example, the two integrands of

$$\int_{\hat{\Gamma}_{h,*}^m} v_h \quad \text{and} \quad \int_{\Gamma_h^m} v_h$$

have the same vector of nodal values, denoted by $\mathbf{v}$, but are defined on different domains $\hat{\Gamma}_{h,*}^m$ and Γ_h^m. When the underlying domain is specified, $\mathbf{v}$ is automatically substantialized to a finite element function v_h on that domain. Since all of the quantitative computations in this paper involve either integrals or norms, our notations for finite element functions will always have a unique and clear meaning. For another example, $\|v_h\|_{\hat{\Gamma}_{h,*}^m}$ and $\|v_h\|_{\Gamma_h^m}$ denote the norms of a finite element function (a nodal vector) on the two different curves $\hat{\Gamma}_{h,*}^m$ and Γ_h^m, respectively.

Correspondingly, the interpolation operator I_h should be interpreted as the determination of the nodal vector which uniquely corresponds to a finite element function after specifying the underlying curve. The lift of a finite element function v_h onto the smooth curve Γ^m is defined as

$$v_h^l = v_h \circ (a^m|_{\hat{\Gamma}_{h,*}^m})^{-1}$$

by first identifying v_h as a finite element function on the interpolated curve $\hat{\Gamma}_{h,*}^m$; see [17, Section 2.4] and [33, Section 3.4]. The inverse lift of $v \in L^2(\Gamma^m)$ onto $\hat{\Gamma}_{h,*}^m$ is defined as $v^{-l} = v \circ a^m$.

Let X_h^m be the finite element function with nodal vector $\mathbf{x}^m$. When X_h^m is considered as a finite element function on $\hat{\Gamma}_{h,*}^m$, it represents the piecewise polynomial of degree $\leq k$ which maps the nodes of $\hat{\Gamma}_{h,*}^m$ to the nodes of Γ_h^m. In order to measure the error between the approximate curve Γ_h^m and the smooth curve Γ^m, we define the lifted error

$$\hat{e}^m = \big(X_h^m - I_h \mathrm{id}_{\Gamma^m}^{-l}\big)^l \in H^1(\Gamma^m),$$

where $X_h^{m,l}$ denotes the lift of X_h^m onto Γ^m through the interpolated curve $\hat{\Gamma}_{h,*}^m$.

The main theoretical result of this article is Theorem 2.1.

Theorem 2.1 (Convergence of the stabilized BGN method)**.** *Suppose that the flow map $X : \Gamma^0 \times [0,T] \to \mathbb{R}^2$ of the curve shortening flow of a closed curve and its inverse map $X(\cdot,t)^{-1} : \Gamma(t) \to \Gamma^0$ are both sufficiently smooth, uniformly with respect to $t \in [0,T]$, and the initial approximation of the curve is sufficiently good, i.e. Γ_h^0 is closed and satisfies (2.1) and $\|\hat{e}^0\|_{\Gamma^0} \leq c_0 h^{k+1}$ for some constant c_0 which is independent of h. Let X_h^m be the finite element solution given by the stablized*

BGN method in (1.5) *with initial condition* $X_h^0 = \mathrm{id}$ *on* Γ_h^0*. Then, for any given constant* c *(independent of* τ *and* h*), there exists a positive constant* h_0 *such that for* $\tau \le ch^{k+1}$ *and* $h \le h_0$ *the following error estimate holds for finite elements of degree* $k \ge 2$*:*

$$\max_{1\le m\le [T/\tau]} \|\hat e^m\|_{L^2(\Gamma^m)}^2 + \sum_{m=1}^{[T/\tau]} \tau \|\nabla_{\Gamma^m} \hat e^m\|_{L^2(\Gamma^m)}^2 \le Ch^{2(k+1)}, \tag{2.3}$$

where the constant C *is independent of* τ *and* h *(but may depend on* c *and* T*).*

Theorem 2.2 (Characterization of the particle trajectories)**.** *Under the assumptions of Theorem* 2.1*, the particle trajectories produced by the stabilized BGN method in* (1.5) *converge to the particle trajectories determined by* (1.10)*.*

Remark 2.3. The stepsize condition $\tau \le ch^{k+1}$ is required to prove the shape regularity of the interpolated curve $\hat\Gamma_{h,*}^m$ and the optimal-order approximation to Γ^m by the interpolated curve $\hat\Gamma_{h,*}^m$; see Section 4.9 and, more specifically, (4.119).

3. Notations and underlying framework

In this section, we present the notation and underlying framework for proving Theorems 2.1 and 2.2. This includes the approximation properties of the interpolated surface $\hat\Gamma_{h,*}^m$ to the smooth surface Γ^m, the mathematical induction assumptions under which we establish the consistency and stability estimates, the superapproximation properties of surface finite elements and Gauss–Lobatto quadrature, the approximation properties of the averaged normal vectors to the original normal vector, the Poincaré inequalities for vector-valued functions on triangulated surfaces, and the geometric relations among the several different definitions of errors.

The underlying framework in this section is a substantial refinement of the general setting presented in [2] for geometric flow of curves with mass lumping parametric FEMs based on Gauss–Lobatto points, and provides a foundation for us to establish optimal-order error estimates of the stabilized BGN method for curve shortening flow.

3.1. **Notations.** The following notations will be frequently used in this article. They are similar to the notations in [2, Section 3.1] and are listed below for the convenience of the readers.

- Γ^m: The exact smooth curve at time level $t = t_m$.
- Γ_h^m: The numerically computed curve at time level $t = t_m$.
- $\mathbf{x}^m$: The nodal vector $\mathbf{x}^m = (x_1^m, \dots, x_J^m)^\top$ consisting of the positions of nodes on Γ_h^m.
- $\hat{\mathbf{x}}_*^m$: The distance projection of $\mathbf{x}^m$ onto the exact curve Γ^m, i.e., $\hat{\mathbf{x}}_*^m = (\hat x_{1,*}^m, \dots, \hat x_{J,*}^m)^\top$ with $\hat x_{j,*}^m = a^m(x_j^m)$.
- $\mathbf{x}_*^{m+1}$: The new position of $\hat{\mathbf{x}}_*^m$ evolving under curve shortening flow (without additional tangential motion) from t_m to t_{m+1}.
- $\hat\Gamma_{h,*}^m$: The piecewise polynomial curve which interpolates Γ^m at the nodes in $\hat{\mathbf{x}}_*^m$.
- $\Gamma_{h,*}^{m+1}$: The piecewise polynomial curve which interpolates Γ^{m+1} at the nodes in $\mathbf{x}_*^{m+1}$.
- X_h^m: The finite element function with nodal vector $\mathbf{x}^m$. It coincides with the identity map, i.e., $\mathrm{id}(x) = x$, when it is considered as a function on Γ_h^m.
- X_h^{m+1}: The finite element function with nodal vector $\mathbf{x}^{m+1}$. When it is considered as a function on Γ_h^m, it represents the local flow map from Γ_h^m to Γ_h^{m+1}.

$\hat X^m_{h,*}$:	The finite element function with nodal vector $\hat{\mathbf{x}}^m_*$. It coincides with the identity map, i.e., $\mathrm{id}(x) = x$, when it is considered as a function on $\hat\Gamma^m_{h,*}$. It coincides with the discrete flow map from $\hat\Gamma^0_{h,*}$ to $\hat\Gamma^m_{h,*}$ when it is considered as a function on $\hat\Gamma^0_{h,*}$.
$X^{m+1}_{h,*}$:	The finite element function with nodal vector $\mathbf{x}^{m+1}_*$. When it is considered as a function on $\hat\Gamma^m_{h,*}$, it represents the local flow map from $\hat\Gamma^m_{h,*}$ to $\Gamma^{m+1}_{h,*}$.
X^{m+1}:	The local flow map from Γ^m to Γ^{m+1} under mean curvature flow.
$\hat e^m_h$:	The finite element error function with nodal vector $\hat{\mathbf{e}}^m = \mathbf{x}^m - \hat{\mathbf{x}}^m_*$.
e^{m+1}_h:	The auxiliary error function with nodal vector $\mathbf{e}^{m+1} = \mathbf{x}^{m+1} - \mathbf{x}^{m+1}_*$.
n^m:	The unit normal vector on Γ^m.
n^m_*:	The unit normal vector of Γ^m inversely lifted to a neighborhood of Γ^m (including $\hat\Gamma^m_{h,*}$), i.e., $n^m_* = n^m \circ a^m$.
$\hat n^m_{h,*}$:	The normal vector on $\hat\Gamma^m_{h,*}$.
$\bar n^m_{h,*}$:	The averaged normal vector on $\hat\Gamma^m_{h,*}$, which is not necessarily unit.
n^m_h:	The normal vector on Γ^m_h.
$\bar n^m_h$:	The averaged normal vector on Γ^m_h, which is not necessarily unit.
$\hat\mu^m_{h,*}$:	The co-normal vector (unit tangent vector) on $\hat\Gamma^m_{h,*}$.
μ^m_h:	The co-normal vector (unit tangent vector) on Γ^m_h.
N^m_*:	The normal projection operator $N^m_* = n^m_*(n^m_*)^\top$ on $\hat\Gamma^m_{h,*}$.
N^m:	The normal projection operator $N^m = n^m(n^m)^\top$ on Γ^m. Thus N^m is the lift of N^m_* onto Γ^m, and N^m_* is the extension of N^m to a neighborhood of Γ^m.
$\hat N^m_{h,*}$:	The normal projection operator $\hat N^m_{h,*} = \hat n^m_{h,*}(\hat n^m_{h,*})^\top$ on $\hat\Gamma^m_{h,*}$.
$\bar N^m_{h,*}$:	The averaged normal projection operator $\bar N^m_{h,*} = \frac{\bar n^m_{h,*}}{\lvert\bar n^m_{h,*}\rvert}(\frac{\bar n^m_{h,*}}{\lvert\bar n^m_{h,*}\rvert})^\top$ on $\hat\Gamma^m_{h,*}$.
T^m_*:	The tangential projection operator $T^m_* = I - n^m_*(n^m_*)^\top$ on $\hat\Gamma^m_{h,*}$.
T^m:	The tangential projection operator $T^m = I - n^m(n^m)^\top$ on Γ^m. Thus T^m is the lift of T^m_* onto Γ^m.
$\hat T^m_{h,*}$:	The tangential projection operator $\hat T^m_{h,*} = I - \hat n^m_{h,*}(\hat n^m_{h,*})^\top$ on $\hat\Gamma^m_{h,*}$.
$\bar T^m_{h,*}$:	The averaged tangential projection operator $\bar T^m_{h,*} = I - \frac{\bar n^m_{h,*}}{\lvert\bar n^m_{h,*}\rvert}(\frac{\bar n^m_{h,*}}{\lvert\bar n^m_{h,*}\rvert})^\top$ on $\hat\Gamma^m_{h,*}$.
$\mathcal{N}(\Gamma^m_h)$:	The collection of nodes of Γ^m_h.
$\mathcal{N}_b(\Gamma^m_h)$:	The collection of endpoints (boundary points) of the elements of Γ^m_h.

For the simplicity of notation, we shall denote by $I_h\bar N^m_{h,*}\phi_h$ and $I_h\bar T^m_{h,*}\phi_h$ the abbreviations of $I_h(\bar N^m_{h,*}\phi_h)$ and $I_h(\bar T^m_{h,*}\phi_h)$, respectively. Similar notations are also adopted for $I_h\hat N^m_{h,*}\phi_h$, $I_hN^m_*\phi_h$, $I_h\hat T^m_{h,*}\phi_h$, $I_hT^m_*\phi_h$, and so on.

If K is an element of $\hat\Gamma^m_{h,*}$ then we denote by $K^0 \subset \Gamma^0_h$ the element which is mapped to K by the discrete flow map $\hat X^m_{h,*} : \Gamma^0_h \to \hat\Gamma^m_{h,*}$, and denote by $F_{K^0} : K^0_{\mathrm f} \to K^0$ the parametrization of the element $K^0 \subset \Gamma^0_h$, where $K^0_{\mathrm f}$ is the flat line segment which has the same endpoints as K^0. The flat line segments $K^0_{\mathrm f}$ form a piecewise linear curve

$$\Gamma^0_{h,\mathrm f} = \bigcup_{K^0\subset\Gamma^0_h} K^0_{\mathrm f}.$$

We still denote by $\hat X^m_{h,*} : \Gamma^0_{h,\mathrm f} \to \hat\Gamma^m_{h,*}$ the unique piecewise polynomial of degree k (with nodal vector $\hat{\mathbf{x}}^m_*$ as before) which maps the Gauss–Lobatto points of every flat segment $K^0_{\mathrm f} \subset \Gamma^0_{h,\mathrm f}$ to the corresponding nodes of element $K \subset \hat\Gamma^m_{h,*}$. Therefore, $\hat X^m_{h,*} : \Gamma^0_{h,\mathrm f} \to \hat\Gamma^m_{h,*}$ is a piecewise polynomial parametrization of $\hat\Gamma^m_{h,*}$. We denote by $\|\hat{X}_{h,*}^m\|_{W_h^{j,\infty}(\Gamma_{h,\mathrm{f}}^0)}$ the piecewise Sobolev norms on $\Gamma_{h,\mathrm{f}}^0$, i.e.,

$$\|\hat{X}_{h,*}^m\|_{W_h^{j,\infty}(\Gamma_{h,\mathrm{f}}^0)} := \max_{K_\mathrm{f}^0\subset\Gamma_{h,\mathrm{f}}^0} \|\hat{X}_{h,*}^m\|_{W^{j,\infty}(K_\mathrm{f}^0)}.$$

Since each piece $K \in \hat{\Gamma}_{h,*}^m$ can be endowed with a canonical smooth structure, the piecewise Sobolev norms can be also defined on $\hat{\Gamma}_{h,*}^m$.

We denote by I_K the interpolation operator on the flat segment K_f^0. Since $F_K = a^m \circ F_K$ at the nodes of K_f^0, it follows that $I_K[a^m \circ F_K] = F_K$. The interpolation of the distance projection $a^m|_{\hat{\Gamma}_{h,*}^m} : \hat{\Gamma}_{h,*}^m \to \Gamma^m$ onto the curved surface $\hat{\Gamma}_{h,*}^m$ is defined as

$$I_h a^m := I_K[a^m \circ F_K] \circ F_K^{-1} = \mathrm{id} \quad \text{on an element } K \subset \hat{\Gamma}_{h,*}^m.$$

For a smooth function f on the smooth curve Γ^m, we denote by $I_h f$ the interpolation of the inversely lifted function $f^{-l} = f \circ a^m$ onto $\hat{\Gamma}_{h,*}^m$, i.e.,

$$I_h f := I_K[f \circ a^m \circ F_K] \circ F_K^{-1} \quad \text{on an element } K \subset \hat{\Gamma}_{h,*}^m.$$

We denote by $(I_h f)^l = (I_h f) \circ (a^m|_{\hat{\Gamma}_{h,*}^m})^{-1}$ the lift of $I_h f$ onto Γ^m. For a piecewise smooth function f on $\hat{\Gamma}_{h,*}^m$ (instead of Γ^m), we use the same notation $I_h f$ denotes the following interpolated function on $\hat{\Gamma}_{h,*}^m$:

$$I_h f := I_K[f \circ F_K] \circ F_K^{-1} \quad \text{on an element } K \subset \hat{\Gamma}_{h,*}^m.$$

3.2. **Approximation properties of the interpolated surface $\hat{\Gamma}_{h,*}^m$.** For the discrete flow map $\hat{X}_{h,*}^m : \Gamma_{h,\mathrm{f}}^0 \to \hat{\Gamma}_{h,*}^m$, we denote

$$\begin{aligned} \kappa_l &:= \max_{0\le m\le l} \big(\|\hat{X}_{h,*}^m\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} + \|(\hat{X}_{h,*}^m)^{-1}\|_{W^{1,\infty}(\hat{\Gamma}_{h,*}^m)}\big) \\ &= \max_{0\le m\le l} \max_{K\subset\hat{\Gamma}_{h,*}^m} \big(\|F_K\|_{W^{k,\infty}(K_\mathrm{f}^0)} + \|F_K^{-1}\|_{W^{1,\infty}(K)}\big). \end{aligned} \tag{3.1}$$

By pulling functions on $\hat{\Gamma}_{h,*}^m$ back to $\Gamma_{h,\mathrm{f}}^0$ via the map $\hat{X}_{h,*}^m : \Gamma_{h,\mathrm{f}}^0 \to \hat{\Gamma}_{h,*}^m$ (and vice visa), one can see that the $W^{1,p}, p \in [1,\infty]$, norms of a finite element function (with a fixed nodal vector) on $\Gamma_{h,\mathrm{f}}^0$ and $\hat{\Gamma}_{h,*}^m$ are equivalent up to constants which depend on κ_l, i.e.,

$$C_{\kappa_l}^{-1}\|v_h\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)} \le \|v_h\|_{W^{1,p}(\Gamma_{h,\mathrm{f}}^0)} \le C_{\kappa_l}\|v_h\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)},$$

for $0 \le m \le l$. Since $\hat{X}_{h,*}^m : \Gamma_{h,\mathrm{f}}^0 \to \hat{\Gamma}_{h,*}^m$ is the Lagrange interpolation of $a^m \circ \hat{X}_{h,*}^m : \Gamma_{h,\mathrm{f}}^0 \to \Gamma^m$ on the piecewise flat curve $\Gamma_{h,\mathrm{f}}^0$, it follows that

$$\|a^m \circ \hat{X}_{h,*}^m - \hat{X}_{h,*}^m\|_{L^\infty(\Gamma_{h,\mathrm{f}}^0)} + h\|a^m \circ \hat{X}_{h,*}^m - \hat{X}_{h,*}^m\|_{W^{1,\infty}(\Gamma_{h,\mathrm{f}}^0)} \le C_{\kappa_l} h^{k+1}. \tag{3.2}$$

Since $I_h a^m = \mathrm{id}$ on $\hat{\Gamma}_{h,*}^m$, inequality (3.2) can be equivalently written as follows by using the norm equivalence on $\Gamma_{h,\mathrm{f}}^0$ and $\hat{\Gamma}_{h,*}^m$:

$$\|a^m - I_h a^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + h\|a^m - I_h a^m\|_{W^{1,\infty}(\hat{\Gamma}_{h,*}^m)} \le C_{\kappa_l} h^{k+1}. \tag{3.3}$$

Moreover, the following estimates hold for any smooth function f on Γ^m:

$$\|f^{-l} - I_h f\|_{L^2(\hat\Gamma^m_{h,*})} + h\|f^{-l} - I_h f\|_{H^1(\hat\Gamma^m_{h,*})} \le C_{\kappa_l}\|f\|_{H^{k+1}(\Gamma^m)} h^{k+1}, \tag{3.4}$$

$$\|f - (I_h f)^l\|_{L^2(\Gamma^m)} + h\|f - (I_h f)^l\|_{H^1(\Gamma^m)} \le C_{\kappa_l}\|f\|_{H^{k+1}(\Gamma^m)} h^{k+1}. \tag{3.5}$$

Similar estimates have been shown in [2, inequalities (3.3) and (3.4)]. The boundedness of κ_l (independent of τ, h and l) will be proved in Section 4.9.

We denote by n^m and H^m the unit normal vector and the mean curvature on Γ^m, respectively, and denote by $n^m_* = n^m \circ a^m$ and $H^m_* = H^m \circ a^m$ the smooth extensions of n^m and H^m to a neighborhood $D_\delta(\Gamma^m)$ of Γ^m. In particular, n^m_* and H^m_* are well defined on $\hat\Gamma^m_{h,*}$ as the inverse lift of n^m and H^m via the distance projection a^m, respectively, with

$$\|n^m_*\|_{W^{j,\infty}(D_\delta(\Gamma^m))} + \|H^m_*\|_{W^{j,\infty}(D_\delta(\Gamma^m))} \le C_j \quad \text{for all } j \ge 0.$$

Moreover, the normal vectors on $\hat\Gamma^m_{h,*}$ and Γ^m (inversely lifted to $\hat\Gamma^m_{h,*}$) have the following expressions (using the parametrizations $\hat X^m_{h,*} : \Gamma^0_{h,\mathrm{f}} \to \hat\Gamma^m_{h,*}$ and $a^m \circ \hat X^m_{h,*} : \Gamma^0_{h,\mathrm{f}} \to \Gamma^m$, respectively):

$$\hat n^m_{h,*} = \frac{(\nabla_{\Gamma^0_{h,\mathrm{f}}} \hat X^m_{h,*})^\perp}{|\nabla_{\Gamma^0_{h,\mathrm{f}}} \hat X^m_{h,*}|} \circ (\hat X^m_{h,*})^{-1} \quad \text{and} \quad n^m_* = \frac{[\nabla_{\Gamma^0_{h,\mathrm{f}}}(a^m \circ \hat X^m_{h,*})]^\perp}{|\nabla_{\Gamma^0_{h,\mathrm{f}}}(a^m \circ \hat X^m_{h,*})|} \circ (\hat X^m_{h,*})^{-1}, \tag{3.6}$$

where $v^\perp := (-v_2, v_1)$ for any vector $v = (v_1, v_2)$. These expressions lead to the following estimates as a result of (3.2):

$$\|\hat n^m_{h,*} - n^m_*\|_{L^\infty(\hat\Gamma^m_{h,*})} \le C_{\kappa_l} h^k. \tag{3.7}$$

The expression of $\hat n^m_{h,*}$ also implies that

$$\|\hat n^m_{h,*}\|_{W^{j,\infty}_h(\hat\Gamma^m_{h,*})} \le C_{\kappa_l,j} \quad \forall j \ge 0, \tag{3.8}$$

which is due to the fact that the $(k+1)$th-order partial derivatives of $\hat X^m_{h,*}$ are zero on $\Gamma^0_{h,\mathrm{f}}$.

Lemma 3.1 was proved in [35, Lemma 4.3]. It shows that norms of the finite element functions with same nodal vectors on the family of surfaces

$$\hat\Gamma^m_{h,\theta} = (1-\theta)\hat\Gamma^m_{h,*} + \theta\Gamma^m_h, \quad \theta \in [0,1],$$

are equivalent, provided that the distance between $\hat\Gamma^m_{h,*}$ and Γ^m_h is small in the $W^{1,\infty}$ norm.

Lemma 3.1. *If $\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \le \frac{1}{2}$ for $\theta \in [0,1]$ then the following equivalence of norms hold for $1 \le p \le \infty$:*

$$\|v_h\|_{L^p(\hat\Gamma^m_{h,*})} \lesssim \|v_h\|_{L^p(\hat\Gamma^m_{h,\theta})} \lesssim \|v_h\|_{L^p(\hat\Gamma^m_{h,*})},$$
$$\|\nabla_{\hat\Gamma^m_{h,*}} v_h\|_{L^p(\hat\Gamma^m_{h,*})} \lesssim \|\nabla_{\hat\Gamma^m_{h,\theta}} v_h\|_{L^p(\hat\Gamma^m_{h,\theta})} \lesssim \|\nabla_{\hat\Gamma^m_{h,*}} v_h\|_{L^p(\hat\Gamma^m_{h,*})}.$$

Lemma 3.2 concerns the difference between integrals on the smooth curve Γ^m and the interpolated curve $\hat\Gamma^m_{h,*}$.

Lemma 3.2 ([32, Lemma 5.6])**.** *The following estimates hold for $f_1, f_2 \in H^1(\hat\Gamma^m_{h,*})$ and their lifts $f_1^l, f_2^l \in H^1(\Gamma^m)$:*

$$\Big|\int_{\hat\Gamma^m_{h,*}} f_1 f_2 - \int_{\Gamma^m} f_1^l f_2^l\Big| \le C_{\kappa_l} h^{k+1} \|f_1\|_{L^2(\hat\Gamma^m_{h,*})}\|f_2\|_{L^2(\hat\Gamma^m_{h,*})},$$

$$\Big|\int_{\hat\Gamma^m_{h,*}} \nabla_{\hat\Gamma^m_{h,*}} f_1 \cdot \nabla_{\hat\Gamma^m_{h,*}} f_2 - \int_{\Gamma^m} \nabla_{\Gamma^m} f_1^l \cdot \nabla_{\Gamma^m} f_2^l\Big|$$
$$\le C_{\kappa_l} h^{k+1} \|\nabla_{\hat\Gamma^m_{h,*}} f_1\|_{L^2(\hat\Gamma^m_{h,*})}\|\nabla_{\hat\Gamma^m_{h,*}} f_2\|_{L^2(\hat\Gamma^m_{h,*})}.$$

In the rest of this article, we denote by C a generic positive constant which may be different at different occurrences, possibly dependent on κ_l and T, but is independent of τ, h and m. We denote by C_0 generic positive constant which is independent of κ_l. For the simplicity of notation, we denote by $A \lesssim B$ the statement "$A \le CB$ for some constant C". The statement "for sufficiently small $h \dots$" means that "there exists a constant C, possibly depending on κ_l, such that for $h \le C^{-1} \dots$".

3.3. **Mathematical induction assumptions.** We assume that the following conditions hold for $m = 0, \dots, l$ (and then prove that these conditions could be recovered for $m = l + 1$):

(1) The numerically computed curve Γ^m_h is in a δ-neighborhood of the exact curve Γ^m. Therefore, the distance projection of the nodes of Γ^m_h onto Γ^m are well defined (thus the interpolated curve $\hat\Gamma^m_{h,*}$ is well defined).

(2) The error $\hat e^m_h = X^m_h - \hat X^m_{h,*}$ satisfies the following estimates:

$$\|\hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})} + h\|\hat e^m_h\|_{H^1(\hat\Gamma^m_{h,*})} \le h^{2.75}. \tag{3.9}$$

Remark 3.3. The exponent 2.75 is required in the derivation of the last inequality in (4.91), which requires $h^{-7/2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2(\hat\Gamma^m_{h,*})} \lesssim 1$.

Based on these induction assumptions, the following results can be obtained from (3.9) by applying the inverse inequality of finite element functions:

$$\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})} \lesssim h^{1.75}, \quad \|\hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \lesssim h^{2.25} \quad \text{and} \quad \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \lesssim h^{1.25}, \tag{3.10}$$

which guarantee the equivalence of L^p and $W^{1,p}$ norms, $1 \le p \le \infty$, of finite element functions v_h with a common nodal vector on the family of curves

$$\hat\Gamma^m_{h,\theta} = (1-\theta)\hat\Gamma^m_{h,*} + \theta\Gamma^m_h, \quad \theta \in [0,1].$$

They are intermediate curves between the interpolated curve $\hat\Gamma^m_{h,*}$ and the approximate curve Γ^m_h given by the numerical solution; see [35, Lemma 4.3]. In particular, the L^p and $W^{1,p}$ norms of a finite element function on $\hat\Gamma^m_{h,*}$ and Γ^m_h (with a common nodal vector) are equivalent.

3.4. **Super-approximation, Gauss–Lobatto quadrature and discrete norms.** The following super-approximation estimates of products of finite element functions were proved in [31, Lemma A] and [2, Lemma 4.4] for parametric finite elements on a surface in the three-dimensional space. The same results and proofs also hold for parametric finite elements on a curve in the two-dimensional plane.

Lemma 3.4 (Super-approximation estimates of type I)**.** *The following estimates hold for any piecewise smooth function f and finite element functions $\phi_h, v_h, w_h \in S_h(\hat\Gamma^m_{h,*})$:*

$$\begin{aligned}
\|(1-I_h)(f\phi_h)\|_{L^2(\hat\Gamma^m_{h,*})} &\lesssim \|f\|_{W_h^{k+1,\infty}(\hat\Gamma^m_{h,*})} h\|\phi_h\|_{L^2(\hat\Gamma^m_{h,*})},\\
\|\nabla_{\hat\Gamma^m_{h,*}}(1-I_h)(f\phi_h)\|_{L^2(\hat\Gamma^m_{h,*})} &\lesssim \|f\|_{W_h^{k+1,\infty}(\hat\Gamma^m_{h,*})} h\|\phi_h\|_{H^1(\hat\Gamma^m_{h,*})},\\
\|(1-I_h)(v_hw_h)\|_{L^2(\hat\Gamma^m_{h,*})} &\lesssim h^2\|v_h\|_{W^{1,\infty}(\hat\Gamma^m_{h,*})}\|w_h\|_{H^1(\hat\Gamma^m_{h,*})},\\
\|\nabla_{\hat\Gamma^m_{h,*}}(1-I_h)(v_hw_h)\|_{L^2(\hat\Gamma^m_{h,*})} &\lesssim h\|v_h\|_{W^{1,\infty}(\hat\Gamma^m_{h,*})}\|w_h\|_{H^1(\hat\Gamma^m_{h,*})}.
\end{aligned}$$

Another super-approximation type of results which has application in the analysis of mass lumping FEMs is based on the Gauss–Lobatto quadrature on each element. Lemma 3.5 is a direct generalization of [30, Lemma 3.6, Eq. (3.15)] to finite element functions on a piecewise polynomial curve (which can be proved by transforming the integrals from curved elements to flat elements).

Lemma 3.5 (Super-approximation estimates of type II)**.** *Let f be a function which is smooth on every element K of $\hat\Gamma^m_{h,*}$, and assume that the pull-back function $f\circ F_K$ vanishes at all the Gauss–Lobatto points of the flat segment $K^0_{\rm f}$ for every element K of $\hat\Gamma^m_{h,*}$. Then the following two types of estimates hold:*

$$\Big|\int_{\hat\Gamma^m_{h,*}} f\,\mathrm{d}\xi\Big| \lesssim h^{2k}\|f\|_{W_h^{2k,1}(\hat\Gamma^m_{h,*})}, \tag{3.11}$$

where $\|\cdot\|_{W_h^{2k,1}(\hat\Gamma^m_{h,*})}$ denotes the piecewise $W^{2k,1}$ norm . If $(f\phi_h)\circ F_K$ vanishes at all the Gauss–Lobatto points of $K^0_{\rm f}$, then the following result follows from Leibniz rule of differentiation and the inverse inequality of finite element functions:

$$\Big|\int_{\hat\Gamma^m_{h,*}} f\phi_h\,\mathrm{d}\xi\Big| \lesssim \|f\|_{H_h^{2k}(\hat\Gamma^m_{h,*})} h^{k+1}\|\phi_h\|_{H^1(\hat\Gamma^m_{h,*})}. \tag{3.12}$$

The result below can be proved similarly as [30, Lemma 3.7] by using integration by parts and the first result of Lemma 3.5.

Lemma 3.6 (Super-approximation estimates of type III)**.** *For a smooth function f on Γ^m the following estimate holds:*

$$\Big|\int_{\Gamma^m} \nabla_{\Gamma^m}(f-(I_hf)^l)\cdot\nabla_{\Gamma^m}\phi_h^l\Big| \lesssim h^{k+1}\|f\|_{H^{2k}(\Gamma^m)}\|\phi_h\|_{H^1(\hat\Gamma^m_{h,*})} \quad \forall\,\phi_h\in S_h(\hat\Gamma^m_{h,*}).$$

Since the weights of the Gauss–Lobatto quadrature are positive, the discrete L^p norm defined by

$$\|v\|_{L_h^p(\hat\Gamma^m_{h,*})} := \Big(\int_{\hat\Gamma^m_{h,*}}^h |v|^p\Big)^{\frac1p} = \Big(\sum_{K\subset\hat\Gamma^m_{h,*}}\int_{K^0_{\rm f}} I_K\big(|v\circ F_K|^p|\nabla_{K^0_{\rm f}}F_K|\big)\Big)^{\frac1p}$$

is indeed a norm on the finite element space $S_h(\hat\Gamma^m_{h,*})$ because $\|v\|_{L_h^p(\hat\Gamma^m_{h,*})}=0$ iff $v=0$ at all the nodes of $\hat\Gamma^m_{h,*}$. In addition, this discrete L^p norm is also well defined for functions which are piecewise continuous on $\hat\Gamma^m_{h,*}$. Its basic properties are summarized below.

Lemma 3.7. *The following relations hold for all finite element functions* $v_h \in S_h(\hat\Gamma^m_{h,*})$ *and piecewise continuous functions* w_1, w_2, w_3 *on* $\hat\Gamma^m_{h,*}$*:*

$$\begin{aligned}\|v_h\|_{L^p_h(\hat\Gamma^m_{h,*})} &\sim \|v_h\|_{L^p(\hat\Gamma^m_{h,*})},\\ \|\nabla_{\hat\Gamma^m_{h,*}} v_h\|_{L^p_h(\hat\Gamma^m_{h,*})} &\sim \|\nabla_{\hat\Gamma^m_{h,*}} v_h\|_{L^p(\hat\Gamma^m_{h,*})},\\ \Big|\int^h_{\hat\Gamma^m_{h,*}} w_1 w_2 w_3\Big| &\lesssim \|w_1\|_{L^\infty(\hat\Gamma^m_{h,*})}\|w_2\|_{L^2_h(\hat\Gamma^m_{h,*})}\|w_3\|_{L^2_h(\hat\Gamma^m_{h,*})}.\end{aligned}$$

The proof of Lemma 3.7 is omitted as these results follow directly from the definition of the discrete L^p norm (analogous results on a bounded interval have been proved in [30]). The first equivalence relation in Lemma 3.7 also holds for piecewise polynomials (not necessarily globally continuous) of degree $\le k$.

Since $\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \lesssim h^{1.25}$, as shown in (3.10), it follows from Lemma 3.1 that for sufficiently small h the quantities $|\nabla_{K^0_{\rm f}} F_K|$ are equivalent for the elements on $\hat\Gamma^m_{h,*}$ and Γ^m_h. Therefore, for any piecewise continuous function v on Γ^m_h the following equivalence relation holds:

$$\|v\|^q_{L^q_h(\Gamma^m_h)} \sim h \sum_{K\subset\Gamma^m_h} \sum_{p\in\mathcal{N}(\Gamma^m_h)\cap K} |v(p)|^q \quad \text{for } 1\le q<\infty. \tag{3.13}$$

Moreover, the following result will be used for finite element functions $v_h, w_h \in S_h(\hat\Gamma^m_{h,*})$:

$$\begin{aligned}\|I_h(v_h w_h)\|_{L^p(\hat\Gamma^m_{h,*})} &\sim \|v_h w_h\|_{L^p_h(\hat\Gamma^m_{h,*})}\\ &\lesssim \|v_h\|_{L^{p_1}_h(\hat\Gamma^m_{h,*})}\|w_h\|_{L^{p_2}_h(\hat\Gamma^m_{h,*})}\\ &\lesssim \|v_h\|_{L^{p_1}(\hat\Gamma^m_{h,*})}\|w_h\|_{L^{p_2}(\hat\Gamma^m_{h,*})}\end{aligned} \tag{3.14}$$

which holds for $1\le p, p_1, p_2\le\infty$ such that $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$.

3.5. **Estimates of the averaged normal vectors.** On the interpolated curve $\hat\Gamma^m_{h,*}$ we can define the averaged normal vector $\bar n^m_{h,*}$ similarly as $\bar n^m_h$ on Γ^m_h, which is defined in (1.6). Namely, we define $\bar n^m_{h,*} \in S_h(\hat\Gamma^m_{h,*})$ to be the unique finite element function satisfying the following relation:

$$\int^h_{\hat\Gamma^m_{h,*}} \bar n^m_{h,*}\cdot\phi_h = \int^h_{\hat\Gamma^m_{h,*}} \hat n^m_{h,*}\cdot\phi_h \quad \forall\,\phi_h\in S_h(\hat\Gamma^m_{h,*}). \tag{3.15}$$

Since (3.15) only involves nodal values, it follows that

$$\int^h_{\hat\Gamma^m_{h,*}} \bar n^m_{h,*}\cdot\phi = \int^h_{\hat\Gamma^m_{h,*}} \hat n^m_{h,*}\cdot\phi \quad \forall\,\phi\in C(\hat\Gamma^m_{h,*})^2. \tag{3.16}$$

It is straightforward to verify the following relations:

$$\begin{aligned}&\bar n^m_{h,*}(p) = \hat n^m_{h,*}(p) && \text{if } p \text{ is an interior node of an element},\\ &\bar n^m_{h,*}(p) = \frac{|w_{K_1}(p)||K^0_{1\rm f}|\ \hat n^m_{h,*}(p-)}{|w_{K_1}(p)||K^0_{1\rm f}| + |w_{K_2}(p)||K^0_{2\rm f}|} + \frac{|w_{K_2}(p)||K^0_{2\rm f}|\ \hat n^m_{h,*}(p+)}{|w_{K_1}(p)||K^0_{1\rm f}| + |w_{K_2}(p)||K^0_{2\rm f}|}\\ &&& \text{if } p = K_1\cap K_2 \text{ for two elements } K_1 \text{ and } K_2,\end{aligned} \tag{3.17}$$

where $w_K(p) = \nabla_{K_{\rm f}^0} F_K \circ F_K^{-1}(p)$ for $p \in K$, with $\hat n^m_{h,*}(p-)$ and $\hat n^m_{h,*}(p+)$ denoting the left (from K_1) and right (from K_2) values of the piecewisely defined normal vector $\hat n^m_{h,*}$ on $\hat\Gamma^m_{h,*}$. Therefore, the amplitude of $\bar n^m_{h,*}$ at the nodes satisfies the following estimates:

(3.18)
$$\begin{aligned}
&|\bar n^m_{h,*}(p)| = 1 && \text{if } p \text{ is an interior node of an element},\\
&|\bar n^m_{h,*}(p)| \le 1, \quad \big||\bar n^m_{h,*}(p)| - 1\big| \le C|\hat n^m_{h,*}(p+) - \hat n^m_{h,*}(p-)|^2 \le C_{\kappa_l} h^{2k} && \\
& && \text{if } p = K_1 \cap K_2 \text{ for two elements } K_1 \text{ and } K_2.
\end{aligned}$$

The estimate of $\big||\bar n^m_{h,*}(p)| - 1\big|$ in (3.18) is obtained by using the expression

$$\bar n^m_{h,*}(p) = \hat n^m_{h,*}(p-) + \lambda(\hat n^m_{h,*}(p+) - \hat n^m_{h,*}(p-)),$$

with $\lambda = |w_{K_2}(p)||K_{2\rm f}^0|/(|w_{K_1}(p)||K_{1\rm f}^0| + |w_{K_2}(p)||K_{2\rm f}^0|)$, and then using the following identity:

$$\begin{aligned}
|\bar n^m_{h,*}(p)|^2 =& 1 + 2\lambda \hat n^m_{h,*}(p-)\cdot(\hat n^m_{h,*}(p+) - \hat n^m_{h,*}(p-)) + \lambda^2|\hat n^m_{h,*}(p+) - \hat n^m_{h,*}(p-)|^2\\
=& 1 + \lambda|\hat n^m_{h,*}(p+) - \hat n^m_{h,*}(p-)|^2 + \lambda^2|\hat n^m_{h,*}(p+) - \hat n^m_{h,*}(p-)|^2
\end{aligned}$$

where we have used the orthogonality between $\hat n_{h,*}(p+) + \hat n_{h,*}(p-)$ and $\hat n_{h,*}(p+) - \hat n_{h,*}(p-)$.

From the expressions of n^m_h (the normal vector on Γ^m_h) and $\hat n^m_{h,*}$ (the normal vector on $\hat\Gamma^m_{h,*}$), as shown in (3.6), one can estimate $n^m_h - \hat n^m_{h,*}$ in terms of the derivative of $\hat e^m_h = X^m_h - \hat X^m_{h,*}$, i.e.,

$$\|n^m_h - \hat n^m_{h,*}\|_{L^2(\hat\Gamma^m_{h,*})} + \|n^m_h - \hat n^m_{h,*}\|_{L^2_h(\hat\Gamma^m_{h,*})} \lesssim \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})}, \tag{3.19}$$

where we have used the equivalence between continuous and discrete L^2 norms in Lemma 3.7. Since $\|\bar n^m_h - \bar n^m_{h,*}\|_{L^2(\hat\Gamma^m_{h,*})}$ can be converted to $\|n^m_h - \hat n^m_{h,*}\|_{L^2_h(\hat\Gamma^m_{h,*})}$ using the nodal expressions of $\bar n^m_h$ and $\hat n^m_{h,*}$ in (1.7) and (3.17), it follows that

$$\|\bar n^m_h - \bar n^m_{h,*}\|_{L^2(\hat\Gamma^m_{h,*})} \lesssim \|n^m_h - \hat n^m_{h,*}\|_{L^2_h(\hat\Gamma^m_{h,*})} \lesssim \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})}. \tag{3.20}$$

Lemma 3.8 shows that the averaged normal vectors approximate the normal vector of Γ^m with the same order of accuracy as the piecewisely defined normal vectors.

Lemma 3.8. *The following approximation properties of $\bar n^m_{h,*}$ and $\bar n^m_h$ hold:*

$$\begin{aligned}
&\|\bar n^m_{h,*} - I_h n^m_*\|_{L^\infty(\hat\Gamma^m_{h,*})} \lesssim h^k,\\
&\|\bar n^m_h - I_h n^m_*\|_{L^2(\hat\Gamma^m_{h,*})} \lesssim h^k + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})},\\
&\|\bar n^m_{h,*} - \hat n^m_{h,*}\|_{L^\infty(\hat\Gamma^m_{h,*})} \lesssim h^k.
\end{aligned} \tag{3.21}$$

Proof. Since $\bar n^m_{h,*}$ is defined as the weighted sum of $\hat n^m_{h,*}$, the L^∞ approximation property

$$\|\hat n^m_{h,*} - I_h n^m_*\|_{L^\infty(\hat\Gamma^m_{h,*})} \lesssim h^k \tag{3.22}$$

implies the first and the third results.

The second result of Lemma 3.8 follows from the application of the triangle inequality, i.e.,

$$\|\bar{n}_h^m - I_h n_*^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \le \|\bar{n}_{h,*}^m - I_h n_*^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|\bar{n}_h^m - \bar{n}_{h,*}^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}, \tag{3.23}$$

where the first term on the right-hand side of (3.23) is bounded by $C_{\kappa_l} h^k$ according to the first result, and the second term on the right-hand side of (3.23) follows from (3.20). □

Lemma 3.8 and the boundedness of n_*^m imply the boundedness of $\bar{n}_{h,*}^m$ and $\bar{n}_h^m$ via the triangle inequality, i.e.,

$$\begin{aligned} &\|\bar{n}_{h,*}^m\|_{H_h^k(\hat{\Gamma}_{h,*}^m)} \lesssim 1, \\ &\|\bar{n}_{h,*}^m\|_{W^{1,\infty}(\hat{\Gamma}_{h,*}^m)} \lesssim 1 + h^{k-1} \lesssim 1, \\ &\|\bar{n}_h^m\|_{W^{1,\infty}(\hat{\Gamma}_{h,*}^m)} \lesssim 1 + h^{k-1} + h^{-1}\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \lesssim 1, \end{aligned} \tag{3.24}$$

where the last inequality follows from the induction assumption in (3.9).

As an application of the discrete norms and the estimates of the normal vectors, we can estimate the following type of endpoint terms arising from integration by parts on each element:

$$h \sum_{p \in \mathcal{N}_b(\Gamma_h^m)} |\mu_h^m(p+) + \mu_h^m(p-)||\varphi_h(p)|,$$

where μ_h^m is the co-normal vector (tangent vector) at an endpoint of an element (pointing to the outward direction) and φ_h is a finite element function on the curve Γ_h^m. Since $|\mu_h^m(p+) + \mu_h^m(p-)| = |n_h^m(p+) - n_h^m(p-)|$, the following result holds:

$$\begin{aligned} &h \sum_{p \in \mathcal{N}_b(\Gamma_h^m)} |\mu_h^m(p+) + \mu_h^m(p-)||\varphi_h(p)| \\ &\le h \sum_{p \in \mathcal{N}_b(\Gamma_h^m)} (|n_h^m(p+) - I_h n_*^m(p+)| + |I_h n_*^m(p-) - n_h^m(p-)|)|\varphi_h(p)| \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } I_h n_*^m(p+) = I_h n_*^m(p-)) \\ &\lesssim \|n_h^m - I_h n_*^m\|_{L_h^2(\Gamma_h^m)} \|\varphi_h\|_{L_h^2(\Gamma_h^m)} \quad (\text{the equivalence relation in (3.13) is used}) \\ &\lesssim (\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + h^k)\|\varphi_h\|_{L^2(\hat{\Gamma}_{h,*}^m)}, \end{aligned} \tag{3.25}$$

where the last inequality uses (3.7) and (3.19).

3.6. **Poincaré inequalities for vector-valued functions.** The following Poincaré type of inequality on a closed finite element curve/surface was proved in [31, Lemma 3.4]:

$$\int_{\hat{\Gamma}_{h,*}^m} |v|^2 \lesssim \int_{\hat{\Gamma}_{h,*}^m} |v \cdot I_h n_*^m|^2 + \int_{\hat{\Gamma}_{h,*}^m} |\nabla_{\hat{\Gamma}_{h,*}^m} v|^2 \quad \forall\, v \in H^1(\hat{\Gamma}_{h,*}^m)^2,$$

which basically says that the full L^2 norm of a vector field can be controlled by the normal component's L^2 norm plus the H^1 semi-norm. By replacing $I_h n_*^m$ with $\bar{n}_{h,*}^m$ and using the first result of Lemma 3.8, we immediately obtain the following Poincaré type inequality with the averaged normal vector $\bar{n}_{h,*}^m$.

Lemma 3.9 (The Poincaré inequality)**.** *For sufficiently small h, the following Poincaré type inequality holds:*

$$\int_{\hat\Gamma_{h,*}^m} |v|^2 \lesssim \int_{\hat\Gamma_{h,*}^m} |v\cdot \bar n_{h,*}^m|^2 + \int_{\hat\Gamma_{h,*}^m} |\nabla_{\hat\Gamma_{h,*}^m} v|^2 \quad \forall\, v\in H^1(\hat\Gamma_{h,*}^m)^2. \tag{3.26}$$

In addition, we can replace the normal component's L^2 norm in the Poincaré inequality by its discrete L^2 norm corresponding to the mass lumping method. This is presented in Lemma 3.10.

Lemma 3.10 (The Poincaré inequality with discrete L^2 norm)**.** *For sufficiently small h, the following Poincaré type inequalitie holds for $v_h\in S_h(\hat\Gamma_{h,*}^m)$:*

$$\int_{\hat\Gamma_{h,*}^m} |v_h|^2 \lesssim \int_{\hat\Gamma_{h,*}^m} I_h(|v_h\cdot \bar n_{h,*}^m|^2) + \int_{\hat\Gamma_{h,*}^m} |\nabla_{\hat\Gamma_{h,*}^m} v_h|^2, \tag{3.27}$$

$$\int_{\hat\Gamma_{h,*}^m} |I_h \bar T_{h,*}^m v_h|^2 \lesssim \int_{\hat\Gamma_{h,*}^m} |\nabla_{\hat\Gamma_{h,*}^m} I_h \bar T_{h,*}^m v_h|^2, \tag{3.28}$$

$$\|v_h\|_{L^\infty(\hat\Gamma_{h,*}^m)} \lesssim \|v_h\cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)} + \|\nabla_{\hat\Gamma_{h,*}^m} v_h\|_{L^2(\hat\Gamma_{h,*}^m)}. \tag{3.29}$$

Proof. From Lemma 3.9 and (3.24) we obtain

$$\begin{aligned}
\|v_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 &\lesssim \|v_h\cdot \bar n_{h,*}^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 + \|\nabla_{\hat\Gamma_{h,*}^m} v_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\
&= \|v_h\cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)}^2 + \|\nabla_{\hat\Gamma_{h,*}^m} v_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \quad (\|\cdot\|_{L^2} \text{ is changed to } \|\cdot\|_{L_h^2}) \\
&\quad + \|v_h\cdot \bar n_{h,*}^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 - \|v_h\cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)}^2 \\
&\lesssim \|v_h\cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)}^2 + \|\nabla_{\hat\Gamma_{h,*}^m} v_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\
&\quad + h^{2k}\||(\bar n_{h,*}^m\cdot v_h)\circ \hat X_{h,*}^m|^2 |\nabla_{\Gamma_{h,\mathrm f}^0} \hat X_{h,*}^m|\|_{W_h^{2k,1}(\Gamma_{h,\mathrm f}^0)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{first result of Lemma 3.5}) \\
&\lesssim \|v_h\cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)}^2 + \|\nabla_{\hat\Gamma_{h,*}^m} v_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\
&\quad + h^2\|\bar n_{h,*}^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}^2 \|v_h\|_{H^1(\hat\Gamma_{h,*}^m)}^2 \quad (\text{inverse inequality}) \\
&\lesssim \|v_h\cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)}^2 + \|\nabla_{\hat\Gamma_{h,*}^m} v_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 + h^2\|v_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\
&\quad + h^2\|\nabla_{\hat\Gamma_{h,*}^m} v_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2,
\end{aligned} \tag{3.30}$$

where the second to last term can be absorbed by the left-hand side. This leads to inequality (3.27).

Inequality (3.28) follows from (3.27) once we note that $I_h \bar T_{h,*}^m v_h \cdot \bar n_{h,*}^m = 0$ at the nodes. Inequality (3.29) also follows from the Sobolev embedding

$$\|v_h\|_{L^\infty(\hat\Gamma_{h,*}^m)} \lesssim \|v_h\|_{L^2(\hat\Gamma_{h,*}^m)} + \|\nabla_{\hat\Gamma_{h,*}^m} v_h\|_{L^2(\hat\Gamma_{h,*}^m)},$$

and that $\|v_h\|_{L^2(\hat\Gamma_{h,*}^m)}$ can be estimated by (3.27). □

Remark 3.11. Since $\bar n_h^m$ differs from $\bar n_{h,*}^m$ by a small quantity in the L^∞ norm as a result of Lemma 3.8 and the induction assumption in (3.9), we can replace $\bar n_{h,*}^m$ by $\bar n_h^m$ in (3.27) and absorb the remainder by the left-hand side. This leads to the following version of Poincaré inequalities in terms of the averaged normal vector $\bar{n}_h^m$:

$$\int_{\hat{\Gamma}_{h,*}^m} |v_h|^2 \lesssim \int_{\hat{\Gamma}_{h,*}^m} I_h(|v_h \cdot \bar{n}_h^m|^2) + \int_{\hat{\Gamma}_{h,*}^m} |\nabla_{\hat{\Gamma}_{h,*}^m} v_h|^2, \tag{3.31}$$

$$\int_{\hat{\Gamma}_{h,*}^m} |I_h \bar{T}_h^m v_h|^2 \lesssim \int_{\hat{\Gamma}_{h,*}^m} |\nabla_{\hat{\Gamma}_{h,*}^m} I_h \bar{T}_h^m v_h|^2. \tag{3.32}$$

3.7. **Geometric relations.** The geometric setting in this article is the same as [2, Section 3.4], including the following relations in (3.33)–(3.38) and Lemma 3.12.

Firstly, by the definition of $\hat{e}_h^{m+1}$ we have the following nodal relation

$$\hat{e}_h^{m+1} = I_h\big[(e_h^{m+1} \cdot n_*^{m+1}) n_*^{m+1}\big] + f_h, \tag{3.33}$$

with

$$|f_h| \lesssim |[1 - n_*^{m+1}(n_*^{m+1})^\top] e_h^{m+1}|^2 \quad \text{at the nodes of } \hat{\Gamma}_{h,*}^{m+1}, \tag{3.34}$$

which means that $\hat{e}_h^{m+1}$ differs from $(e_h^{m+1} \cdot n_*^{m+1}) n_*^{m+1}$ by a much smaller quantity.

Secondly, we denote by $X_{h,*}^{m+1} : \hat{\Gamma}_{h,*}^m \to \Gamma_{h,*}^{m+1}$ the local flow map under which the nodes of $\hat{\Gamma}_{h,*}^m$ move exactly according to curve shortening flow without tangential motion, and denote by $X^{m+1} : \Gamma^m \to \Gamma^{m+1}$ the local flow map of curve shortening flow. Since $X_{h,*}^{m+1} - \hat{X}_{h,*}^m = X^{m+1} - \mathrm{id}$ at the finite element nodes on Γ^m, it follows that

$$X_{h,*}^{m+1} - \hat{X}_{h,*}^m = I_h(X^{m+1} - \mathrm{id}) \qquad \text{on } \hat{\Gamma}_{h,*}^m, \tag{3.35}$$

$$X^{m+1} - \mathrm{id} = \tau(-H^m n^m + g^m) \qquad \text{on } \Gamma^m, \tag{3.36}$$

where $-H^m n^m$ is the exact velocity of curve shortening flow without tangential motion at time level $t = t_m$, and g^m is the smooth correction from the Taylor expansion, satisfying the following estimate:

$$\|g^m\|_{W^{1,\infty}(\Gamma^m)} \le C\tau. \tag{3.37}$$

Therefore, we obtain

$$\begin{aligned} X_h^{m+1} - X_h^m &= e_h^{m+1} - \hat{e}_h^m + X_{h,*}^{m+1} - \hat{X}_{h,*}^m \\ &= e_h^{m+1} - \hat{e}_h^m + \tau I_h(-H^m n^m + g^m). \end{aligned} \tag{3.38}$$

This relation plays an important role in estimating the numerical displacement $X_h^{m+1} - X_h^m$.

The definition of $\hat{e}_h^m$ (i.e., orthogonal to Γ^m at the nodes) guarantees that the tangential component of $\hat{e}_h^m$ (at points which are not nodes) is much smaller than its normal component in the L^2 and H^1 norms. As a result, the full L^2 and H^1 norms of $\hat{e}_h^m$ can be controlled by the normal component's L^2 and H^1 norms, respectively. These results are presented in Lemma 3.12 and will play important roles in the recovery of H^1 full parabolicity of the curve shortening flow. The proof of this lemma can be found in [2, Section 3.5].

Lemma 3.12. *For sufficiently small h, the following estimates hold:*

$$\|(I - n_*^m (n_*^m)^\top)\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \lesssim h\|(\hat{e}_h^m \cdot n_*^m)n_*^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}, \tag{3.39}$$

$$\|(I - n_*^m (n_*^m)^\top)\hat{e}_h^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim h\|(\hat{e}_h^m \cdot n_*^m)n_*^m\|_{H^1(\hat{\Gamma}_{h,*}^m)}, \tag{3.40}$$

$$\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \leq 2\|(\hat{e}_h^m \cdot n_*^m)n_*^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}, \tag{3.41}$$

$$\|\hat{e}_h^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} \leq 2\|(\hat{e}_h^m \cdot n_*^m)n_*^m\|_{H^1(\hat{\Gamma}_{h,*}^m)}. \tag{3.42}$$

The similar results hold if n_*^m is replaced by the averaged normal vector $\bar{n}_{h,*}^m$ on $\hat{\Gamma}_{h,*}^m$ (thus $T_*^m = I - n_*^m(n_*^m)^\top$ is replaced by $\bar{T}_{h,*}^m = I - \frac{\bar{n}_{h,*}^m}{|\bar{n}_{h,*}^m|}(\frac{\bar{n}_{h,*}^m}{|\bar{n}_{h,*}^m|})^\top$), as shown in Lemma 3.13.

Lemma 3.13. *For sufficiently small h, the following estimates hold:*

$$\|I_h \bar{T}_{h,*}^m \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \lesssim h\|\hat{e}_h^m \cdot \bar{n}_{h,*}^m\|_{L_h^2(\hat{\Gamma}_{h,*}^m)}, \tag{3.43}$$

$$\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \leq 2\|\hat{e}_h^m \cdot \bar{n}_{h,*}^m\|_{L_h^2(\hat{\Gamma}_{h,*}^m)}. \tag{3.44}$$

Proof. Since $T_*^m = I - n_*^m(n_*^m)^\top$ is piecewise smooth on $\hat{\Gamma}_{h,*}^m$, the first superapproximation result in Lemma 3.4 implies that

$$\|(1 - I_h)T_*^m \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \lesssim h\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}.$$

By using the two results above and the smoothness of n_*^m in a neighborhood of Γ^m and the first result of Lemma 3.8, as well as the L^∞-stability of the interpolation operator I_h, we have

$$\begin{aligned}
&\|I_h(\bar{T}_{h,*}^m - T_*^m)\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\lesssim \|\bar{T}_{h,*}^m - T_*^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \quad \text{((3.14) is used)} \\
&\leq \|\bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top - n_*^m(n_*^m)^\top\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\quad + \|\bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top - \frac{\bar{n}_{h,*}^m}{|\bar{n}_{h,*}^m|}(\frac{\bar{n}_{h,*}^m}{|\bar{n}_{h,*}^m|})^\top\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\leq \|(\bar{n}_{h,*}^m - I_h n_*^m)(\bar{n}_{h,*}^m)^\top\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\quad + \|(I_h n_*^m)(\bar{n}_{h,*}^m - I_h n_*^m)^\top\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\quad + \|(I_h n_*^m)(I_h n_*^m)^\top - n_*^m(n_*^m)^\top\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\quad + \|\bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top - \frac{\bar{n}_{h,*}^m}{|\bar{n}_{h,*}^m|}(\frac{\bar{n}_{h,*}^m}{|\bar{n}_{h,*}^m|})^\top\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\lesssim h^{k-\frac{1}{2}}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + h^{k-\frac{1}{2}}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + h^{k+1}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + h^{2k}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\lesssim h^{k-\frac{1}{2}}\|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}.
\end{aligned}$$

By using the three results above and the triangle inequality, as well as the first result of Lemma 3.12, we obtain

$$
\begin{aligned}
&\|I_h \bar{T}^m_{h,*} \hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} \\
&\lesssim \|I_h(\bar{T}^m_{h,*} - T^m_*)\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} + \|(1-I_h)T^m_*\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} + \|T^m_*\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} \\
&\lesssim h^{k-\frac12}\|\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} + h\|\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} + h\|\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} \\
&\lesssim h\|I_h\bar{T}^m_{h,*}\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} + h\|I_h\bar{N}^m_{h,*}\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} \\
&\lesssim h\|I_h\bar{T}^m_{h,*}\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} + h\|\hat{e}^m_h\cdot\bar{n}^m_{h,*}\|_{L^2_h(\hat{\Gamma}^m_{h,*})},
\end{aligned} \tag{3.45}
$$

where, in the last inequality, we have used the following norm equivalence

$$
\begin{aligned}
\|I_h\bar{N}^m_{h,*}\hat{e}^m_h\|_{L^2(\hat{\Gamma}^m_{h,*})} \sim \|I_h\bar{N}^m_{h,*}\hat{e}^m_h\|_{L^2_h(\hat{\Gamma}^m_{h,*})} &= \Big\|\hat{e}^m_h\cdot\frac{\bar{n}^m_{h,*}}{|\bar{n}^m_{h,*}|}\Big\|_{L^2_h(\hat{\Gamma}^m_{h,*})} \\
&\sim \|\hat{e}^m_h\cdot\bar{n}^m_{h,*}\|_{L^2_h(\hat{\Gamma}^m_{h,*})},
\end{aligned} \tag{3.46}
$$

and the last equivalence follows from (3.18). Since the first term on the right-hand side of (3.45) can be absorbed by its left-hand side, we obtain the first result of Lemma 3.13. The second result of Lemma 3.13 follows immediately. □

3.8. **Surface calculus formulas.** Given a smooth curve Γ (with or without boundary) in $\mathbb{R}^2$ and $u\in C^\infty(\Gamma)$, we denote by $\underline{D}_i u, i=1,2$, the ith component of the tangent vector $\nabla_\Gamma u$ in $\mathbb{R}^2$. The corresponding Leibniz rule, chain rule, integration-by-parts formula, commutators, and the evolution equation of normal vector are summarized below.

Lemma 3.14. *Let Γ and Γ' be two smooth curves that are possibly open, such as smooth pieces of some finite element curves, and let $f,h\in C^\infty(\Gamma)$ and $g\in C^\infty(\Gamma';\Gamma)$ be given functions. Then the following results hold.*

(1) $\underline{D}_i(fh) = \underline{D}_i f h + f\underline{D}_i h$ *on* Γ.
(2) $\underline{D}_i(g\circ f) = (\underline{D}_j g\circ f)\,\underline{D}_i f$ *on* Γ'.
(3) $\int_\Gamma f\underline{D}_i h = -\int_\Gamma \underline{D}_i f h + \int_\Gamma fhHn_i + \int_{\partial\Gamma} fh\mu_i$ *where n,μ are the normal and co-normal (tangential) direction, respectively, and $H:=\underline{D}_i n_i$ (with the Einstein notation) is the mean curvature, i.e. the trace of the second fundamental form.*
(4) $\underline{D}_i\underline{D}_j f = \underline{D}_j\underline{D}_i f + n_iH_{jl}\underline{D}_l f - n_jH_{il}\underline{D}_l f$, *where* $H_{ij} := \underline{D}_i n_j = \underline{D}_j n_i$.
(5) *If Γ evolves under the velocity field v, and $G_T := \bigcup_{t\in[0,T]}\Gamma(t)\times\{t\}$, then*

$$\partial_t^\bullet(\underline{D}_i f) = \underline{D}_i(\partial_t^\bullet f) - (\underline{D}_i v_j - n_in_l\underline{D}_j v_l)\underline{D}_j f \quad \forall\, f\in C^2(G_T),$$

where $\partial_t^\bullet$ to denote the material derivative with respect to v.
(6) *If $f,h\in C^2(G_T)$ then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma fh = \int_\Gamma \partial_t^\bullet f h + \int_\Gamma f\partial_t^\bullet h + \int_\Gamma fh(\nabla_\Gamma\cdot v).$$

The divergence is defined as $\nabla_\Gamma\cdot v := \underline{D}_i v_i$, which coincides with the intrinsic divergence on the curve if v is a tangential vector field on Γ. Since the Lagrange interpolation commutes with the material time derivative, it *is straightforward to check in the local coordinates that an analogous result also holds for the mass lumping integral, i.e.,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h}^h \tilde f\tilde h = \int_{\Gamma_h}^h \partial_t^\bullet \tilde f\tilde h + \int_{\Gamma_h}^h \tilde f\partial_t^\bullet \tilde h + \int_{\Gamma_h}^h \tilde f\tilde h(\nabla_{\Gamma_h}\cdot v_h),$$

where Γ_h is a finite element curve moving with polynomial velocity $v_h \in S_h(\Gamma_h)$ (mass lumping is well defined on Γ_h), and $\tilde f, \tilde h$ are continuous functions defined on $\bigcup_{t\in[0,T]}\Gamma_h(t)\times\{t\}$.

(7) *The evolution of the unit normal vector n of the curve Γ with respect to the velocity field v satisfies the following relation:*

$$\partial_t^\bullet n_i = -\underline{D}_i v_j n_j.$$

Proof. The first two relations are obvious from the local formula of $\underline{D}$ (cf. [2, Eq. (5.1)]). The third relation is shown in [24, Theorem 2.10]. The fourth and fifth equalities are proved in [25, Lemma 2.4 and 2.6], and the proof of the sixth and last formulae can be found in [22, Appendix A] and [39, p. 33] respectively. □

The following formula can be derived by using the fundamental theorem of calculus and the formulas in Lemma 3.14, item 5 and item 6 (proof is straightforward and omitted). In the case $\partial_\theta^\bullet w_h^\theta = \partial_\theta^\bullet z_h^\theta = 0$, this formula was proved in [33, Lemma 7.1].

Lemma 3.15. *For two family of finite element functions w_h^θ and z_h^θ defined on the intermediate curve $\hat\Gamma_{h,\theta}^m = (1-\theta)\hat\Gamma_{h,*}^m + \theta\Gamma_h^m$, the following identity holds:*

$$\begin{aligned}
&\int_{\Gamma_h^m}\nabla_{\Gamma_h^m} w_h^\theta\cdot\nabla_{\Gamma_h^m} z_h^\theta - \int_{\hat\Gamma_{h,*}^m}\nabla_{\hat\Gamma_{h,*}^m} w_h^\theta\cdot\nabla_{\hat\Gamma_{h,*}^m} z_h^\theta\\
&= \int_0^1\int_{\hat\Gamma_{h,\theta}^m}\nabla_{\hat\Gamma_{h,\theta}^m} w_h^\theta\cdot(D\hat e_h^m)\nabla_{\hat\Gamma_{h,\theta}^m} z_h^\theta \mathrm{d}\theta + \int_0^1\int_{\hat\Gamma_{h,\theta}^m}\nabla_{\hat\Gamma_{h,\theta}^m}\partial_\theta^\bullet w_h^\theta\cdot\nabla_{\hat\Gamma_{h,\theta}^m} z_h^\theta \mathrm{d}\theta\\
&\quad + \int_0^1\int_{\hat\Gamma_{h,\theta}^m}\nabla_{\hat\Gamma_{h,\theta}^m} w_h^\theta\cdot\nabla_{\hat\Gamma_{h,\theta}^m}\partial_\theta^\bullet z_h^\theta \mathrm{d}\theta,
\end{aligned}\tag{3.47}$$

where $(Dv)_{rl} := -\underline{D}_l v_r - \underline{D}_r v_l + \delta_{rl}\underline{D}_m v_m$.

4. Convergence of the stabilized BGN method (Proof of Theorem 2.1)

4.1. **Consistency error.** The optimal-order consistency estimates in this section use the following result.

Lemma 4.1. *For any $\mathbb{R}^2$-valued function f on $\hat\Gamma_{h,*}^m$ which is smooth on each element of $\hat\Gamma_{h,*}^m$, the following estimate holds:*

$$\Big|\int_{\hat\Gamma_{h,*}^m} f\cdot(\hat n_{h,*}^m - n_*^m)\Big| \lesssim h^{k+1}\|f\|_{H^1(\hat\Gamma_{h,*}^m)}.\tag{4.1}$$

Proof. By using the triangle inequality, we have

$$
\begin{aligned}
(4.2)\qquad & \Big|\int_{\hat\Gamma^m_{h,*}} f\cdot(\hat n^m_{h,*}-n^m_*)\Big| \\
&\le \Big|\int_{\Gamma^m} f^l\cdot((\hat n^m_{h,*})^l-n^m)\Big| \\
&\quad + \Big|\int_{\hat\Gamma^m_{h,*}} f\cdot(\hat n^m_{h,*}-n^m_*) - \int_{\Gamma^m} f^l\cdot(\hat n^m_{h,*}-n^m_*)^l\Big| \\
&\lesssim \Big|\int_{\Gamma^m} f^l\cdot((\hat n^m_{h,*})^l-n^m)\Big| + h^{k+1}\|\hat n^m_{h,*}-n^m_*\|_{L^2(\hat\Gamma^m_{h,*})}\|f\|_{L^2(\hat\Gamma^m_{h,*})} \\
&\lesssim \Big|\int_{\Gamma^m} f^l\cdot((\hat n^m_{h,*})^l-n^m)\Big| + h^{k+1}\|f\|_{H^1(\hat\Gamma^m_{h,*})}.
\end{aligned}
$$

In order to estimate the first term on the right-hand side above, we define the intermediate curve $\hat\Gamma^{m,\theta}_{h,*}=(1-\theta)\Gamma^m+\theta\hat\Gamma^m_{h,*}$, which can be parametrized by $\hat X^{m,\theta}_{h,*}:\Gamma^m\to\hat\Gamma^{m,\theta}_{h,*}$ with $\hat X^{m,\theta}_{h,*}=(1-\theta)a^m+\theta(I_ha^m)^l$ and transport velocity $\partial_\theta\hat X^{m,\theta}_{h,*}=(I_ha^m)^l-a^m$, where I_h denotes the interpolation onto $\hat\Gamma^m_{h,*}$. We denote by $\hat n^{m,\theta}_{h,*}$ the unit normal vector of $\hat\Gamma^{m,\theta}_{h,*}$, and denote by v^{l_θ} (and v^{-l_θ}) the lift (and the inverse lift) of a function v from $\hat\Gamma^{m,\theta}_{h,*}$ to Γ^m (and Γ^m to $\hat\Gamma^{m,\theta}_{h,*}$) via this transport velocity. Then

$$
(\hat n^m_{h,*})^l-n^m=\int_0^1\partial^\bullet_\theta(\hat n^{m,\theta}_{h,*})^{l_\theta}\mathrm{d}\theta
$$

and Lemma 3.14 (item 7) implies that

$$
\partial^\bullet_\theta(\hat n^{m,\theta}_{h,*})^{l_\theta}=\big(-\nabla_{\hat\Gamma^{m,\theta}_{h,*}}((I_ha^m)^l-a^m)^{-l_\theta}\,\hat n^{m,\theta}_{h,*}\big)^{l_\theta}. \tag{4.3}
$$

By using the fundamental theorem of calculus and the commutator formula in Lemma 3.14, we have

$$
\begin{aligned}
&\int_{\Gamma^m} f^l\cdot((\hat n^m_{h,*})^l-n^m) \\
&=\int_{\Gamma^m} f^l\cdot\int_0^1\partial^\bullet_\theta(\hat n^{m,\theta}_{h,*})^{l_\theta}\mathrm{d}\theta \\
&=\int_{\Gamma^m} f^l\cdot\int_0^1\Big(-\nabla_{\hat\Gamma^{m,\theta}_{h,*}}((I_ha^m)^l-a^m)^{-l_\theta}\,\hat n^{m,\theta}_{h,*}\Big)^{l_\theta}\mathrm{d}\theta \\
&=-\int_{\Gamma^m} f^l\cdot\Big(\nabla_{\Gamma^m}((I_ha^m)^l-a^m)\,n^m\Big) \\
&\quad-\int_{\Gamma^m} f^l\cdot\int_0^1\Big[\Big(\nabla_{\hat\Gamma^{m,\theta}_{h,*}}((I_ha^m)^l-a^m)^{-l_\theta}\,\hat n^{m,\theta}_{h,*}\Big)^{l_\theta}-\nabla_{\Gamma^m}((I_ha^m)^l-a^m)\,n^m\Big]\mathrm{d}\theta \\
&=-\int_{\Gamma^m} f^l\cdot\Big(\nabla_{\Gamma^m}((I_ha^m)^l-a^m)\,n^m\Big) \\
&\quad-\int_{\Gamma^m} f^l\cdot\int_0^1\int_0^\theta\partial^\bullet_\alpha\Big(\nabla_{\hat\Gamma^{m,\alpha}_{h,*}}((I_ha^m)^l-a^m)^{-l_\alpha}\,\hat n^{m,\alpha}_{h,*}\Big)^{l_\alpha}\mathrm{d}\alpha\mathrm{d}\theta \\
&=-\int_{\Gamma^m} f^l\cdot\Big(\nabla_{\Gamma^m}((I_ha^m)^l-a^m)\,n^m\Big)
\end{aligned}
$$

$$+2\int_{\Gamma^m} f^l\cdot\int_0^1\int_0^\theta\Big(\nabla_{\hat\Gamma^{m,\alpha}_{h,*}}((I_ha^m)^l-a^m)^{-l_\alpha}\nabla_{\hat\Gamma^{m,\alpha}_{h,*}}((I_ha^m)^l-a^m)^{-l_\alpha}\,\hat n^{m,\alpha}_{h,*}\Big)^{l_\alpha}\mathrm{d}\alpha\mathrm{d}\theta$$

(Lemma 3.14, item 7 is used)

$$-\int_{\Gamma^m} f^l\cdot\int_0^1\int_0^\theta\Big(\hat n^{m,\alpha}_{h,*}\Big|\nabla_{\hat\Gamma^{m,\alpha}_{h,*}}((I_ha^m)^l-a^m)^{-l_\alpha}\,\hat n^{m,\alpha}_{h,*}\Big|^2\Big)^{l_\alpha}\mathrm{d}\alpha\mathrm{d}\theta$$

(Lemma 3.14, item 5 is used)

$$=: D_1+D_2+D_3.$$

Via integration by parts on each piece where I_ha^m is smooth, and using Lemma 3.14 (item 3) as well as the property that $(I_ha^m)^l-a^m$ vanishes at the endpoints of these smooth piecewises, we have the following estimate of D_1:

$$\begin{aligned}|D_1| &= \Big|-\int_{\Gamma^m}(\nabla_{\Gamma^m}\cdot f^l)((I_ha^m)^l-a^m)\cdot n^m\\ &\quad-\int_{\Gamma^m} f^l\cdot\Big(\nabla_{\Gamma^m}n^m\,((I_ha^m)^l-a^m)\Big)\\ &\quad+\int_{\Gamma^m}H^mn^m\cdot f^l\,((I_ha^m)^l-a^m)\cdot n^m\Big|\\ &\lesssim h^{k+1}\|f\|_{H^1(\hat\Gamma^m_{h,*})}.\end{aligned}\tag{4.4}$$

The two terms D_2 and D_3 contain squares of the interpolation errors and therefore can be estimated to higher-order, i.e.,

$$|D_2|+|D_3|\lesssim\|\nabla_{\Gamma^m}((I_ha^m)^l-a^m)\|^2_{L^2(\Gamma^m)}\|f\|_{L^\infty(\hat\Gamma^m_{h,*})}\lesssim h^{2k}\|f\|_{H^1(\hat\Gamma^m_{h,*})}.\tag{4.5}$$

The estimates of D_1, D_2 and D_3 lead to

$$\Big|\int_{\Gamma^m} f^l\cdot((\hat n^m_{h,*})^l-n^m)\Big|\lesssim h^{k+1}\|f\|_{H^1(\hat\Gamma^m_{h,*})}.$$

The result of Lemma 4.1 can be obtained by substituting the above inequality into (4.2). □

In view of the stabilized BGN method in (1.5), we define the remainder (consistency error) at the time level t_m to be the following linear functional on $S_h(\hat\Gamma^m_{h,*})$:

$$\begin{aligned}d^m(\phi_h) &:= \int^h_{\hat\Gamma^m_{h,*}}\frac{X^{m+1}_{h,*}-\mathrm{id}}{\tau}\cdot\bar n^m_{h,*}\,\bar n^m_{h,*}\cdot\phi_h+\int_{\hat\Gamma^m_{h,*}}\nabla_{\hat\Gamma^m_{h,*}}X^{m+1}_{h,*}\cdot\nabla_{\hat\Gamma^m_{h,*}}\phi_h\\ &\quad-\int_{\hat\Gamma^m_{h,*}}\nabla_{\hat\Gamma^m_{h,*}}\hat X^m_{h,*}\cdot\nabla_{\hat\Gamma^m_{h,*}}I_h[(I-\bar n^m_{h,*}(\bar n^m_{h,*})^\top)\phi_h]\\ &= \int^h_{\hat\Gamma^m_{h,*}}\frac{X^{m+1}_{h,*}-\mathrm{id}}{\tau}\cdot\bar n^m_{h,*}\,\bar n^m_{h,*}\cdot\phi_h+\int_{\Gamma^m}H^mn^m\cdot\phi_h^l\\ &\quad-\int_{\Gamma^m}\nabla_{\Gamma^m}\mathrm{id}\cdot\nabla_{\Gamma^m}\phi_h^l+\int_{\hat\Gamma^m_{h,*}}\nabla_{\hat\Gamma^m_{h,*}}X^{m+1}_{h,*}\cdot\nabla_{\hat\Gamma^m_{h,*}}\phi_h\\ &\quad-\int_{\hat\Gamma^m_{h,*}}\nabla_{\hat\Gamma^m_{h,*}}\hat X^m_{h,*}\cdot\nabla_{\hat\Gamma^m_{h,*}}I_h[(I-\bar n^m_{h,*}(\bar n^m_{h,*})^\top)\phi_h]\\ &=: d_1^m(\phi_h)+d_2^m(\phi_h)+d_3^m(\phi_h),\end{aligned}\tag{4.6}$$

where we have used the identity $\int_{\Gamma^m} \nabla_{\Gamma^m}\mathrm{id}\cdot\nabla_{\Gamma^m}\phi_h^l = \int_{\Gamma^m} H^m n^m\cdot\phi_h^l$.

Proposition 4.2. *The remainder defined in (4.6) satisfies the following estimate:*

$$|d^m(\phi_h)| \lesssim \tau\|\phi_h\|_{L^2(\hat\Gamma^m_{h,*})} + h^{k+1}\|\phi_h\|_{H^1(\hat\Gamma^m_{h,*})} \quad \forall\,\phi_h\in S_h(\hat\Gamma^m_{h,*}). \tag{4.7}$$

Proof. By using relation $(X_{h,*}^{m+1}-\mathrm{id})/\tau = I_h(X^{m+1}-\mathrm{id})/\tau$, the first term on the right-hand side of (4.6) can be decomposed into six parts as follows:

$$\begin{aligned}
d_1^m(\phi_h) &= \int_{\hat\Gamma^m_{h,*}}^h \frac{X_{h,*}^{m+1}-\mathrm{id}}{\tau}\cdot\bar n^m_{h,*}\,\bar n^m_{h,*}\cdot\phi_h + \int_{\Gamma^m} H^m n^m\cdot\phi_h^l\\
&= \int_{\hat\Gamma^m_{h,*}}^h I_h\Big(\frac{X^{m+1}-\mathrm{id}}{\tau} + H^m n^m\Big)\cdot\bar n^m_{h,*}\,\bar n^m_{h,*}\cdot\phi_h\\
&\quad - \int_{\hat\Gamma^m_{h,*}}^h (I_h(H^m n^m) - H^{m,-l}n^{m,-l})\cdot\bar n^m_{h,*}\,\bar n^m_{h,*}\cdot\phi_h\\
&\quad - \int_{\hat\Gamma^m_{h,*}}^h (H^{m,-l}n^{m,-l}\cdot\bar n^m_{h,*}\,\bar n^m_{h,*} - H^{m,-l}n^{m,-l}\cdot\hat n^m_{h,*}\,\hat n^m_{h,*})\cdot\phi_h\\
&\quad - \Big(\int_{\hat\Gamma^m_{h,*}}^h - \int_{\hat\Gamma^m_{h,*}}\Big) H^{m,-l}n^{m,-l}\cdot\hat n^m_{h,*}\,\hat n^m_{h,*}\cdot\phi_h\\
&\quad - \int_{\hat\Gamma^m_{h,*}} (H^{m,-l}n^{m,-l}\cdot\hat n^m_{h,*}\,\hat n^m_{h,*} - H^{m,-l}n^{m,-l})\cdot\phi_h\\
&\quad - \int_{\hat\Gamma^m_{h,*}} H^{m,-l}n^{m,-l}\cdot\phi_h + \int_{\Gamma^m} H^m n^m\cdot\phi_h^l\\
&=: \sum_{i=1}^6 d_{1i}^m(\phi_h),
\end{aligned} \tag{4.8}$$

where we have used the abbreviation $\big(\int_{\hat\Gamma^m_{h,*}}^h - \int_{\hat\Gamma^m_{h,*}}\big)f = \int_{\hat\Gamma^m_{h,*}}^h f - \int_{\hat\Gamma^m_{h,*}} f$ for any function f defined on $\hat\Gamma^m_{h,*}$. The first and second terms on the right-hand side of (4.8) can be estimated by using relations (3.36)–(3.37) and the nodal relation, respectively, i.e.,

$$\begin{aligned}
&|d_{11}^m(\phi_h)| \lesssim \tau\|\phi_h\|_{L^2(\hat\Gamma^m_{h,*})},\\
&d_{12}^m(\phi_h) = 0.
\end{aligned}$$

The third term on the right-hand side of (4.8) can be rewritten as

$$
\begin{aligned}
&d_{13}^m(\phi_h)\\
&\quad= -\int_{\hat\Gamma_{h,*}^m}^h H^{m,-l} n^{m,-l} \cdot \bar n_{h,*}^m (\bar n_{h,*}^m - \hat n_{h,*}^m) \cdot \phi_h\\
&\qquad - \int_{\hat\Gamma_{h,*}^m}^h H^{m,-l} n^{m,-l} \cdot (\bar n_{h,*}^m - \hat n_{h,*}^m)\, \hat n_{h,*}^m \cdot \phi_h\\
&\quad= -\int_{\hat\Gamma_{h,*}^m}^h H^{m,-l} n^{m,-l} \cdot \bar n_{h,*}^m (\bar n_{h,*}^m - \hat n_{h,*}^m) \cdot \phi_h\\
&\qquad - \int_{\hat\Gamma_{h,*}^m}^h H^{m,-l} n^{m,-l} \cdot (\bar n_{h,*}^m - \hat n_{h,*}^m) \bar n_{h,*}^m \cdot \phi_h\\
&\qquad + \int_{\hat\Gamma_{h,*}^m}^h H^{m,-l} n^{m,-l} \cdot (\bar n_{h,*}^m - \hat n_{h,*}^m)\, (\bar n_{h,*}^m - \hat n_{h,*}^m) \cdot \phi_h\\
&\quad= \int_{\hat\Gamma_{h,*}^m}^h H^{m,-l} n^{m,-l} \cdot (\bar n_{h,*}^m - \hat n_{h,*}^m)\, (\bar n_{h,*}^m - \hat n_{h,*}^m) \cdot \phi_h,
\end{aligned}
$$

where we have used the following identities:

$$
\int_{\hat\Gamma_{h,*}^m}^h H^{m,-l} n^{m,-l} \cdot \bar n_{h,*}^m (\bar n_{h,*}^m - \hat n_{h,*}^m) \cdot \phi_h = 0
$$

and

$$
\int_{\hat\Gamma_{h,*}^m}^h H^{m,-l} n^{m,-l} \cdot (\bar n_{h,*}^m - \hat n_{h,*}^m) \bar n_{h,*}^m \cdot \phi_h = 0,
$$

which follow from property (3.16). Therefore, $d_{13}^m(\phi_h)$ can be estimated by using the third result of Lemma 3.8, which implies that

$$
|d_{13}^m(\phi_h)| \lesssim h^{2k} \|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}.
$$

The fourth term on the right-hand side of (4.8) can be estimated by the superconvergence of Gauss–Lobatto quadrature in Lemma 3.5, i.e.,

$$
|d_{14}^m(\phi_h)| \lesssim h^{k+1} \|H^{m,-l} n^{m,-l} \cdot \hat n_{h,*}^m\, \hat n_{h,*}^m\|_{H_h^{2k}(\hat\Gamma_{h,*}^m)} \|\phi_h\|_{H^1(\hat\Gamma_{h,*}^m)} \lesssim h^{k+1} \|\phi_h\|_{H^1(\hat\Gamma_{h,*}^m)},
$$

where we have used the result $\|\hat n_{h,*}^m\|_{W_h^{2k,\infty}(\hat\Gamma_{h,*}^m)} \lesssim 1$, which is shown in (3.8).

Since $n^{m,-l} = n^m \circ a^m = n_*^m$ and $H^{m,-l} = H^m \circ a^m = H_*^m$, the fifth term on the right-hand side of (4.8) can bedecomposed into the following three parts:

$$\begin{aligned} d_{15}^m(\phi_h) &= -\int_{\hat\Gamma_{h,*}^m} (H_*^m n_*^m \cdot \hat n_{h,*}^m \hat n_{h,*}^m - H_*^m n_*^m \cdot n_*^m n_*^m) \cdot \phi_h \\ &= -\int_{\hat\Gamma_{h,*}^m} H_*^m n_*^m \cdot (\hat n_{h,*}^m - n_*^m)(\hat n_{h,*}^m - n_*^m) \cdot \phi_h \\ &\quad - \int_{\hat\Gamma_{h,*}^m} H_*^m n_*^m \cdot (\hat n_{h,*}^m - n_*^m) n_*^m \cdot \phi_h \\ &\quad - \int_{\hat\Gamma_{h,*}^m} H_*^m n_*^m \cdot n_*^m (\hat n_{h,*}^m - n_*^m) \cdot \phi_h, \end{aligned}$$

which can be estimated by using (3.7) (for the first part) and Lemma 4.1 (for the second and third parts), i.e.,

$$|d_{15}^m(\phi_h)| \lesssim h^{2k} \|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)} + h^{k+1} \|\phi_h\|_{H^1(\hat\Gamma_{h,*}^m)}.$$

The last term on the right-hand side of (4.8) can be estimated by using the geometric perturbation estimate in Lemma 3.2 and the norm equivalence, which implies that

$$|d_{16}^m(\phi_h)| \lesssim h^{k+1} \|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}.$$

The estimates of $d_{1i}^m(\phi_h)$, $i = 1, \dots, 6$, lead to the following result:

$$|d_1^m(\phi_h)| \lesssim \tau \|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)} + h^{k+1} \|\phi_h\|_{H^1(\hat\Gamma_{h,*}^m)}.$$

We can decompose $d_2^m(\phi_h)$, which is defined in (4.6), into three parts in the same way as [2, Lemma 4.3], i.e.,

$$\begin{aligned} d_2^m(\phi_h) &= \int_{\hat\Gamma_{h,*}^m} \nabla_{\hat\Gamma_{h,*}^m} X_{h,*}^{m+1} \cdot \nabla_{\hat\Gamma_{h,*}^m} \phi_h - \int_{\Gamma^m} \nabla_{\Gamma^m} \mathrm{id} \cdot \nabla_{\Gamma^m} \phi_h^l \\ &= \int_{\hat\Gamma_{h,*}^m} \nabla_{\hat\Gamma_{h,*}^m} (X_{h,*}^{m+1} - \hat X_{h,*}^m) \cdot \nabla_{\hat\Gamma_{h,*}^m} \phi_h \\ &\quad + \int_{\hat\Gamma_{h,*}^m} \nabla_{\hat\Gamma_{h,*}^m} \hat X_{h,*}^m \cdot \nabla_{\hat\Gamma_{h,*}^m} \phi_h - \int_{\Gamma^m} \nabla_{\Gamma^m} (\hat X_{h,*}^m)^l \cdot \nabla_{\Gamma^m} \phi_h^l \\ &\quad + \int_{\Gamma^m} \nabla_{\Gamma^m} [(I_h a^m)^l - a^m] \cdot \nabla_{\Gamma^m} \phi_h^l \\ &= d_{21}^m(\phi_h) + d_{22}^m(\phi_h) + d_{23}^m(\phi_h), \end{aligned} \tag{4.9}$$

where we have used the following relations in the derivation of the second to last equality:

$$(\hat X_{h,*}^m)^l = (I_h a^m)^l \quad \text{and} \quad \mathrm{id} = a^m \text{ on } \Gamma^m.$$

The two terms $d_{21}^m(\phi_h)$ and $d_{22}^m(\phi_h)$ are estimated in [2, Lemma 4.3] with the following results:

$$|d_{21}^m(\phi_h)| \lesssim \tau \|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)} \quad \text{and} \quad |d_{22}^m(\phi_h)| \lesssim h^{k+1} \|\phi_h\|_{H^1(\hat\Gamma_{h,*}^m)}.$$

By using the super-convergence result in Lemma 3.6, we can obtain the following estimate of d_{23}^m (which is better than the result in [2, Lemma 4.3]):

$$|d_{23}^m(\phi_h)| \lesssim h^{k+1} \|\phi_h\|_{H^1(\hat\Gamma_{h,*}^m)}.$$

The estimates of $d_{2i}^m(\phi_h)$, $i = 1, 2, 3$, lead to the following result:

$$|d_2^m(\phi_h)| \lesssim \tau\|\phi_h\|_{L^2(\hat{\Gamma}_{h,*}^m)} + h^{k+1}\|\phi_h\|_{H^1(\hat{\Gamma}_{h,*}^m)}.$$

We can decompose $d_3^m(\phi_h)$, which is defined in (4.6), into several parts by using integration by parts (Lemma 3.14, item 3) and identity $\Delta_{\hat{\Gamma}_{h,*}^m}\hat{X}_{h,*}^m = -\hat{H}_{h,*}^m\hat{n}_{h,*}^m$ on any element of $\hat{\Gamma}_{h,*}^m$, as well as the mass lumping approximation of the integral, i.e.,

$$\begin{aligned}
|d_3^m(\phi_h)| &= \Big|\int_{\hat{\Gamma}_{h,*}^m} \nabla_{\hat{\Gamma}_{h,*}^m}\hat{X}_{h,*}^m \cdot \nabla_{\hat{\Gamma}_{h,*}^m} I_h[(I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h]\Big| \\
&\le \Big|\int_{\hat{\Gamma}_{h,*}^m}^h \hat{H}_{h,*}^m\hat{n}_{h,*}^m \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h\Big| \\
&\quad + \Big|\Big(\int_{\hat{\Gamma}_{h,*}^m} - \int_{\hat{\Gamma}_{h,*}^m}^h\Big)\hat{H}_{h,*}^m\hat{n}_{h,*}^m \cdot I_h[(I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h]\Big| \\
&\quad + \Big|\sum_{p\in\mathcal{N}_b(\hat{\Gamma}_{h,*}^m)} \Big(\hat{\mu}_{h,*}^m(p+)^\top\nabla_{\hat{\Gamma}_{h,*}^m}\hat{X}_{h,*}^m(p+) + \hat{\mu}_{h,*}^m(p-)^\top\nabla_{\hat{\Gamma}_{h,*}^m}\hat{X}_{h,*}^m(p-)\Big) \\
&\qquad\qquad \cdot I_h\big[(I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h(p)\big]\Big|.
\end{aligned}$$

The first term on the right-hand side of the inequality above can be rewritten as

$$\begin{aligned}
&\int_{\hat{\Gamma}_{h,*}^m}^h \hat{H}_{h,*}^m\hat{n}_{h,*}^m \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h \\
&\quad = \int_{\hat{\Gamma}_{h,*}^m}^h (\hat{H}_{h,*}^m - H_*^m)(\hat{n}_{h,*}^m - \bar{n}_{h,*}^m) \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h \\
&\qquad + \int_{\hat{\Gamma}_{h,*}^m}^h (\hat{H}_{h,*}^m - H_*^m)\bar{n}_{h,*}^m \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h \\
&\qquad + \int_{\hat{\Gamma}_{h,*}^m}^h H_*^m\hat{n}_{h,*}^m \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h \\
&\quad = \int_{\hat{\Gamma}_{h,*}^m}^h (\hat{H}_{h,*}^m - H_*^m)(\hat{n}_{h,*}^m - \bar{n}_{h,*}^m) \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h,
\end{aligned}$$

where the last equality uses $\bar{n}_{h,*}^m \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h = 0$ and the following relation as a result of (3.16):

$$\int_{\hat{\Gamma}_{h,*}^m}^h H_*^m\hat{n}_{h,*}^m \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h = \int_{\hat{\Gamma}_{h,*}^m}^h H_*^m\bar{n}_{h,*}^m \cdot (I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top)\phi_h.$$

Therefore, using the identity $\hat{\mu}^m_{h,*}(p\pm)^\top \nabla_{\hat{\Gamma}^m_{h,*}} \hat{X}^m_{h,*}(p\pm) = \hat{\mu}^m_{h,*}(p\pm)^\top$, we have

$$
\begin{aligned}
|d_3^m(\phi_h)| \le& \Big| \int_{\hat{\Gamma}^m_{h,*}}^h (\hat{H}^m_{h,*} - H^m_*)(\hat{n}^m_{h,*} - \bar{n}^m_{h,*}) \cdot (I - \bar{n}^m_{h,*}(\bar{n}^m_{h,*})^\top)\phi_h \Big| \\
&+ \Big| \Big(\int_{\hat{\Gamma}^m_{h,*}} - \int_{\hat{\Gamma}^m_{h,*}}^h \Big) \hat{H}^m_{h,*}\hat{n}^m_{h,*} \cdot I_h[(I - \bar{n}^m_{h,*}(\bar{n}^m_{h,*})^\top)]\phi_h \Big| \\
&+ \Big| \sum_{p\in\mathcal{N}_b(\hat{\Gamma}^m_{h,*})} \Big(\hat{\mu}^m_{h,*}(p+) + \hat{\mu}^m_{h,*}(p-) \Big) \cdot (I - \bar{n}^m_{h,*}(\bar{n}^m_{h,*})^\top)\phi_h(p) \Big| \\
\lesssim& \; h^{2k-1}\|\phi_h\|_{L^2(\hat{\Gamma}^m_{h,*})} + h^{k+1}\|\phi_h\|_{H^1(\hat{\Gamma}^m_{h,*})} \\
&+ \Big| \sum_{p\in\mathcal{N}_b(\hat{\Gamma}^m_{h,*})} \Big(\hat{\mu}^m_{h,*}(p+) + \hat{\mu}^m_{h,*}(p-) \Big) \cdot (I - \bar{n}^m_{h,*}(\bar{n}^m_{h,*})^\top)\phi_h(p) \Big|,
\end{aligned}
$$

where the the second term on the right-hand side of the inequality above is obtained by using the second super-approximation result in Lemma 3.5, and the first term on the right-hand side follows from the estimates $\|\hat{H}^m_{h,*} - H^m_*\|_{L^\infty(\hat{\Gamma}^m_{h,*})} \lesssim h^{k-1}$ (property of approximating Γ^m by $\hat{\Gamma}^m_{h,*}$) and $\|\hat{n}^m_{h,*} - \bar{n}^m_{h,*}\|_{L^\infty(\hat{\Gamma}^m_{h,*})} \lesssim h^k$ (the third result of Lemma 3.8).

Since $\hat{\mu}^m_{h,*}(p+) + \hat{\mu}^m_{h,*}(p-)$ is the jump of tangential vector at the endpoint p of an element, it has magnitude $O(h^k)$ and in the direction of $(\hat{n}^m_{h,*}(p+) + \hat{n}^m_{h,*}(p-))/2$. Therefore,

$$
\begin{aligned}
&\Big| \sum_{p\in\mathcal{N}_b(\hat{\Gamma}^m_{h,*})} \big(\hat{\mu}^m_{h,*}(p+) + \hat{\mu}^m_{h,*}(p-) \big) \cdot (I - \bar{n}^m_{h,*}(\bar{n}^m_{h,*})^\top)\phi_h(p) \Big| \\
&\lesssim h^k \Big| \sum_{p\in\mathcal{N}_b(\hat{\Gamma}^m_{h,*})} (\hat{n}^m_{h,*}(p+) + \hat{n}^m_{h,*}(p-)) \cdot (I - \bar{n}^m_{h,*}(\bar{n}^m_{h,*})^\top)\phi_h(p) \Big| \\
&= h^k \Big| \sum_{p\in\mathcal{N}_b(\hat{\Gamma}^m_{h,*})} (\hat{n}^m_{h,*}(p+) + \hat{n}^m_{h,*}(p-) - 2\bar{n}^m_{h,*}) \cdot (I - \bar{n}^m_{h,*}(\bar{n}^m_{h,*})^\top)\phi_h(p) \Big| \\
&\lesssim h^{k-1}\|\hat{n}^m_{h,*} - \bar{n}^m_{h,*}\|_{L^2_h(\hat{\Gamma}^m_{h,*})}\|\phi_h\|_{L^2_h(\hat{\Gamma}^m_{h,*})} \\
&\lesssim h^{2k-1}\|\phi_h\|_{L^2(\hat{\Gamma}^m_{h,*})},
\end{aligned}
$$

where we have used the estimate $\|\hat{n}^m_{h,*} - \bar{n}^m_{h,*}\|_{L^\infty(\hat{\Gamma}^m_{h,*})} \lesssim h^k$ (third result of Lemma 3.8) and the norm equivalence in Lemma 3.7. This proves $|d_3^m(\phi_h)| \lesssim h^{k+1}\|\phi_h\|_{H^1(\hat{\Gamma}^m_{h,*})}$ for $k \ge 2$.

Finally, combining the estimates of $d_1^m(\phi_h)$, $d_2^m(\phi_h)$ and $d_3^m(\phi_h)$, we obtain the result of Proposition 4.2. □

4.2. **The error equation and the H^1 parabolicity.** The following error equation is obtained by subtracting (4.6) from (1.5):

$$\begin{aligned}
&\int_{\Gamma_h^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot \bar n_h^m\,\bar n_h^m\cdot\phi_h - \int_{\hat\Gamma_{h,*}^m} \frac{X_{h,*}^{m+1}-\hat X_{h,*}^m}{\tau}\cdot \bar n_{h,*}^m\,\bar n_{h,*}^m\cdot\phi_h\\
&\quad+\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}X_h^{m+1}\cdot\nabla_{\Gamma_h^m}\phi_h-\int_{\hat\Gamma_{h,*}^m}\nabla_{\hat\Gamma_{h,*}^m}X_{h,*}^{m+1}\cdot\nabla_{\hat\Gamma_{h,*}^m}\phi_h\\
&\quad-\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}X_h^m\cdot\nabla_{\Gamma_h^m}I_h[I-\bar n_h^m(\bar n_h^m)^\top)\phi_h]\\
&\quad+\int_{\hat\Gamma_{h,*}^m}\nabla_{\hat\Gamma_{h,*}^m}X_{h,*}^m\cdot\nabla_{\hat\Gamma_{h,*}^m}I_h[(I-\bar n_{h,*}^m(\bar n_{h,*}^m)^\top)\phi_h]\\
&\quad=-d^m(\phi_h),
\end{aligned}\tag{4.10}$$

where the first two terms on the left-hand side can be written as

$$\begin{aligned}
&\int_{\Gamma_h^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot \bar n_h^m\,\bar n_h^m\cdot\phi_h - \int_{\hat\Gamma_{h,*}^m}^h \frac{X_{h,*}^{m+1}-\hat X_{h,*}^m}{\tau}\cdot \bar n_{h,*}^m\,\bar n_{h,*}^m\cdot\phi_h\\
&\quad=:\int_{\hat\Gamma_{h,*}^m}^h \frac{e_h^{m+1}-\hat e_h^m}{\tau}\cdot \bar n_{h,*}^m\,\bar n_{h,*}^m\cdot\phi_h + J^m(\phi_h),
\end{aligned}\tag{4.11}$$

with

$$J^m(\phi_h)=\int_{\Gamma_h^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot \bar n_h^m\,\bar n_h^m\cdot\phi_h - \int_{\hat\Gamma_{h,*}^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot \bar n_{h,*}^m\,\bar n_{h,*}^m\cdot\phi_h. \tag{4.12}$$

In [2, Section 5.2] we see that the third and fourth terms on the left-hand side of (4.10) can be rewritten into a H^1 bilinear form plus lower-order terms by using the following notations for any two $\mathbb{R}^2$-valued functions u and v on Γ:

$$\begin{aligned}
A_\Gamma(u,v) &:= \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma v,\\
A_\Gamma^N(u,v) &:= \int_\Gamma [(\nabla_\Gamma u)n]\cdot[(\nabla_\Gamma v)n],\\
A_\Gamma^T(u,v) &:= \int_\Gamma \operatorname{tr}\left[(\nabla_\Gamma u)(I-nn^\top)(\nabla_\Gamma v)^T\right],\\
B_\Gamma(u,v) &:= \int_\Gamma (\nabla_\Gamma\cdot u)(\nabla_\Gamma\cdot v)-\operatorname{tr}(\nabla_\Gamma u\nabla_\Gamma v),
\end{aligned}\tag{4.13}$$

with $A_\Gamma(u,v)=A_\Gamma^N(u,v)+A_\Gamma^T(u,v)$. These bilinear forms can also be defined on the approximate curves $\hat\Gamma_{h,*}^m$, Γ_h^m and $\hat\Gamma_{h,\theta}^m$. The following identity was shown in [2, Eq. (5.8)]:

$$\int_\Gamma \nabla_\Gamma \mathrm{id}\cdot(D_\Gamma u)\nabla_\Gamma v=-A_\Gamma^T(u,v)+B_\Gamma(u,v), \tag{4.14}$$

which also holds for the approximate curves $\hat\Gamma_{h,*}^m$, Γ_h^m and $\hat\Gamma_{h,\theta}^m$. It is shown in [1, Eq. (2.1)] that (with integration by parts), if the underlying curve is sufficiently smooth, then the symmetric bilinear form $B_\Gamma(u,v)$ can be written as

$$\begin{aligned} B_\Gamma(u,v) = &\int_\Gamma u_j \underline{D}_i v_i H n_j - \int_\Gamma u_j \underline{D}_j v_i H n_i \\ &+ \int_\Gamma u_j \underline{D}_k v_i n_i H_{jk} - \int_\Gamma u_j \underline{D}_k v_i H_{ik} n_j \quad \forall\, u,v \in H^1(\Gamma). \end{aligned} \tag{4.15}$$

We define $\hat{X}^m_{h,\theta} := (1-\theta)\hat{X}^m_{h,*} + \theta X^m_h$ and $X^{m+1}_{h,\theta} := (1-\theta)X^{m+1}_{h,*} + \theta X^{m+1}_h$ in the sense of nodal vectors. Then the third and fourth terms on the left-hand side of (4.10) can be decomposed as follows (as shown in [2, Eq. (5.10)])

$$\begin{aligned} &\int_{\Gamma^m_h} \nabla_{\Gamma^m_h} X^{m+1}_h \cdot \nabla_{\Gamma^m_h} \phi_h - \int_{\hat{\Gamma}^m_{h,*}} \nabla_{\hat{\Gamma}^m_{h,*}} X^{m+1}_{h,*} \cdot \nabla_{\hat{\Gamma}^m_{h,*}} \phi_h \\ &= A^N_{h,*}(e^{m+1}_h, \phi_h) + A^T_{h,*}(e^{m+1}_h - \hat{e}^m_h, \phi_h) + B^m(\hat{e}^m_h, \phi_h) + K^m(\phi_h), \end{aligned} \tag{4.16}$$

where we have used the following notations for simplicity:

(4.17)

$$A^N_{h,*}(u_h, v_h) := A^N_{\hat{\Gamma}^m_{h,*}}(u_h, v_h) \quad \text{and} \quad A^T_{h,*}(u_h, v_h) := A^T_{\hat{\Gamma}^m_{h,*}}(u_h, v_h),$$

(4.18)

$$A_{h,*}(u_h, v_h) := A^N_{h,*}(u_h, v_h) + A^T_{h,*}(u_h, v_h) \quad \text{and} \quad B^m(u_h, v_h) = B_{\Gamma^m}(u^l_h, v^l_h)$$

$$\begin{aligned} K^m(\phi_h) = &\int_0^1 \big[A^N_{\hat{\Gamma}^m_{h,\theta}}(e^{m+1}_h, \phi_h) - A^N_{\hat{\Gamma}^m_{h,*}}(e^{m+1}_h, \phi_h)\big] \mathrm{d}\theta \\ &+ \int_0^1 \big[A^T_{\hat{\Gamma}^m_{h,\theta}}(e^{m+1}_h - \hat{e}^m_h, \phi_h) - A^T_{\hat{\Gamma}^m_{h,*}}(e^{m+1}_h - \hat{e}^m_h, \phi_h)\big] \mathrm{d}\theta \\ &+ \int_0^1 \big[B_{\hat{\Gamma}^m_{h,\theta}}(\hat{e}^m_h, \phi_h) - B_{\hat{\Gamma}^m_{h,*}}(\hat{e}^m_h, \phi_h)\big] \mathrm{d}\theta \\ &+ B_{\hat{\Gamma}^m_{h,*}}(\hat{e}^m_h, \phi_h) - B_{\Gamma^m}(\hat{e}^m_h, \phi_h) \\ &+ \int_0^1 \int_{\hat{\Gamma}^m_{h,\theta}} \nabla_{\hat{\Gamma}^m_{h,\theta}} (X^{m+1}_{h,\theta} - \hat{X}^m_{h,\theta}) \cdot D_{\hat{\Gamma}^m_{h,\theta}} \hat{e}^m_h \nabla_{\hat{\Gamma}^m_{h,\theta}} \phi_h \mathrm{d}\theta. \end{aligned} \tag{4.19}$$

The last two terms on the left-hand side of (4.10), which arise from the stabilization introduced in this article, can be decomposed into the following several parts:

$$
\begin{aligned}
&- \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^m \cdot \nabla_{\Gamma_h^m} I_h[(I - \bar n_h^m (\bar n_h^m)^\top)\phi_h] \\
&+ \int_{\hat\Gamma_{h,*}^m} \nabla_{\hat\Gamma_{h,*}^m} X_{h,*}^m \cdot \nabla_{\hat\Gamma_{h,*}^m} I_h[(I - \bar n_{h,*}^m (\bar n_{h,*}^m)^\top)\phi_h] \\
= &- \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^m \cdot \nabla_{\Gamma_h^m} I_h[(I - \bar n_h^m (\bar n_h^m)^\top - \bar T_h^m)\phi_h] \\
&+ \int_{\hat\Gamma_{h,*}^m} \nabla_{\hat\Gamma_{h,*}^m} X_{h,*}^m \cdot \nabla_{\hat\Gamma_{h,*}^m} I_h[(I - \bar n_{h,*}^m (\bar n_{h,*}^m)^\top - \bar T_{h,*}^m)\phi_h] \\
&- \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^m \cdot \nabla_{\Gamma_h^m} I_h[(\bar T_h^m - \bar T_{h,*}^m)\phi_h] \\
&- \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^m \cdot \nabla_{\Gamma_h^m} I_h(\bar T_{h,*}^m\phi_h) + \int_{\hat\Gamma_{h,*}^m} \nabla_{\hat\Gamma_{h,*}^m} X_{h,*}^m \cdot \nabla_{\hat\Gamma_{h,*}^m} I_h(\bar T_{h,*}^m\phi_h) \\
=: &\, F_1^m(\phi_h) + F_2^m(\phi_h) + F_3^m(\phi_h) \\
&- A_{h,*}^N(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h) - B^m(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h) - Q^m(I_h\bar T_{h,*}^m\phi_h), \qquad (4.20)
\end{aligned}
$$

where $\bar T_h^m = I - \bar n_h^m (\bar n_h^m)^\top / |\bar n_h^m|^2$ and $\bar T_{h,*}^m = I - \bar n_{h,*}^m (\bar n_{h,*}^m)^\top / |\bar n_{h,*}^m|^2$, and the last three terms are obtained from the following relation (cf. [2, Eq. (5.10)]):

$$
\begin{aligned}
&\int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^m \cdot \nabla_{\Gamma_h^m} I_h\bar T_{h,*}^m\phi_h - \int_{\hat\Gamma_{h,*}^m} \nabla_{\hat\Gamma_{h,*}^m} X_{h,*}^m \cdot \nabla_{\hat\Gamma_{h,*}^m} I_h\bar T_{h,*}^m\phi_h \\
&= \int_0^1 \int_{\hat\Gamma_{h,\theta}^m} \nabla_{\hat\Gamma_{h,\theta}^m} \hat e_h^m \cdot \nabla_{\hat\Gamma_{h,\theta}^m} I_h\bar T_{h,*}^m\phi_h \mathrm{d}\theta \\
&\quad + \int_0^1 \int_{\hat\Gamma_{h,\theta}^m} \nabla_{\hat\Gamma_{h,\theta}^m} \hat X_{h,\theta}^m \cdot D_{\hat\Gamma_{h,\theta}^m} \hat e_h^m \nabla_{\hat\Gamma_{h,\theta}^m} I_h\bar T_{h,*}^m\phi_h \mathrm{d}\theta \\
&\qquad\qquad \text{(Lemma 3.15 is used)} \\
&= A_{h,*}(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h) - A_{h,*}^T(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h) + B^m(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h) + Q^m(I_h\bar T_{h,*}^m\phi_h) \\
&\qquad\qquad \text{(relation (4.14) and notations (4.17)–(4.18) are used)} \\
&= A_{h,*}^N(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h) + B^m(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h) + Q^m(I_h\bar T_{h,*}^m\phi_h),
\end{aligned}
$$

with

$$
\begin{aligned}
Q^m(\phi_h) := &\int_0^1 \big[A_{\hat\Gamma_{h,\theta}^m}^N(\hat e_h^m, \phi_h) - A_{\hat\Gamma_{h,*}^m}^N(\hat e_h^m, \phi_h)\big]\mathrm{d}\theta \\
&+ \int_0^1 \big[B_{\hat\Gamma_{h,\theta}^m}(\hat e_h^m, \phi_h) - B_{\hat\Gamma_{h,*}^m}(\hat e_h^m, \phi_h)\big]\mathrm{d}\theta \\
&+ B_{\hat\Gamma_{h,*}^m}(\hat e_h^m, \phi_h) - B_{\Gamma^m}(\hat e_h^m, \phi_h).
\end{aligned}
$$

In summary, by substituting (4.11), (4.16) and (4.20) into (4.10), we can rewrite the error equation into the following form:

$$\begin{aligned}\int_{\hat\Gamma^m_{h,*}}^h & \frac{e_h^{m+1}-\hat e_h^m}{\tau}\cdot \bar n^m_{h,*}\,\phi_h\cdot \bar n^m_{h,*} + J^m(\phi_h)\\ &+ A^N_{h,*}(e_h^{m+1},\phi_h) + A^T_{h,*}(e_h^{m+1}-\hat e_h^m,\phi_h) + B^m(\hat e_h^m,\phi_h) + K^m(\phi_h)\\ &+ \sum_{i=1}^3 F_i^m(\phi_h) - A^N_{h,*}(\hat e_h^m, I_h\bar T^m_{h,*}\phi_h) - B^m(\hat e_h^m, I_h\bar T^m_{h,*}\phi_h) - Q^m(I_h\bar T^m_{h,*}\phi_h)\\ &= -d^m(\phi_h). \end{aligned} \tag{4.21}$$

By choosing $\phi_h = e_h^{m+1}$ in the error equation we can obtain the following inequality (which is proved in [2, Eq. (5.15)]):

$$A^N_{h,*}(e_h^{m+1},e_h^{m+1}) + A^T_{h,*}(e_h^{m+1}-\hat e_h^m, e_h^{m+1}) \ge \frac12 A_{h,*}(e_h^{m+1},e_h^{m+1}) - \frac12 A^T_{h,*}(\hat e_h^m,\hat e_h^m). \tag{4.22}$$

The full H^1 parabolicity stems from the property that $\frac12 A^T_{h,*}(\hat e_h^m,\hat e_h^m)$ is much smaller than $\frac12 A_{h,*}(e_h^{m+1},e_h^{m+1})$ due to the orthogonality between $\hat e_h^m$ and the tangent plane of Γ^m at the nodes. This means on the left-hand side of the error equation (4.22) we have a very good H^1 positive definite term. In particular, the following estimates were shown in [2, Eqs. (5.16), (5.17), (5.22)]:

$$|A^T_{h,*}(\hat e_h^m,\hat e_h^m)| \lesssim \epsilon^{-1}\|\hat e_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})} + \epsilon\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}, \tag{4.23}$$

$$|B^m(\hat e_h^m,e_h^{m+1})| \lesssim \epsilon^{-1}\|\hat e_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})} + \epsilon\|\nabla_{\hat\Gamma^m_{h,*}} e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})}, \tag{4.24}$$

$$\begin{aligned}|K^m(\phi_h)| \lesssim\ & \|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^\infty(\hat\Gamma^m_{h,*})}\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^2(\hat\Gamma^m_{h,*})}\|\nabla_{\hat\Gamma^m_{h,*}}\phi_h\|_{L^2(\hat\Gamma^m_{h,*})}\\ &+ \|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^\infty(\hat\Gamma^m_{h,*})}\|\nabla_{\hat\Gamma^m_{h,*}} e_h^{m+1}\|_{L^2(\hat\Gamma^m_{h,*})}\|\nabla_{\hat\Gamma^m_{h,*}}\phi_h\|_{L^2(\hat\Gamma^m_{h,*})}\\ &+ (\tau+h^k)\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^2(\hat\Gamma^m_{h,*})}\|\nabla_{\hat\Gamma^m_{h,*}}\phi_h\|_{L^2(\hat\Gamma^m_{h,*})}. \end{aligned} \tag{4.25}$$

Remark 4.3. The factor $(\tau + h^k)$ in the last term of (4.25) is better than the the factor $(\tau + h^{k-1})$ in [2, Eq. (5.22)] because we can use a better geometric perturbation estimate (i.e., Lemma 3.2 with L^2 norms on both f_1 and f_2) than that in [2, Lemma 4.2] (with L^∞ norm on f_1 and L^2 norm on f_2). The reason that we have a better geometric perturbation estimate in Lemma 3.2 to use in this article is that we allow the generic constant C to depend on the $W_h^{k,\infty}$ norm of the map $\hat X^m_{h,*} : \Gamma^0_{h,\mathrm{f}} \to \hat\Gamma^m_{h,*}$ defined in (3.1), while [2] only allows the generic constant C to depend on the H_h^k norm of this map.

Moreover, from the expression of $B^m(\cdot,\cdot)$ in (4.15) and the geometric perturbation estimates, we can obtain the following estimates similarly as [2, inequality

(5.20)]:

$$|B^m(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h)| \lesssim \|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|I_h\bar T_{h,*}^m\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}$$

$$\lesssim \epsilon^{-1}\|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 + \epsilon\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2, \tag{4.26}$$

$$|Q^m(I_h\bar T_{h,*}^m\phi_h)| \lesssim (\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^\infty(\hat\Gamma_{h,*}^m)}+h^k)\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|\nabla_{\hat\Gamma_{h,*}^m}\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}$$

$$\lesssim \epsilon^{-1}\|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}^2 + \epsilon\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2, \tag{4.27}$$

where the last inequality uses (3.10) and the inverse inequality to remove the derivative from ϕ_h. The estimation of $J^m(\phi_h)$, $F_i^m(\phi_h)$ and $A_{h,*}^N(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h)$ in (4.21) is presented in the next subsection.

Remark 4.4. By choosing $\phi_h = e_h^{m+1}$ in the error equation and a sufficiently small ϵ, the terms $\epsilon\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2$ arising from (4.23)–(4.27) can be absorbed by the first term on the right-hand side of (4.22). This benefits from the recovery of full H^1 parabolicity in (4.22)–(4.23).

4.3. **Estimates for $J^m(\phi_h)$, $F_i^m(\phi_h)$ and $A_{h,*}^N(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h)$.** Let $\bar n_{h,\theta}^m$ be the averaged normal vector on curve $\hat\Gamma_{h,\theta}^m = (1-\theta)\hat\Gamma_{h,*}^m + \theta\Gamma_h^m$, with $\theta\in[0,1]$, defined in the same way as (3.15) in terms of the piecewise normal vector $\hat n_{h,\theta}^m$ on $\hat\Gamma_{h,\theta}^m$. Thus $\bar n_{h,\theta}^m$ is not necessarily of unit length. The curve $\hat\Gamma_{h,\theta}^m$ moves with velocity $\hat e_h^m$ as θ increases, and any finite element function v_h with a fixed nodal vector independent of $\theta\in[0,1]$ has the transport property $\partial_\theta^\bullet v_h = 0$ on $\hat\Gamma_{h,\theta}^m$. The functional $J^m(\phi_h)$ defined in (4.12) can be rewritten into the following form using the fundamental theorem of calculus:

$$\begin{aligned}
J^m(\phi_h) &= \int_{\hat\Gamma_{h,\theta}^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot\bar n_{h,\theta}^m\,\phi_h\cdot\bar n_{h,\theta}^m\Big|_{\theta=0}^{\theta=1} \\
&= \int_0^1\frac{\mathrm d}{\mathrm d\theta}\int_{\hat\Gamma_{h,\theta}^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot\bar n_{h,\theta}^m\,\phi_h\cdot\bar n_{h,\theta}^m\mathrm d\theta \\
&= \int_0^1\int_{\hat\Gamma_{h,\theta}^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot\partial_\theta^\bullet\bar n_{h,\theta}^m\,\phi_h\cdot\bar n_{h,\theta}^m\mathrm d\theta \\
&\quad+\int_0^1\int_{\hat\Gamma_{h,\theta}^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot\bar n_{h,\theta}^m\,\phi_h\cdot\partial_\theta^\bullet\bar n_{h,\theta}^m\mathrm d\theta \\
&\quad+\int_0^1\int_{\hat\Gamma_{h,\theta}^m}^h \frac{X_h^{m+1}-X_h^m}{\tau}\cdot\bar n_{h,\theta}^m\,\phi_h\cdot\bar n_{h,\theta}^m(\nabla_{\hat\Gamma_{h,\theta}^m}\cdot\hat e_h^m)\mathrm d\theta \quad \text{(Lemma 3.14, item 6).}
\end{aligned}\tag{4.28}$$

From Lemma 3.14, item 7, we know that

$$\partial_\theta^\bullet\hat n_{h,\theta}^m = -\nabla_{\Gamma_{h,\theta}^m}\hat e_h^m\cdot\hat n_{h,\theta}^m \quad \text{(piecewisely defined on each element).} \tag{4.29}$$

The relation between $\bar n^m_{h,\theta}(p)$ and $\hat n^m_{h,\theta}(p)$ at a node p, as shown in (3.17), implies the following results:

$$|\partial^\bullet_\theta \bar n^m_{h,\theta}(p)| = |\nabla_{\Gamma^m_{h,\theta}} \hat e^m_h(p)\cdot n^m_{h,\theta}(p)| \qquad \text{if } p \text{ is an interior node of an element,} \tag{4.30}$$

$$\begin{aligned}|\partial^\bullet_\theta \bar n^m_{h,\theta}(p)| &\lesssim |\nabla_{\Gamma^m_{h,\theta}} \hat e^m_h(p+)\cdot n^m_{h,\theta}(p+)| + |\nabla_{\Gamma^m_{h,\theta}} \hat e^m_h(p+)| \\ &\quad + |\nabla_{\Gamma^m_{h,\theta}} \hat e^m_h(p-)\cdot n^m_{h,\theta}(p-)| + |\nabla_{\Gamma^m_{h,\theta}} \hat e^m_h(p-)| \\ &\qquad \text{if } p = K_1\cap K_2 \text{ for two elements } K_1 \text{ and } K_2,\end{aligned} \tag{4.31}$$

where the first and third terms on the right-hand side of (4.31) are generated from taking material derivative of $\hat n^m_{h,\theta}|_{K_1}(p)$ and $\hat n^m_{h,\theta}|_{K_2}(p)$, respectively, while the second and the fourth terms arise from taking material derivative of the weights $w_K(p) = \nabla_{K^0_{\rm f}} F_K\circ F_K^{-1}(p)$ for $K = K_1$ and $K = K_2$, respectively. Hence, by using Hölder's inequality, we obtain the following estimate:

$$\begin{aligned}|J^m(\phi_h)| &\lesssim \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})}\|\phi_h\|_{L^2(\hat\Gamma^m_{h,*})} \\ &\lesssim \epsilon\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2(\hat\Gamma^m_{h,*})} + \epsilon^{-1}\|\phi_h\|^2_{L^2(\hat\Gamma^m_{h,*})},\end{aligned} \tag{4.32}$$

where ϵ is an arbitrary small number arising from Young's inequality.

By using the expression of $\bar n^m_h$ in (1.7), we can estimate the amplitude of $\bar n^m_h$ at the nodes similarly as (3.18), i.e.,

$$\begin{aligned}&|\bar n^m_h(p)| = 1 && \text{if } p \text{ is an interior node of an element,}\\ &|\bar n^m_h(p)|\le 1, \quad \big||\bar n^m_h(p)|-1\big| \lesssim |n^m_h(p+)-n^m_h(p-)|^2 && \\ & && \text{if } p \text{ is an endpoint of an element.}\end{aligned} \tag{4.33}$$

This implies, in view of the norm equivalence relation in (3.13),

$$\begin{aligned}\||\bar n^m_h|-1\|_{L^1_h(\hat\Gamma^m_{h,*})} &\lesssim \sum_{p\in\mathcal N_b(\hat\Gamma^m_{h,*})} h|n^m_h(p+)-n^m_h(p-)|^2 \\ &\lesssim \sum_{p\in\mathcal N_b(\hat\Gamma^m_{h,*})} h(|n^m_h(p+)-I_hn^m_*(p)|^2 + |n^m_h(p-)-I_hn^m_*(p)|^2) \\ &\lesssim \|n^m_h - I_hn^m_*\|^2_{L^2_h(\hat\Gamma^m_{h,*})} \\ &\lesssim \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2_h(\hat\Gamma^m_{h,*})} + h^{2k},\end{aligned}$$

where (3.7), (3.19) and the triangle inequality are used in deriving the last inequality. By using this result and the inverse inequality, we obtain the following result for the $F^m_1(\phi_h)$ defined in (4.20):

$$\begin{aligned}|F^m_1(\phi_h)| &= \Big|\int_{\Gamma^m_h} \nabla_{\Gamma^m_h} X^m_h\cdot\nabla_{\Gamma^m_h} I_h[(I-\bar n^m_h(\bar n^m_h)^\top - \bar T^m_h)\phi_h]\Big| \\ &\lesssim h^{-1}\||\bar n^m_h|-1\|_{L^1_h(\hat\Gamma^m_{h,*})}\|\phi_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \qquad \text{(inverse inequality)} \\ &\lesssim h^{-1}(\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2_h(\hat\Gamma^m_{h,*})} + h^{2k})(\epsilon^{-1}\|\phi_h\|_{L^2(\hat\Gamma^m_{h,*})} + \epsilon\|\nabla_{\hat\Gamma^m_{h,*}}\phi_h\|_{L^2(\hat\Gamma^m_{h,*})}).\end{aligned} \tag{4.34}$$

Similarly, using the estimate in (3.18), we have

$$
\begin{aligned}
|F_2^m(\phi_h)| &= \Big| \int_{\hat\Gamma_{h,*}^m} \nabla_{\hat\Gamma_{h,*}^m} X_{h,*}^m \cdot \nabla_{\hat\Gamma_{h,*}^m} I_h(1-\bar n_{h,*}^m(\bar n_{h,*}^m)^\top - \bar T_{h,*}^m)\phi_h \Big| \\
&\lesssim h^{2k-1}(\epsilon^{-1}\|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)} + \epsilon\|\nabla_{\hat\Gamma_{h,*}^m}\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}). \qquad (4.35)
\end{aligned}
$$

The term $|F_3^m(\phi_h)|$ can be estimated by using integration by parts similarly as $d_3^m(\phi_h)$, i.e.,

$$
\begin{aligned}
F_3^m(\phi_h) &= \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} X_h^m \cdot \nabla_{\Gamma_h^m} I_h[(\bar T_h^m - \bar T_{h,*}^m)\phi_h] \\
&= -\int_{\Gamma_h^m} \Delta_{\Gamma_h^m} X_h^m \cdot I_h[(\bar T_h^m - \bar T_{h,*}^m)\phi_h] \\
&\quad + \sum_{p\in\mathcal N_b(\Gamma_h^m)} \Big(\mu_h^m(p+)\cdot(\nabla_{\Gamma_h^m}X_h^m)(p+) + \mu_h^m(p-)\cdot(\nabla_{\Gamma_h^m}X_h^m)(p-)\Big)(\bar T_h^m(p) - \bar T_{h,*}^m(p))\phi_h(p) \\
&= \int_{\Gamma_h^m} H_h^m n_h^m \cdot I_h[(\bar T_h^m - \bar T_{h,*}^m)\phi_h] + \sum_{p\in\mathcal N_b(\Gamma_h^m)} (\mu_h^m(p+) + \mu_h^m(p-))\cdot(\bar T_h^m(p) - \bar T_{h,*}^m(p))\phi_h(p),
\end{aligned}
$$

where the first term on the right-hand side can be estimated by using the equivalence between the discrete and continuous norms, i.e.,

$$
\begin{aligned}
&\Big| \int_{\Gamma_h^m} H_h^m n_h^m \cdot I_h[(\bar T_h^m - \bar T_{h,*}^m)\phi_h] \Big| \\
&\quad \lesssim \|I_h[(\bar T_h^m - \bar T_{h,*}^m)\phi_h]\|_{L^1(\hat\Gamma_{h,*}^m)} \\
&\quad \lesssim \|(\bar T_h^m - \bar T_{h,*}^m)\phi_h\|_{L_h^1(\hat\Gamma_{h,*}^m)} \\
&\quad \lesssim \|(\bar T_h^m - \bar T_{h,*}^m)\phi_h\|_{L^1(\hat\Gamma_{h,*}^m)} \\
&\quad \lesssim \|\bar T_h^m - \bar T_{h,*}^m\|_{L^2\hat\Gamma_{h,*}^m)}\|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)} \\
&\quad \lesssim \|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)} \quad \text{(here (3.20) is used).}
\end{aligned}
$$

The second term in the expression of $F_3^m(\phi_h)$ can be estimated by using (3.25) with $\varphi_h = I_h(\bar T_h^m - \bar T_{h,*}^m)$ and (3.20). This leads to the following estimate:

$$
\begin{aligned}
&\sum_{p\in\mathcal N_b(\Gamma_h^m)} (\mu_h^m(p+) + \mu_h^m(p-))\cdot(\bar T_h^m(p) - \bar T_{h,*}^m(p))\phi_h(p) \\
&\lesssim h^{-1}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\Gamma_h^m)}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|\phi_h\|_{L^\infty(\hat\Gamma_{h,*}^m)} \\
&\lesssim h^{-\frac32}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\Gamma_h^m)}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}.
\end{aligned}
$$

Therefore,

$$
|F_3^m(\phi_h)| \lesssim (1 + h^{-\frac32}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\Gamma_h^m)})\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}. \qquad (4.36)
$$

Analogous to [2, Eqs. (5.41), (5.49)], the following estimate can be established by using the nodal orthogonality relation (the details are omitted):

$$
\begin{aligned}
|A_{h,*}^N(\hat e_h^m, I_h\bar T_{h,*}^m\phi_h)| &\le |A_{h,*}^N(\hat e_h^m, I_h T_*^m\phi_h)| + |A_{h,*}^N(\hat e_h^m, I_h(\bar T_{h,*}^m - T_*^m)\phi_h)| \\
&\lesssim \|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|I_h\bar T_{h,*}^m\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)} \\
&\quad + h^{k-1}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}.
\end{aligned}
$$

4.4. **Stability of the tangential motion.** For an arbitrarily prescribed smooth velocity field u on Γ which is not necessarily tangential, we consider the velocity v whose tangential motion is specified by the following elliptic system on Γ

$$v \cdot n = u \cdot n, \tag{4.37a}$$

$$-\Delta_\Gamma v = \kappa n. \tag{4.37b}$$

This system can be reformulated as the Euler-Lagrange equation of the energy functional $\int_\Gamma |\nabla_\Gamma v|^2$ under the pointwise constraint $v \cdot n = u \cdot n$. Formally the elliptic system (4.37) with $u = -Hn$ is underlying the PDE to which the BGN method converges; see [31, Section 1]. Since $\int_\Gamma |\nabla_\Gamma \cdot|^2$ indicates the infinitesimal distortion of the mesh and v is the minimizer of this functional, this correspondence explains why the tangential velocity endowed by the BGN method helps to improve the mesh quality.

Lemma 4.5. *If the underlying closed surface Γ is smooth, then the elliptic velocity system (4.37) has a unique solution $(v, \kappa) \in H^1(\Gamma) \times H^{-1}(\Gamma)$ with $v \cdot n = u \cdot n$ almost everywhere, and moreover this solution (v, κ) is smooth.*

Proof. We first consider the following energy functional $I : H^1(\Gamma) \to \mathbb{R}$

$$I(v) = \int_\Gamma |\nabla_\Gamma v|^2 \tag{4.38a}$$

with v in the convex admissible set

$$\mathcal{H} = \{v \in H^1(\Gamma) : v \cdot n = u \cdot n \quad \text{a.e.}\}. \tag{4.38b}$$

We know $\mathcal{H} \neq \emptyset$ because u is in $\mathcal{H}$. Then we define $I_0 = \inf_{v \in \mathcal{H}} I(v) \geq 0$ and pick out a minimizing sequence $v_i \in \mathcal{H}$ such that $I(v_i) \to I_0$. From the vectorial Poincaré inequality, it follows that $\{v_i\}$ is bounded in $H^1(\Gamma)$. Therefore, by the compactness, we can extract a subsequence, also denoted by $\{v_i\}$ for simplicity, such that $v_i \to v$ in $L^2(\Gamma)$, $v_i \rightharpoonup v$ in $H^1(\Gamma)$ and $v \cdot n = u \cdot n$ a.e.. According to the weak lower semi-continuity of the norm, it holds that $I(v) \leq \inf_i I(v_i) = I_0$, which means the infimum of energy I_0 can be indeed reached at v.

For the uniqueness of the minimizer, if $I(v_1) = I(v_2) = I_0$ for some $v_1 \neq v_2$, by the strict convexity of I, we have $I(\frac{v_1+v_2}{2}) < \frac{1}{2}I(v_1) + \frac{1}{2}I(v_2) = I_0$ contradicting the minimality.

To obtain the Euler-Lagrange equation for the variational problem (4.38), we take the variation $v + \epsilon\varphi$ with $\varphi \in C^\infty(\Gamma; T\Gamma)$ being any smooth vector field on Γ. Since φ is tangential, $v + \epsilon\varphi \in \mathcal{H}$ is also admissible. Using the minimality of v, we derive

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0} I(v + \epsilon\varphi) \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0} \int_\Gamma |\nabla_\Gamma (v + \epsilon\varphi)|^2 \\ &= 2\int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \varphi. \end{aligned}$$

Then we differentiate the constraint $v \cdot n = u \cdot n$ twice and get the following distributional identity

$$\Delta_\Gamma v \cdot n = \Delta_\Gamma (u \cdot n) - 2\nabla_\Gamma v \cdot \nabla_\Gamma n - v \cdot \Delta_\Gamma n \in L^2(\Gamma).$$

If we define $\kappa := -\Delta_\Gamma v \cdot n \in L^2(\Gamma)$ and denote by $P := I - nn^\top$ the pointwise orthogonal tangential projection, then it follows that for any $\varphi \in H^1(\Gamma)$

$$\begin{aligned}(-\Delta_\Gamma v, \varphi) &= \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \varphi \\ &= \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma (P\varphi + n(n\cdot\varphi)) \\ &= \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma (n(n\cdot\varphi)) \\ &= (-\Delta_\Gamma v \cdot n, n\cdot\varphi) \\ &= (\kappa n, \varphi).\end{aligned}$$

Thus $-\Delta_\Gamma v = \kappa n \in L^2(\Gamma)$ holds in the sense of distribution. By the elliptic regularity theory of the Laplace-Beltrami operator Δ_Γ, we know $v \in H^2(\Gamma)$ and hence $\kappa = -\Delta_\Gamma v \cdot n = -\Delta_\Gamma (u \cdot n) + 2\nabla_\Gamma v \cdot \nabla_\Gamma n + v \cdot \Delta_\Gamma n \in H^1(\Gamma)$. Therefore, by applying this procedure recursively, we conclude that (v, κ) is smooth.

To complete the proof, it remains to show that the PDE system (4.37) has a unique solution. If $(\tilde v, \tilde\kappa) \in H^1(\Gamma) \times H^{-1}(\Gamma)$ with $\tilde v \neq v$ is another solution of (4.37), then by testing arbitrary smooth vector field we know $\tilde v$ is the local minimizer of (4.38). By the convexity of I and the fact that v is the unique minimality of I, we have

$$I((1-\theta)\tilde v + \theta v) < (1-\theta) I(\tilde v) + \theta I(v) < I(\tilde v)$$

for all $\theta \in (0, 1]$, which contradicts the local minimality of $\tilde v$ when θ is sufficiently small. So we have $\tilde v = v$ and $\tilde\kappa = -\Delta_\Gamma \tilde v \cdot n = -\Delta_\Gamma v \cdot n = \kappa$, and the proof is complete. □

Applying the above lemma with $u = -Hn$, let (v, κ) be the unique smooth solution to the following elliptic system on the smooth curve $\Gamma = \Gamma(t)$:

$$v \cdot n = -H, \tag{4.39a}$$

$$-\Delta_\Gamma v = \kappa n. \tag{4.39b}$$

In this subsection we present the stability estimates for the tangential velocity produced by the stabilized BGN method by comparing the velocity $v_h^m := (X_h^{m+1} - X_h^m)/\tau$ of the numerical solution with the velocity $v^m = v(t_m)$ determined by the elliptic system (4.39). The estimates of the function $w_h^m := v_h^m - I_h v^m \in S_h(\Gamma_h^m)$ in this subsection essentially characterize the limit of the tangential motion produced by the stabilized BGN method.

Since (4.39a) implies that $v^m = -H^m n^m + T^m v^m$, where $T^m = I - n^m (n^m)^\top$ is the tangential projection matrix on Γ^m, the following relation follows from (3.38) and the nodal relation $T^m = T_*^m$:

$$\begin{aligned}X_h^{m+1} - X_h^m - \tau I_h v^m &= X_h^{m+1} - X_h^m - \tau I_h(-H^m n^m) - \tau I_h T_*^m v^m \\ &= e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v^m + \tau I_h g^m \quad \text{on } \hat\Gamma_{h,*}^m.\end{aligned} \tag{4.40}$$

The following relation can be obtained by subtracting integral $\tau\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}I_hv^m\cdot\nabla_{\Gamma_h^m}\phi_h$ from the both sides of the numerical scheme in (1.5):

$$\begin{aligned}
&\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}(X_h^{m+1}-X_h^m-\tau I_hv^m)\cdot\nabla_{\Gamma_h^m}\phi_h\\
&=-\int_{\Gamma_h^m}^h\frac{X_h^{m+1}-\mathrm{id}}{\tau}\cdot\bar n_h^m\,\phi_h\cdot\bar n_h^m-\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}X_h^m\cdot\nabla_{\Gamma_h^m}I_h[(\phi_h\cdot\bar n_h^m)\bar n_h^m]\\
&\quad-\tau\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}I_hv^m\cdot\nabla_{\Gamma_h^m}\phi_h\\
&=-\int_{\Gamma_h^m}^h\frac{X_h^{m+1}-\mathrm{id}}{\tau}\cdot\bar n_h^m\,\phi_h\cdot\bar n_h^m\\
&\quad-\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}X_h^m\cdot\nabla_{\Gamma_h^m}I_h[(\phi_h\cdot\bar n_h^m)\bar n_h^m]\\
&\quad-\tau\int_{\Gamma^m}\nabla_{\Gamma^m}v^m\cdot\nabla_{\Gamma^m}\phi_h^l\\
&\quad+\tau\int_{\Gamma^m}\nabla_{\Gamma^m}(v^m-(I_hv^m)^l)\cdot\nabla_{\Gamma^m}\phi_h^l\\
&\quad-\tau\int_{\hat\Gamma_{h,*}^m}\nabla_{\hat\Gamma_{h,*}^m}I_hv^m\cdot\nabla_{\hat\Gamma_{h,*}^m}\phi_h+\tau\int_{\Gamma^m}\nabla_{\Gamma^m}(I_hv^m)^l\cdot\nabla_{\Gamma^m}\phi_h^l\\
&\quad-\tau\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}I_hv^m\cdot\nabla_{\Gamma_h^m}\phi_h+\tau\int_{\hat\Gamma_{h,*}^m}\nabla_{\hat\Gamma_{h,*}^m}I_hv^m\cdot\nabla_{\hat\Gamma_{h,*}^m}\phi_h\\
&=:\sum_{i=1}^6L_i(\phi_h).
\end{aligned}\tag{4.41}$$

For any function $\phi_h\in S_h(\hat\Gamma_{h,*}^m)$, due to the orthogonality between $\bar n_h^m$ and $I_h\bar T_h^m\phi_h$ at the nodes, the two terms $L_1(I_h\bar T_h^m\phi_h)$ and $L_2(I_h\bar T_h^m\phi_h)$ vanish. The three terms $L_4(\phi_h)$, $L_5(\phi_h)$ and $L_6(\phi_h)$ can be estimated by using the superconvergence of Gauss–Lobatto quadrature (Lemma 3.6), the geometric perturbation estimate (Lemma 3.2), and the fundamental theorem of calculus (Lemma 3.15), respectively:

$$|L_4(\phi_h)|\lesssim\tau h^{k+1}\|\phi_h\|_{H^1(\hat\Gamma_{h,*}^m)},\tag{4.42}$$

$$|L_5(\phi_h)|\lesssim\tau h^{k+1}\|\nabla_{\hat\Gamma_{h,*}^m}I_hv^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|\nabla_{\hat\Gamma_{h,*}^m}\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)},\tag{4.43}$$

$$|L_6(\phi_h)|\lesssim\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|\nabla_{\hat\Gamma_{h,*}^m}I_hv^m\|_{L^\infty(\hat\Gamma_{h,*}^m)}\|\nabla_{\hat\Gamma_{h,*}^m}\phi_h\|_{L^2(\hat\Gamma_{h,*}^m)}.\tag{4.44}$$

We can estimate $L_3(I_h\bar{T}_h^m\phi_h)$ with integration by parts and relation (4.39b) as follows, using the identities $I_h\bar{T}_h^m\phi_h = I_h\bar{T}_h^m I_h\bar{T}_h^m\phi_h$:

$$
\begin{aligned}
L_3(I_h\bar{T}_h^m\phi_h) &= \tau\int_{\Gamma^m}\Delta_{\Gamma^m}v^m\cdot(I_h\bar{T}_h^m I_h\bar{T}_h^m\phi_h)^l \qquad \text{(integration by parts)}\\
&= -\tau\int_{\Gamma^m}\kappa^m n^m\cdot(I_h\bar{T}_h^m I_h\bar{T}_h^m\phi_h)^l \qquad \text{(relation (4.39b) is used)}\\
&= \tau\int_{\Gamma^m}(1-I_h^l)\Big(\kappa^m n^m\cdot\big((1-I_h)T_*^m I_h\bar{T}_h^m\phi_h\big)^l\Big)\\
&\quad -\tau\int_{\Gamma^m}\kappa^m n^m\cdot(I_h(\bar{T}_h^m-\bar{T}_{h,*}^m)I_h\bar{T}_h^m\phi_h)^l\\
&\quad -\tau\int_{\Gamma^m}\kappa^m n^m\cdot(I_h(\bar{T}_{h,*}^m-T_*^m)I_h\bar{T}_h^m\phi_h)^l
\end{aligned}\tag{4.45}
$$

where the first term on the right-hand side is obtained by using the following identity:

$$
-\kappa^m n^m\cdot\big(I_h T_*^m I_h\bar{T}_h^m\phi_h\big)^l = (1-I_h^l)\big(\kappa^m n^m\cdot\big((1-I_h)T_*^m I_h\bar{T}_h^m\phi_h\big)^l\big).
$$

We can further decompose $L_3(I_h\bar{T}_h^m\phi_h)$ into the following seven parts:

$$
\begin{aligned}
L_3(I_h\bar{T}_h^m\phi_h) &= \tau\int_{\Gamma^m}(1-I_h^l)\Big(\kappa^m n^m\cdot\big((1-I_h)T_*^m I_h\bar{T}_h^m\phi_h\big)^l\Big)\\
&\quad -\tau\int_{\Gamma^m}\kappa^m n^m\cdot(I_h(\bar{T}_h^m-\bar{T}_{h,*}^m)I_h\bar{T}_h^m\phi_h)^l\\
&\quad -\tau\Big(\int_{\Gamma^m}\kappa^m n^m\cdot(I_h(\bar{T}_{h,*}^m-T_*^m)I_h\bar{T}_h^m\phi_h)^l\\
&\qquad\quad -\int_{\hat{\Gamma}_{h,*}^m}\kappa^{m,-l}n^{m,-l}\cdot I_h(\bar{T}_{h,*}^m-T_*^m)I_h\bar{T}_h^m\phi_h\Big)\\
&\quad -\tau\Big(\int_{\hat{\Gamma}_{h,*}^m}-\int_{\hat{\Gamma}_{h,*}^m}^h\Big)\kappa^{m,-l}n^{m,-l}\cdot I_h(\bar{T}_{h,*}^m-T_*^m)I_h\bar{T}_h^m\phi_h\\
&\quad -\tau\int_{\hat{\Gamma}_{h,*}^m}^h\kappa^{m,-l}n^{m,-l}\cdot(\bar{T}_{h,*}^m-\hat{T}_{h,*}^m)I_h\bar{T}_h^m\phi_h\\
&\quad -\tau\Big(\int_{\hat{\Gamma}_{h,*}^m}^h-\int_{\hat{\Gamma}_{h,*}^m}\Big)\kappa^{m,-l}n^{m,-l}\cdot(\hat{T}_{h,*}^m-T_*^m)I_h\bar{T}_h^m\phi_h\\
&\quad -\tau\int_{\hat{\Gamma}_{h,*}^m}\kappa^{m,-l}n^{m,-l}\cdot(\hat{T}_{h,*}^m-T_*^m)I_h\bar{T}_h^m\phi_h\\
&=: \sum_{i=1}^{7}L_{3i}(\phi_h).
\end{aligned}\tag{4.46}
$$

The super-convergence result of the Gauss–Lobatto quadrature (i.e., Lemma 3.5) can be used to prove the following estimates (the details are omitted):

$$
|L_{31}(\phi_h)|+|L_{34}(\phi_h)|+|L_{36}(\phi_h)| \lesssim \tau h^{k+1}\|I_h\bar{T}_h^m\phi_h\|_{H^1(\hat{\Gamma}_{h,*}^m)}.
$$

$L_{32}(\phi_h)$ can be estimated by using the expressions $\bar{T}_h^m = I - \bar{n}_h^m(\bar{n}_h^m)^\top/|\bar{n}_h^m|^2$ and $\bar{T}_{h,*}^m = I - \bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top/|\bar{n}_{h,*}^m|^2$ and (3.20), which lead to the following result:

$$|L_{32}(\phi_h)| \lesssim \tau \|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \|I_h \bar{T}_h^m \phi_h\|_{L^2(\hat{\Gamma}_{h,*}^m)}.$$

$L_{33}(\phi_h)$ can be estimated by the geometric perturbation estimate in Lemma 3.2 and Lemma 3.8:

$$\begin{aligned} |L_{33}(\phi_h)| &\lesssim \tau h^{k+1} \|I_h(\bar{T}_{h,*}^m - T_*^m)\|_{L^2(\hat{\Gamma}_{h,*}^m)} \|I_h \bar{T}_h^m \phi_h\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\ &\lesssim \tau h^{2k+1} \|I_h \bar{T}_h^m \phi_h\|_{L^2(\hat{\Gamma}_{h,*}^m)}. \end{aligned}$$

We can rewrite $L_{35}(\phi_h)$ as

$$\begin{aligned} L_{35}(\phi_h) &= \tau \int_{\hat{\Gamma}_{h,*}^m}^h \kappa^{m,-l} n^{m,-l} \cdot \Big(\frac{\bar{n}_{h,*}^m(\bar{n}_{h,*}^m)^\top}{|\bar{n}_{h,*}^m|^2} - \hat{n}_{h,*}^m(\hat{n}_{h,*}^m)^\top \Big) I_h \bar{T}_h^m \phi_h \\ &= 2\tau \int_{\hat{\Gamma}_{h,*}^m}^h \kappa^{m,-l} n^{m,-l} \cdot (\bar{n}_{h,*}^m - \hat{n}_{h,*}^m) \frac{1}{|\bar{n}_{h,*}^m|^2} \bar{n}_{h,*}^m \cdot I_h \bar{T}_h^m \phi_h \\ &\quad - \tau \int_{\hat{\Gamma}_{h,*}^m}^h \kappa^{m,-l} n^{m,-l} \cdot (\bar{n}_{h,*}^m - \hat{n}_{h,*}^m) \frac{1}{|\bar{n}_{h,*}^m|^2} (\bar{n}_{h,*}^m - \hat{n}_{h,*}^m) \cdot I_h \bar{T}_h^m \phi_h \\ &\quad + \tau \int_{\hat{\Gamma}_{h,*}^m}^h \kappa^{m,-l} n^{m,-l} \cdot \hat{n}_{h,*}^m \Big(\frac{1}{|\bar{n}_{h,*}^m|^2} - 1 \Big) \hat{n}_{h,*}^m \cdot I_h \bar{T}_h^m \phi_h \\ &= -\tau \int_{\hat{\Gamma}_{h,*}^m}^h \kappa^{m,-l} n^{m,-l} \cdot \frac{1}{|\bar{n}_{h,*}^m|^2} (\bar{n}_{h,*}^m - \hat{n}_{h,*}^m)(\bar{n}_{h,*}^m - \hat{n}_{h,*}^m) \cdot I_h \bar{T}_h^m \phi_h \\ &\quad + \tau \int_{\hat{\Gamma}_{h,*}^m}^h \kappa^{m,-l} n^{m,-l} \cdot \hat{n}_{h,*}^m \Big(\frac{1}{|\bar{n}_{h,*}^m|^2} - 1 \Big) \hat{n}_{h,*}^m \cdot I_h \bar{T}_h^m \phi_h, \end{aligned}$$

where the last equality follows from (3.15). Then we can estimate $\|\bar{n}_{h,*}^m - \hat{n}_{h,*}^m\|_{L_h^2(\hat{\Gamma}_{h,*}^m)}$ by using the equivalence between discrete and continuous norms as well as the estimates in (3.7) and Lemma 3.8 with the triangle inequality, and estimate $|\bar{n}_{h,*}^m| - 1$ by using (3.18). This leads to the following estimate:

$$|L_{35}(\phi_h)| \lesssim \tau h^{2k} \|I_h \bar{T}_h^m \phi_h\|_{H^1(\hat{\Gamma}_{h,*}^m)}.$$

Similarly, we can rewrite $L_{37}(\phi_h)$ as

$$\begin{aligned} L_{37}(\phi_h) &= \tau \int_{\hat{\Gamma}_{h,*}^m} \kappa^{m,-l} n^{m,-l} \cdot (\hat{n}_{h,*}^m(\hat{n}_{h,*}^m)^\top - n_*^m(n_*^m)^\top) I_h \bar{T}_h^m \phi_h \\ &= 2\tau \int_{\hat{\Gamma}_{h,*}^m}^h \kappa^{m,-l} n^{m,-l} \cdot (\hat{n}_{h,*}^m - n_*^m) n_*^m \cdot I_h \bar{T}_h^m \phi_h \\ &\quad + \tau \int_{\hat{\Gamma}_{h,*}^m}^h \kappa^{m,-l} n^{m,-l} \cdot (\hat{n}_{h,*}^m - n_*^m)(\hat{n}_{h,*}^m - n_*^m) \cdot I_h \bar{T}_h^m \phi_h, \end{aligned}$$

where the last term is the same as the right-hand side of $L_{35}(\phi_h)$ and therefore has already been estimated, and the second to last term can be estimated by using Lemma 4.1. This leads to the following result:

$$|L_{37}(\phi_h)| \lesssim \tau h^{k+1} \|I_h \bar{T}_h^m \phi_h\|_{H^1(\hat{\Gamma}_{h,*}^m)} + \tau h^{2k} \|I_h \bar{T}_h^m \phi_h\|_{H^1(\hat{\Gamma}_{h,*}^m)}.$$

In summary, since $2k \geq k+1$, we have

$$|L_3(I_h\bar{T}_h^m\phi_h)| \lesssim \tau(h^{k+1} + \|\nabla_{\hat{\Gamma}_{h,*}^m}\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)})\|I_h\bar{T}_h^m\phi_h\|_{H^1(\hat{\Gamma}_{h,*}^m)}. \tag{4.47}$$

By using the above estimates of $|L_j(I_h\bar{T}_h^m\phi_h)|$, $j = 1,\dots,6$, choosing $\phi_h = I_h\bar{T}_h^m(X_h^{m+1} - X_h^m - \tau I_h v^m)$ in (4.41) leads to

$$\begin{aligned}
&\int_{\Gamma_h^m} \nabla_{\Gamma_h^m}(X_h^{m+1} - X_h^m - \tau I_h v^m)\cdot\nabla_{\Gamma_h^m} I_h\bar{T}_h^m(X_h^{m+1} - X_h^m - \tau I_h v^m)\\
&= \sum_{i=1}^{6} L_i(I_h\bar{T}_h^m(X_h^{m+1} - X_h^m - \tau I_h v^m))\\
&\lesssim \tau(\|\nabla_{\hat{\Gamma}_{h,*}^m}\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + h^{k+1})\|I_h\bar{T}_h^m(X_h^{m+1} - X_h^m - \tau I_h v^m)\|_{H^1(\hat{\Gamma}_{h,*}^m)}.
\end{aligned} \tag{4.48}$$

Utilizing the orthogonality between $\bar{N}_h^m$ and $\bar{T}_h^m$, we can prove the following result, which essentially controls the H^1 bilinear form $\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}\bar{N}_h^m f_{h,1}\cdot\nabla_{\Gamma_h^m}\bar{T}_h^m f_{h,2}$ by the L^2 norm of $f_{h,1}$ and the H^1 seminorm of $f_{h,2}$ for any two functions $f_{h,1}, f_{h,2} \in S_h(\Gamma_h^m)$.

Lemma 4.6. *The following estimate for the displacement $X_h^{m+1} - X_h^m - \tau I_h v^m$ holds:*

$$\begin{aligned}
&\Big|\int_{\Gamma_h^m}\nabla_{\Gamma_h^m}[I_h\bar{N}_h^m(X_h^{m+1} - X_h^m - \tau I_h v^m)]\cdot\nabla_{\Gamma_h^m}[I_h\bar{T}_h^m(X_h^{m+1} - X_h^m - \tau I_h v^m)]\Big|\\
&\lesssim \epsilon^{-1}(1 + h^{-4}\|\nabla_{\hat{\Gamma}_{h,*}^m}\hat{e}_h^m\|^2_{L^2(\hat{\Gamma}_{h,*}^m)})\|I_h\bar{N}_h^m(X_h^{m+1} - X_h^m - \tau I_h v^m)\|^2_{L^2(\hat{\Gamma}_{h,*}^m)}\\
&\quad + (\epsilon + \epsilon^{-1}h^2)\|\nabla_{\hat{\Gamma}_{h,*}^m} I_h\bar{T}_h^m(X_h^{m+1} - X_h^m - \tau I_h v^m)\|^2_{L^2(\hat{\Gamma}_{h,*}^m)} \qquad \forall\,\epsilon > 0.
\end{aligned} \tag{4.49}$$

Proof. For the simplicity of notation, we denote the displacement $\delta X_h^m := X_h^{m+1} - X_h^m - \tau I_h v^m \in S_h(\Gamma_h^m)$. From the fundamental theorem of calculus, geometric perturbation estimates and the mathematical induction assumptions, we have

$$\begin{aligned}
&\Big|\int_{\Gamma_h^m}\nabla_{\Gamma_h^m} I_h\bar{N}_h^m\delta X_h^m\cdot\nabla_{\Gamma_h^m} I_h\bar{T}_h^m\delta X_h^m\Big|\\
&= \Big|\Big(\int_{\Gamma_h^m}\nabla_{\Gamma_h^m} I_h\bar{N}_h^m\delta X_h^m\cdot\nabla_{\Gamma_h^m} I_h\bar{T}_h^m\delta X_h^m - \int_{\hat{\Gamma}_{h,*}^m}\nabla_{\hat{\Gamma}_{h,*}^m} I_h\bar{N}_h^m\delta X_h^m\cdot\nabla_{\hat{\Gamma}_{h,*}^m} I_h\bar{T}_h^m\delta X_h^m\Big)\\
&\quad + \Big(\int_{\hat{\Gamma}_{h,*}^m}\nabla_{\hat{\Gamma}_{h,*}^m} I_h\bar{N}_h^m\delta X_h^m\cdot\nabla_{\hat{\Gamma}_{h,*}^m} I_h\bar{T}_h^m\delta X_h^m - \int_{\Gamma^m}\nabla_{\Gamma^m}(I_h\bar{N}_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(I_h\bar{T}_h^m\delta X_h^m)^l\Big)\\
&\quad + \int_{\Gamma^m}\nabla_{\Gamma^m}(I_h\bar{N}_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(I_h\bar{T}_h^m\delta X_h^m)^l\Big|\\
&\lesssim (h^{k+1} + \|\nabla_{\hat{\Gamma}_{h,*}^m}\hat{e}_h^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)})\|\nabla_{\hat{\Gamma}_{h,*}^m} I_h\bar{N}_h^m\delta X_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}\|\nabla_{\hat{\Gamma}_{h,*}^m} I_h\bar{T}_h^m\delta X_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}\\
&\quad + \Big|\int_{\Gamma^m}\nabla_{\Gamma^m}(I_h\bar{N}_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(I_h\bar{T}_h^m\delta X_h^m)^l\Big|\\
&\lesssim \epsilon^{-1}\|I_h\bar{N}_h^m\delta X_h^m\|^2_{L^2(\hat{\Gamma}_{h,*}^m)} + \epsilon\|\nabla_{\hat{\Gamma}_{h,*}^m} I_h\bar{T}_h^m\delta X_h^m\|^2_{L^2(\hat{\Gamma}_{h,*}^m)}\\
&\quad + \Big|\int_{\Gamma^m}\nabla_{\Gamma^m}(I_h\bar{N}_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(I_h\bar{T}_h^m\delta X_h^m)^l\Big|,
\end{aligned} \tag{4.50}$$

where the last inequality uses the induction assumption $\|\nabla_{\hat{\Gamma}_{h,*}^m}\hat{e}_h^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \lesssim h$ in (3.10) and the inverse inequality which removes the gradient in front of $I_h\bar{N}_h^m\delta X_h^m$.

By the super-convergence estimates (Lemma 3.4) and (3.24), it follows that

$$
\begin{aligned}
&\Big|\int_{\Gamma^m} \nabla_{\Gamma^m}(I_h\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(I_h\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\Big|\\
&=\Big|\int_{\Gamma^m} \nabla_{\Gamma^m}(\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\\
&\quad-\int_{\Gamma^m} \nabla_{\Gamma^m}((1-I_h)\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(I_h\bar T_h^m\delta X_h^m)^l\\
&\quad-\int_{\Gamma^m} \nabla_{\Gamma^m}(I_h\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}((1-I_h)\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\\
&\quad-\int_{\Gamma^m} \nabla_{\Gamma^m}((1-I_h)\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}((1-I_h)\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\Big|\\
&\lesssim\Big|\int_{\Gamma^m} \nabla_{\Gamma^m}(\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\Big|\\
&\quad+h\|\delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)}\|\nabla I_h\bar T_h^m\delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\quad+h\|\nabla_{\hat\Gamma_{h,*}^m} I_h\bar N_h^m\delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|I_h\bar T_h^m\delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)}\\
&\quad+h^2\|\delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)}\|I_h\bar T_h^m\delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)},
\end{aligned}\tag{4.51}
$$

where we have used the following estimates (which follow from Lemma 3.4):

$$
\begin{aligned}
\|\nabla_{\Gamma^m}((1-I_h)\bar N_h^m\delta X_h^m)^l\|_{L^2(\Gamma^m)}&\lesssim h\|\bar n_h^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}\|\delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)},\\
\|\nabla_{\Gamma^m}((1-I_h)\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\|_{L^2(\Gamma^m)}&\lesssim h\|\bar n_h^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}\|I_h\bar T_h^m\delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)}.
\end{aligned}\tag{4.52}
$$

(In our notation, $I_h\bar T_h^m\delta X_h^m = I_h[\bar T_h^m\delta X_h^m]$ is a finite element function and therefore satisfies the requirement of Lemma 3.4.) The boundedness of $\|\bar N_h^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}$ and $\|\bar T_h^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}$ follows from the definitions of $\bar N_h^m$ and $\bar T_h^m$ in terms of $\bar n_h^m$ as well as the $W^{1,\infty}$ estimate of $\bar n_h^m$ in (3.24).

By decomposing δX_h^m into $I_h\bar N_h^m\delta X_h^m$ plus $I_h\bar T_h^m\delta X_h^m$ on the right-hand side of (4.51), applying the inverse inequality to $\|I_h\bar N_h^m\delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)}$ and the Poincaré inequality (3.32), we obtain

$$
\begin{aligned}
&\Big|\int_{\Gamma^m} \nabla_{\Gamma^m}(I_h\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(I_h\bar T_h^m\delta X_h^m)^l\Big|\\
&\lesssim\Big|\int_{\Gamma^m} \nabla_{\Gamma^m}(\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\Big|\\
&\quad+\epsilon^{-1}\|I_h\bar N_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma_{h,*}^m)}+\epsilon\|\nabla_{\hat\Gamma_{h,*}^m} I_h\bar T_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma_{h,*}^m)},
\end{aligned}\tag{4.53}
$$

where the L^2 norm of $I_h\bar T_h^m\delta X_h^m$ (arising from decomposing δX_h^m into $I_h\bar N_h^m\delta X_h^m$ plus $I_h\bar T_h^m\delta X_h^m$) is converted to its H^1 semi-norm by using the Poincaré type of inequality in (3.32). The first term on the right-hand side of (4.53) can be further decomposed into

$$
\begin{aligned}
&\Big|\int_{\Gamma^m} \nabla_{\Gamma^m}(\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\Big|\\
&=\Big|\int_{\Gamma^m} \nabla_{\Gamma^m}(N_*^m\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}(T_*^m\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\Gamma^m} \nabla_{\Gamma^m}((\bar N_h^m - N_*^m)\bar N_h^m \delta X_h^m)^l \cdot \nabla_{\Gamma^m}(T_*^m \bar T_h^m I_h \bar T_h^m \delta X_h^m)^l \\
&+ \int_{\Gamma^m} \nabla_{\Gamma^m}(\bar N_h^m \bar N_h^m \delta X_h^m)^l \cdot \nabla_{\Gamma^m}((\bar T_h^m - T_*^m)\bar T_h^m I_h \bar T_h^m \delta X_h^m)^l \Big| \\
\lesssim &\Big| \int_{\Gamma^m} \nabla_{\Gamma^m}(N_*^m \bar N_h^m \delta X_h^m)^l \cdot \nabla_{\Gamma^m}(T_*^m \bar T_h^m I_h \bar T_h^m \delta X_h^m)^l \Big| \\
&+ \Big(\|\nabla_{\hat\Gamma_{h,*}^m}(\bar N_h^m - N_*^m)\|_{L^2(\hat\Gamma_{h,*}^m)} \|\bar N_h^m \delta X_h^m\|_{L^\infty(\hat\Gamma_{h,*}^m)} \\
&+ \|\bar N_h^m - N_*^m\|_{L^\infty(\hat\Gamma_{h,*}^m)} \|\nabla_{\hat\Gamma_{h,*}^m} \bar N_h^m \delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \Big) \|\bar T_h^m I_h \bar T_h^m \delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)} \\
&+ \Big(\|\nabla_{\hat\Gamma_{h,*}^m}(\bar T_h^m - T_*^m)\|_{L^2(\hat\Gamma_{h,*}^m)} \|\bar T_h^m I_h \bar T_h^m \delta X_h^m\|_{L^\infty(\hat\Gamma_{h,*}^m)} \\
&+ \|\bar T_h^m - T_*^m\|_{L^\infty(\hat\Gamma_{h,*}^m)} \|\nabla_{\hat\Gamma_{h,*}^m} \bar T_h^m I_h \bar T_h^m \delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \Big) \|\bar N_h^m \delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)} \\
&\qquad\qquad \text{(product rule of differentiation is used)} \\
\lesssim &\Big| \int_{\Gamma^m} \nabla_{\Gamma^m}(N_*^m \bar N_h^m \delta X_h^m)^l \cdot \nabla_{\Gamma^m}(T_*^m \bar T_h^m I_h \bar T_h^m \delta X_h^m)^l \Big| \\
&+ h^{-1}(h^k + \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}) \|\bar N_h^m \delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)} \|\bar T_h^m I_h \bar T_h^m \delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)} \\
\lesssim &\Big| \int_{\Gamma^m} \nabla_{\Gamma^m}[N^m(\bar N_h^m \delta X_h^m)^l] \cdot \nabla_{\Gamma^m}[T^m(\bar T_h^m I_h \bar T_h^m \delta X_h^m)^l] \Big| \\
&+ \epsilon^{-1} h^{-2}(h^k + \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)})^2 \|I_h \bar N_h^m \delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)}^2 \\
&+ \epsilon \|I_h \bar T_h^m \delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)}^2 + h^2 \|\delta X_h^m\|_{H^1(\hat\Gamma_{h,*}^m)}^2 \qquad \text{(here (4.52) is used)} \\
\lesssim &\Big| \int_{\Gamma^m} \nabla_{\Gamma^m}[N^m(\bar N_h^m \delta X_h^m)^l] \cdot \nabla_{\Gamma^m}[T^m(\bar T_h^m I_h \bar T_h^m \delta X_h^m)^l] \Big| \\
&+ \epsilon^{-1} h^{-4}(h^k + \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)})^2 \|I_h \bar N_h^m \delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\
&+ (\epsilon + h^2) \|\nabla_{\hat\Gamma_{h,*}^m} I_h \bar T_h^m \delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 + \|I_h \bar N_h^m \delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\
&\quad \text{(inverse inequality and Poincaré inequality in (3.32) are used)} \\
\lesssim &\Big| \int_{\Gamma^m} \nabla_{\Gamma^m}[N^m(\bar N_h^m \delta X_h^m)^l] \cdot \nabla_{\Gamma^m}[T^m(\bar T_h^m I_h \bar T_h^m \delta X_h^m)^l] \Big| \\
&+ \epsilon^{-1}(1 + h^{-4} \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2) \|I_h \bar N_h^m \delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\
&+ (\epsilon + h^2) \|\nabla_{\hat\Gamma_{h,*}^m} I_h \bar T_h^m \delta X_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \quad (k \ge 2 \text{ is used}).
\end{aligned} \tag{4.54}
$$

For the first term on the right-hand side of (4.54), we consider the following further decomposition:

$$
\begin{aligned}
&\Big| \int_{\Gamma^m} \nabla_{\Gamma^m}[N^m(\bar N_h^m \delta X_h^m)^l] \cdot \nabla_{\Gamma^m}[T^m(\bar T_h^m I_h \bar T_h^m \delta X_h^m)^l] \Big| \\
&= \Big| \int_{\Gamma^m} (\nabla_{\Gamma^m} N^m) N^m (\bar N_h^m \delta X_h^m)^l \cdot (\nabla_{\Gamma^m} T^m) T^m (\bar T_h^m I_h \bar T_h^m \delta X_h^m)^l \\
&\quad + \int_{\Gamma^m} N^m \nabla_{\Gamma^m}[N^m(\bar N_h^m \delta X_h^m)^l] \cdot T^m \nabla_{\Gamma^m}[T^m(\bar T_h^m I_h \bar T_h^m \delta X_h^m)^l]
\end{aligned}
$$

$$+\int_{\Gamma^m}(\nabla_{\Gamma^m}N^m)N^m(\bar N_h^m\delta X_h^m)^l\cdot T^m\nabla_{\Gamma^m}[T^m(\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l]$$
$$+\int_{\Gamma^m}N^m\nabla_{\Gamma^m}[N^m(\bar N_h^m\delta X_h^m)^l]\cdot(\nabla_{\Gamma^m}T^m)T^m(\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\Big|,$$

where the second term on the right-hand side is zero due to the orthogonality between the two projections N^m and T^m. For the last term on the right-hand side, we can remove the gradient from $N^m(\bar N_h^m\delta X_h^m)^l$ via integration by parts. This leads to the following estimate:

$$\begin{aligned}
&\Big|\int_{\Gamma^m}\nabla_{\Gamma^m}N^m(\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}T^m(\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\Big|\\
&\lesssim\epsilon^{-1}\|\bar N_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}+\epsilon\|\bar T_h^m I_h\bar T_h^m\delta X_h^m\|^2_{H^1(\hat\Gamma^m_{h,*})}\\
&\lesssim\epsilon^{-1}\|I_h\bar N_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}+\varepsilon^{-1}h^2\|\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}\\
&\quad+\epsilon\|\nabla_{\hat\Gamma^m_{h,*}}I_h\bar T_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}+\epsilon h^2\|\delta X_h^m\|^2_{H^1(\hat\Gamma^m_{h,*})},
\end{aligned}\tag{4.55}$$

where the last inequality follows from the triangle inequality and (4.52), as well as the following result which is similar as (4.52):

$$\|(1-I_h)\bar N_h^m\delta X_h^m\|_{L^2(\Gamma^m)}\lesssim h\|\delta X_h^m\|_{L^2(\hat\Gamma^m_{h,*})}.$$

The terms $h^2\|\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}$ and $h^2\|\delta X_h^m\|^2_{H^1(\hat\Gamma^m_{h,*})}$ on the right-hand side of (4.55) can be furthermore decomposed into the normal and tangential parts, respectively, e.g.,

$$\begin{aligned}
h^2\|\delta X_h^m\|^2_{H^1(\hat\Gamma^m_{h,*})}&\lesssim h^2\|I_h\bar N_h^m\delta X_h^m\|^2_{H^1(\hat\Gamma^m_{h,*})}+h^2\|I_h\bar T_h^m\delta X_h^m\|^2_{H^1(\hat\Gamma^m_{h,*})}\\
&\lesssim\|I_h\bar N_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}+h^2\|\nabla_{\hat\Gamma^m_{h,*}}I_h\bar T_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})},
\end{aligned}$$

where the inverse inequality and the Poincaré inequality in (3.32) are used. Therefore, (4.55) can be reduced to the following one:

$$\begin{aligned}
&\Big|\int_{\Gamma^m}\nabla_{\Gamma^m}N^m(\bar N_h^m\delta X_h^m)^l\cdot\nabla_{\Gamma^m}T^m(\bar T_h^m I_h\bar T_h^m\delta X_h^m)^l\Big|\\
&\lesssim\epsilon^{-1}\|I_h\bar N_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}+(\epsilon+\epsilon^{-1}h^2)\|\nabla_{\hat\Gamma^m_{h,*}}I_h\bar T_h^m\delta X_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}.
\end{aligned}\tag{4.56}$$

The result of Lemma 4.6 follows from (4.50)–(4.56). ☐

Remark 4.7. The same proof leads to the following result, with $\bar N_h^m$ and $\bar T_h^m$ replaced by $\bar N_{h,*}^m$ and $\bar T_{h,*}^m$, respectively, and Γ_h^m replaced by $\hat\Gamma_{h,*}^m$:

$$\begin{aligned}
&\Big|\int_{\hat\Gamma^m_{h,*}}\nabla_{\hat\Gamma^m_{h,*}}I_h\bar N_{h,*}^m(X_h^{m+1}-X_h^m-\tau I_hv^m)\cdot\nabla_{\hat\Gamma^m_{h,*}}I_h\bar T_{h,*}^m(X_h^{m+1}-X_h^m-\tau I_hv^m)\Big|\\
&\lesssim\epsilon^{-1}\|I_h\bar N_{h,*}^m(X_h^{m+1}-X_h^m-\tau I_hv^m)\|^2_{L^2(\hat\Gamma^m_{h,*})}\\
&\quad+(\epsilon+\epsilon^{-1}h^2)\|\nabla_{\hat\Gamma^m_{h,*}}I_h\bar T_{h,*}^m(X_h^{m+1}-X_h^m-\tau I_hv^m)\|^2_{L^2(\hat\Gamma^m_{h,*})}.
\end{aligned}\tag{4.57}$$

Compared with (4.49), the right-hand side of the above inequality does not contain the term $\epsilon^{-1}h^{-4}\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|^2_{L^2(\hat\Gamma^m_{h,*})}$ because only the consistency error is involved here.

By using the estimates in (4.48) and (4.49), we have

$$
\begin{aligned}
&\int_{\Gamma_h^m} \nabla_{\Gamma_h^m} I_h\bar T_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\cdot\nabla_{\Gamma_h^m} I_h\bar T_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\\
&=\int_{\Gamma_h^m} \nabla_{\Gamma_h^m}(X_h^{m+1}-X_h^m-\tau I_h v^m)\cdot\nabla_{\Gamma_h^m} I_h\bar T_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\\
&\quad-\int_{\Gamma_h^m} \nabla_{\Gamma_h^m} I_h\bar N_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\cdot\nabla_{\Gamma_h^m} I_h\bar T_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\\
&\lesssim \tau(\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}+h^{k+1})\|I_h\bar T_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\|_{H^1(\hat\Gamma_{h,*}^m)}\\
&\quad+\epsilon^{-1}(1+h^{-4}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2)\|I_h\bar N_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\|_{L^2(\hat\Gamma_{h,*}^m)}^2\\
&\quad+\epsilon\|\nabla_{\hat\Gamma_{h,*}^m} I_h\bar T_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\|_{L^2(\hat\Gamma_{h,*}^m)}^2\\
&\lesssim \epsilon^{-1}\tau^2h^{2k+2}+\epsilon^{-1}\tau^2\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2\\
&\quad+\epsilon^{-1}(1+h^{-4}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2)\|I_h\bar N_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\|_{L^2(\hat\Gamma_{h,*}^m)}^2\\
&\quad+\epsilon\|\nabla_{\hat\Gamma_{h,*}^m} I_h\bar T_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\|_{L^2(\hat\Gamma_{h,*}^m)}^2.
\end{aligned}
\tag{4.58}
$$

Since the L^2 norms of a finite element function on $\hat\Gamma_{h,*}^m$ and Γ_h^m are equivalent, by choosing a sufficiently small ϵ, the last term on the right-hand side of (4.58) can be absorbed by its left-hand side. As a result, we obtain the following inequality:

$$
\begin{aligned}
&\|\nabla_{\hat\Gamma_{h,*}^m} I_h\bar T_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\lesssim \tau h^{k+1}+\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\quad+(1+h^{-2}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)})\|I_h\bar N_h^m(X_h^{m+1}-X_h^m-\tau I_h v^m)\|_{L^2(\hat\Gamma_{h,*}^m)}.
\end{aligned}
\tag{4.59}
$$

Hence, by using the relation $e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m = X_h^{m+1}-X_h^m-\tau I_hv^m-\tau I_hg^m$ from (4.40) and the estimate in (4.59), we have

$$
\begin{aligned}
&\|\nabla_{\hat\Gamma_{h,*}^m} I_h\bar T_{h,*}^m(e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m)\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\lesssim \tau(\tau+h^{k+1})+\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\quad+(1+h^{-2}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)})\|I_h\bar N_h^m(e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m)\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\quad+\|\nabla_{\hat\Gamma_{h,*}^m} I_h[(\bar T_h^m-\bar T_{h,*}^m)(e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m)]\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\lesssim \tau(\tau+h^{k+1})+\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\quad+(1+h^{-2}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)})\|I_h\bar N_h^m(e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m)\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\quad+h^{-1}\|\bar T_h^m-\bar T_{h,*}^m\|_{L^2(\hat\Gamma_{h,*}^m)}\|e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m\|_{L^\infty(\hat\Gamma_{h,*}^m)}\\
&\qquad\text{(inverse inequality and equivalence between discrete and continuous norms)}\\
&\lesssim \tau(\tau+h^{k+1})+\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}\\
&\quad+(1+h^{-2}\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)})\|I_h\bar N_{h,*}^m(e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m)\|_{L^2(\hat\Gamma_{h,*}^m)}
\end{aligned}
$$

$$+ h^{-1}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})}\|\nabla_{\hat\Gamma^m_{h,*}} I_h\bar T^m_{h,*}(e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m)\|_{L^2(\hat\Gamma^m_{h,*})}, \tag{4.60}$$

where we have estimated $\|\bar T^m_h - \bar T^m_{h,*}\|_{L^2(\hat\Gamma^m_{h,*})}$ by using (3.20) and decomposed the term $\|e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m\|_{L^\infty(\hat\Gamma^m_{h,*})}$ into its normal and tangential parts, respectively, and have changed $\bar N^m_h$ to $\bar N^m_{h,*}$ by using estimate

$$\|\bar N^m_h - \bar N^m_{h,*}\|_{L^2(\hat\Gamma^m_{h,*})} \lesssim \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})} \lesssim h^{1.75}.$$

This estimate follows from (3.20) and (3.10), and can be used to absorb the additional perturbation term caused by changing $\bar N^m_h$ to $\bar N^m_{h,*}$. Since $\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})} \lesssim h^{1.75}$, as shown in (3.10), the last term on the right-hand side of (4.60) can be absorbed by the left-hand side. This leads to the following result:

$$\begin{aligned}
&\|\nabla_{\hat\Gamma^m_{h,*}} I_h\bar T^m_{h,*}(e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m)\|_{L^2(\hat\Gamma^m_{h,*})}\\
&\lesssim \tau(\tau + h^{k+1}) + \tau\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})}\\
&\quad + (1 + h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})})\|I_h\bar N^m_{h,*}(e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m)\|_{L^2(\hat\Gamma^m_{h,*})}.
\end{aligned} \tag{4.61}$$

Then, by applying the Poincaré inequality with $v_h = I_h\bar T^m_{h,*}(e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m)$ satisfying $I_h(v_h \cdot \bar n^m_{h,*}) = 0$ in Lemma 3.10, we can control the L^2 norm of the tangential component $I_h\bar T^m_{h,*}(e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m)$ by the left-hand side of (4.61). Since the L^2 norm of the normal component $I_h\bar N^m_{h,*}(e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m)$ already appears on the right-hand side of (4.61), by summing up the L^2 norms of the tangential and normal components of $e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m$ we obtain the following result:

$$\begin{aligned}
&\|e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m\|_{L^2(\hat\Gamma^m_{h,*})}\\
&\lesssim \tau(\tau + h^{k+1}) + \tau\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})}\\
&\quad + (1 + h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})})\|I_h\bar N^m_{h,*}(e^{m+1}_h - \hat e^m_h - \tau I_h T^m_* v^m)\|_{L^2(\hat\Gamma^m_{h,*})}.
\end{aligned} \tag{4.62}$$

4.5. **Velocity estimates.** The last term on the right-hand side (4.62) can be estimated by testing the error equation (4.21) with $\phi_h = e^m_{v,h} := \frac{1}{\tau}(e^{m+1}_h - \hat e^m_h) -$

$I_hT_*^m v^m$. This leads to the following estimate:

$$
\begin{aligned}
&\int_{\hat\Gamma_{h,*}^m}^h \Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m \Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m \\
&= -\int_{\hat\Gamma_{h,*}^m}^h I_hT_*^m v^m\cdot \bar n_{h,*}^m \Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m \\
&\quad + \int_{\hat\Gamma_{h,*}^m}^h \frac{e_h^{m+1}-\hat e_h^m}{\tau}\cdot \bar n_{h,*}^m \Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m \\
&= -\int_{\hat\Gamma_{h,*}^m}^h I_hT_*^m v^m\cdot \bar n_{h,*}^m \Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m \\
&\quad - d^m(e_{v,h}^m) - J^m(e_{v,h}^m) - B^m(\hat e_h^m, e_{v,h}^m) - K^m(e_{v,h}^m) \\
&\quad - A_{h,*}^N\Big(e_h^{m+1}, \frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big) - A_{h,*}^T\Big(e_h^{m+1}-\hat e_h^m, \frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big) \\
&\quad - \sum_{i=1}^3 F_i^m(e_{v,h}^m) + A_{h,*}^N(\hat e_h^m, I_h\bar T_{h,*}^m e_{v,h}^m) + B^m(\hat e_h^m, I_h\bar T_{h,*}^m e_{v,h}^m) + Q^m(I_h\bar T_{h,*}^m e_{v,h}^m) \\
&\le -\int_{\hat\Gamma_{h,*}^m}^h \Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m\, I_hT_*^m v^m\cdot \bar n_{h,*}^m \\
&\quad - d^m(e_{v,h}^m) - J^m(e_{v,h}^m) - B^m(\hat e_h^m, e_{v,h}^m) - K^m(e_{v,h}^m) \\
&\quad - A_{h,*}^N\Big(\hat e_h^m + \tau I_hT_*^m v^m, \frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big) \\
&\quad - A_{h,*}^T\Big(\tau I_hT_*^m v^m, \frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big) \\
&\quad - \sum_{i=1}^3 F_i^m(e_{v,h}^m) + A_{h,*}^N(\hat e_h^m, I_h\bar T_{h,*}^m e_{v,h}^m) + B^m(\hat e_h^m, I_h\bar T_{h,*}^m e_{v,h}^m) + Q^m(I_h\bar T_{h,*}^m e_{v,h}^m),
\end{aligned}
\tag{4.63}
$$

where we have dropped the following two non-positive terms from the right-hand side of the last inequality:

$$
\begin{aligned}
&-\tau A_{h,*}^N\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m, \frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big) \\
\text{and}\qquad &-\tau A_{h,*}^T\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m, \frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big).
\end{aligned}
$$

The first term on the right-hand side of (4.63) can be estimated by using the orthogonality between $I_h\bar T_{h,*}^m v^m$ and $\bar n_{h,*}^m$ at the nodes, which implies that $I_hT_*^m v^m\cdot \bar n_{h,*}^m = (I_hT_*^m v^m - \bar T_{h,*}^m v^m)\cdot \bar n_{h,*}^m$ at nodes and therefore

$$
\begin{aligned}
&\Big|\int_{\hat\Gamma_{h,*}^m}^h \Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m\, I_hT_*^m v^m\cdot \bar n_{h,*}^m\Big| \\
&\lesssim \|I_hT_*^m - \bar T_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)} \Big\|\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m\Big\|_{L_h^2(\hat\Gamma_{h,*}^m)} \\
&\lesssim h^k \Big\|\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau} - I_hT_*^m v^m\Big)\cdot \bar n_{h,*}^m\Big\|_{L_h^2(\hat\Gamma_{h,*}^m)},
\end{aligned}
\tag{4.64}
$$

where the last inequality follows from Lemma 3.8. The second and third terms on the right-hand side of (4.63) can be estimated by using the results in (4.7) and

(4.32), i.e.,

$$\Big|d^m\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big| \lesssim \tau\Big\|\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big\|_{L^2(\hat\Gamma^m_{h,*})}+h^{k+1}\Big\|\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big\|_{H^1(\hat\Gamma^m_{h,*})}, \tag{4.65}$$

$$\Big|J^m\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big| \lesssim \|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^2(\hat\Gamma^m_{h,*})}\Big\|\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big\|_{L^2(\hat\Gamma^m_{h,*})}. \tag{4.66}$$

The following result can be obtained by using the triangle inequality and the boundedness of $I_hT_*^mv^m$:

$$\|\nabla_{\hat\Gamma^m_{h,*}}e_h^{m+1}\|_{L^2(\hat\Gamma^m_{h,*})}\lesssim\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^2(\hat\Gamma^m_{h,*})} +\tau\Big\|\nabla_{\hat\Gamma^m_{h,*}}\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big\|_{L^2(\hat\Gamma^m_{h,*})}+\tau.$$

By substituting the above inequality into the right-hand side of (4.25) and (4.27), the following estimates of K^m and Q^m can be verified:

$$\begin{aligned}
&\Big|K^m\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big|+\Big|Q^m\Big(I_h\bar T^m_{h,*}\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big)\Big|\\
&\lesssim\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^\infty(\hat\Gamma^m_{h,*})}\Big(\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^2(\hat\Gamma^m_{h,*})}+\tau\Big\|\nabla_{\hat\Gamma^m_{h,*}}\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big\|_{L^2(\hat\Gamma^m_{h,*})}+\tau\Big)\\
&\quad\times\Big\|\nabla_{\hat\Gamma^m_{h,*}}\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big\|_{L^2(\hat\Gamma^m_{h,*})}\\
&\quad+(\tau+h^k)\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^2(\hat\Gamma^m_{h,*})}\Big\|\nabla_{\hat\Gamma^m_{h,*}}\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big\|_{L^2(\hat\Gamma^m_{h,*})}\\
&\lesssim(h^{-1/2}\tau+h^k+\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^\infty(\hat\Gamma^m_{h,*})})\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^2(\hat\Gamma^m_{h,*})}\Big\|\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big\|_{H^1(\hat\Gamma^m_{h,*})}\\
&\quad+\tau\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^\infty(\hat\Gamma^m_{h,*})}\Big\|\nabla_{\hat\Gamma^m_{h,*}}\Big(\frac{e_h^{m+1}-\hat e_h^m}{\tau}-I_hT_*^mv^m\Big)\Big\|^2_{L^2(\hat\Gamma^m_{h,*})}.
\end{aligned}\tag{4.67}$$

By the definitions of the bilinear forms $A^N_{h,*}(\cdot,\cdot)$, $A^T_{h,*}(\cdot,\cdot)$ and $B^m(\cdot,\cdot)$ in (4.17)–(4.18), we have

$$|A^N_{h,*}(u_h,v_h)|+|A^T_{h,*}(u_h,v_h)|+|B^m(u_h,v_h)|\lesssim\|\nabla_{\hat\Gamma^m_{h,*}}u_h\|_{L^2(\hat\Gamma^m_{h,*})}\|v_h\|_{H^1(\hat\Gamma^m_{h,*})}, \tag{4.68}$$

for any $u_h,v_h\in S_h(\hat\Gamma^m_{h,*})$. By substituting these estimates together with the estimates of $F_i^m(e^m_{v,h})$ from (4.34)–(4.36) into the right-hand side of (4.63), and then using the estimate in (4.61), we obtain

$$\begin{aligned}
&\Big\|\Big(\frac{e_h^{m+1}-\hat e^m_{h,*}}{\tau}-I_hT_*^mv^m\Big)\cdot\bar n^m_{h,*}\Big\|^2_{L^2_h(\hat\Gamma^m_{h,*})}\\
&\lesssim(\tau+h^{k+1}+\|\nabla_{\hat\Gamma^m_{h,*}}\hat e_h^m\|_{L^2(\hat\Gamma^m_{h,*})})\Big\|\frac{e_h^{m+1}-\hat e^m_{h,*}}{\tau}-I_hT_*^mv^m\Big\|_{H^1(\hat\Gamma^m_{h,*})}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \tau \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \Big\|\nabla_{\hat\Gamma^m_{h,*}} \Big(\frac{e^{m+1}_h - \hat e^m_h}{\tau} - I_h T^m_* v^m\Big)\Big\|^2_{L^2(\hat\Gamma^m_{h,*})} \\
&\quad + h^k \Big\|\Big(\frac{e^{m+1}_h - \hat e^m_h}{\tau} - I_h T^m_* v^m\Big)\cdot \bar n^m_{h,*}\Big\|_{L^2_h(\hat\Gamma^m_{h,*})} \\
&\lesssim h^{-1}\Big(\tau + h^{k+1} + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})}\Big)\Big\| I_h \bar N^m_{h,*}\big(\frac{e^{m+1}_h - \hat e^m_{h,*}}{\tau} - I_h T^m_* v^m\big)\Big\|_{L^2(\hat\Gamma^m_{h,*})} \\
&\quad + (\tau + h^{k+1} + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})})\Big\|\nabla_{\hat\Gamma^m_{h,*}} I_h \bar T^m_{h,*}\Big(\frac{e^{m+1}_h - \hat e^m_{h,*}}{\tau} - I_h T^m_* v^m\Big)\Big\|_{L^2(\hat\Gamma^m_{h,*})} \\
&\quad + \tau \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \Big\|\nabla_{\hat\Gamma^m_{h,*}} I_h \bar T^m_{h,*}\Big(\frac{e^{m+1}_h - \hat e^m_h}{\tau} - I_h T^m_* v^m\Big)\Big\|^2_{L^2(\hat\Gamma^m_{h,*})} \\
&\quad + h^{-2}\tau \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \Big\| I_h \bar N^m_{h,*}\Big(\frac{e^{m+1}_h - \hat e^m_h}{\tau} - I_h T^m_* v^m\Big)\Big\|^2_{L^2(\hat\Gamma^m_{h,*})} \quad \text{(inverse inequality)} \\
&\quad + h^k \Big\|\Big(\frac{e^{m+1}_h - \hat e^m_h}{\tau} - I_h T^m_* v^m\Big)\cdot \bar n^m_{h,*}\Big\|_{L^2_h(\hat\Gamma^m_{h,*})} \\
&\lesssim (h^{-1}\tau + h^k + h^{-1}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})})\Big\| I_h \bar N^m_{h,*}\Big(\frac{e^{m+1}_h - \hat e^m_{h,*}}{\tau} - I_h T^m_* v^m\Big)\Big\|_{L^2(\hat\Gamma^m_{h,*})} \\
&\quad + (\tau + h^{k+1} + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})})\Big[(\tau + h^{k+1}) + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})} \\
&\qquad\qquad + (1 + h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})})\Big\| I_h \bar N^m_{h,*}\Big(\frac{e^{m+1}_h - \hat e^m_{h,*}}{\tau} - I_h T^m_* v^m\Big)\Big\|_{L^2(\hat\Gamma^m_{h,*})}\Big] \\
&\quad + \tau \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})} \Big[(\tau + h^{k+1})^2 + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2(\hat\Gamma^m_{h,*})} \\
&\qquad\qquad + (1 + h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})})^2\Big\| I_h \bar N^m_{h,*}\Big(\frac{e^{m+1}_h - \hat e^m_{h,*}}{\tau} - I_h T^m_* v^m\Big)\Big\|^2_{L^2(\hat\Gamma^m_{h,*})}\Big] \\
&\quad + h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})}\tau \Big\| I_h \bar N^m_{h,*}\Big(\frac{e^{m+1}_h - \hat e^m_h}{\tau} - I_h T^m_* v^m\Big)\Big\|^2_{L^2(\hat\Gamma^m_{h,*})} \\
&\lesssim \epsilon^{-1}\Big(h^{-1}\tau + h^k + h^{-1}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})} + h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2(\hat\Gamma^m_{h,*})}\Big)^2 \\
&\quad + (\tau + h^{k+1})^2 + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2(\hat\Gamma^m_{h,*})} \\
&\quad + (\epsilon + h^{-2}\tau\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})})\Big\| I_h \bar N^m_{h,*}\Big(\frac{e^{m+1}_h - \hat e^m_h}{\tau} - I_h T^m_* v^m\Big)\Big\|^2_{L^2(\hat\Gamma^m_{h,*})} \\
&\lesssim \epsilon^{-1}h^{-2}(\tau + h^{k+1})^2 + \epsilon^{-1}h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2(\hat\Gamma^m_{h,*})} + \epsilon^{-1}h^{-4}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^4_{L^2(\hat\Gamma^m_{h,*})} \\
&\quad + (\epsilon + h^{-2}\tau\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^\infty(\hat\Gamma^m_{h,*})})\Big\|\Big(\frac{e^{m+1}_h - \hat e^m_{h,*}}{\tau} - I_h T^m_* v^m\Big)\cdot \bar n^m_{h,*}\Big\|^2_{L^2_h(\hat\Gamma^m_{h,*})},
\end{aligned}
$$

where, in the derivation of the last inequality, we have used the equivalence between continuous and discrete norms for finite element functions, as shown in (3.46). Under the stepsize condition $\tau \le c h^{k+1}$, for sufficiently small h and ϵ, the last term on the right-hand side above can be absorbed by the left-hand side. Then we obtain

$$
\begin{aligned}
&\Big\|\Big(\frac{e^{m+1}_h - \hat e^m_{h,*}}{\tau} - I_h T^m_* v^m\Big)\cdot \bar n^m_{h,*}\Big\|^2_{L^2_h(\hat\Gamma^m_{h,*})} \\
&\lesssim h^{-2}(\tau + h^{k+1})^2 + h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2(\hat\Gamma^m_{h,*})},
\end{aligned}
\tag{4.69}
$$

where we have absorbed $h^{-4}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^4_{L^2(\hat\Gamma^m_{h,*})}$ into $h^{-2}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|^2_{L^2(\hat\Gamma^m_{h,*})}$ by using the estimate $\|\nabla_{\hat\Gamma^m_{h,*}} \hat e^m_h\|_{L^2(\hat\Gamma^m_{h,*})} \lesssim h$ in (3.10). By considering the square root of (4.69) and using the norm equivalence relation in (3.46) again, we obtain the following result:

$$\Big\| I_h \bar N_{h,*}^m \Big(\frac{e_h^{m+1} - \hat e_{h,*}^m}{\tau} - I_h T_*^m v^m \Big) \Big\|_{L^2(\hat\Gamma_{h,*}^m)} \lesssim h^{-1}(\tau + h^{k+1}) + h^{-1} \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}. \tag{4.70}$$

From (4.61) and (4.70), we get an H^1 estimate for the tangential velocity, i.e.,

$$\begin{aligned} &\|\nabla_{\hat\Gamma_{h,*}^m} I_h \bar T_{h,*}^m (e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v^m)\|_{L^2(\hat\Gamma_{h,*}^m)} \\ &\lesssim h^{-1}\tau(\tau + h^{k+1}) + h^{-1}\tau \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \\ &\quad + h^{-3}\|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \tau(\tau + h^{k+1}) + h^{-3}\tau \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\ &\lesssim h^{-1}\tau(\tau + h^{k+1}) + h^{-1}\tau \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} + h^{-3}\tau \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2, \end{aligned} \tag{4.71}$$

where we have used $h^{-2}(\tau + h^{k+1}) \lesssim 1$ in the last inequality.

Furthermore, by decomposing the velocity into its normal and tangential components, we can obtain an H^1 estimate of the full velocity, i.e.,

$$\begin{aligned} &\|e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v^m\|_{H^1(\hat\Gamma_{h,*}^m)} \\ &\lesssim h^{-1}\|I_h \bar N_{h,*}^m (e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v^m)\|_{L^2(\hat\Gamma_{h,*}^m)} + \|I_h \bar T_{h,*}^m (e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v^m)\|_{H^1(\hat\Gamma_{h,*}^m)} \\ &\lesssim h^{-2}\tau(\tau + h^{k+1}) + h^{-2}\tau \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}, \end{aligned} \tag{4.72}$$

where the last inequality follows from (4.71), the Poincaré inequality in Lemma 3.10 and the estimate $\|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \lesssim h^{1.75}$.

An important application of the velocity estimates in (4.62) and (4.70)–(4.71) is the following estimate of $\|e_h^{m+1}\|_{L^2(\hat\Gamma_{h,*}^m)}$:

$$\begin{aligned} &\|e_h^{m+1}\|_{L^2(\hat\Gamma_{h,*}^m)} \\ &\quad \le \|\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} + \|e_h^{m+1} - \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \\ &\quad \lesssim \|\hat e_h^m \cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)} + \|e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v^m\|_{L^2(\hat\Gamma_{h,*}^m)} + \tau \\ &\qquad\qquad \text{((3.44) and the triangle inequality are used)} \\ &\quad \lesssim \|\hat e_h^m \cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)} + h^{-1}\tau \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} + h^{-3}\tau \|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 + \tau \\ &\quad \lesssim \|\hat e_h^m \cdot \bar n_{h,*}^m\|_{L_h^2(\hat\Gamma_{h,*}^m)} + \tau, \end{aligned} \tag{4.73}$$

where we have used the induction assumption $\|\nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \lesssim h^{1.75}$ in the last inequality. Inequality (4.73) will help us to convert $\|e_h^{m+1}\|_{L^2(\hat\Gamma_{h,*}^m)}^2$ to $\|\hat e_h^m \cdot \bar n_{h,*}^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2$ on the right-hand side of the error estimates. The latter will be absorbed by the left-hand side by using the discrete version of Grönwall's inequality.

Via the inverse inequality and the stepsize condition $\tau \le ch^{k+1}$, inequality (4.73) also implies that

$$\|e_h^{m+1}\|_{L^\infty(\hat\Gamma_{h,*}^m)} \lesssim h^{1.25}. \tag{4.74}$$

From (3.33)–(3.34) we also see that

$$\|\hat{e}_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \lesssim \|e_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|^2_{L^\infty(\hat{\Gamma}_{h,*}^m)} \lesssim h^{1.25}. \tag{4.75}$$

Furthermore, the following inequalities can be proved by using (3.33)–(3.34) and (4.74):

$$\begin{aligned} &\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} \lesssim \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}, \\ &\|\hat{e}_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|\nabla_{\hat{\Gamma}_{h,*}^m} e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}. \end{aligned} \tag{4.76}$$

By substituting (4.73) into the right-hand side of (4.76), we can obtain the following result:

$$\|\hat{e}_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim \tau + \|\hat{e}_h^m \cdot \bar{n}_{h,*}^m\|_{L_h^2(\hat{\Gamma}_{h,*}^m)} + \|\nabla_{\hat{\Gamma}_{h,*}^m} e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}. \tag{4.77}$$

We can also prove the stability in the other way round (cf. [2, Eqs. (5.68), (5.69)]):

$$\|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} \lesssim \|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \tau, \tag{4.78}$$

$$\|e_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim \|\hat{e}_h^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} + \tau. \tag{4.79}$$

They can be shown by the velocity estimate (4.72) and the stepsize condition $\tau \le ch^{k+1}$:

$$\begin{aligned} \|e_h^{m+1} - \hat{e}_h^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} &\lesssim \tau + \|e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} \\ &\lesssim \tau + h^{-2}\tau(\tau + h^{k+1}) + h^{-2}\tau\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\ &\lesssim \|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \tau, \end{aligned}$$

and then (4.78)–(4.79) follow imminently from the triangle inequality.

4.6. **Norm equivalence on the curves Γ_h^m, Γ_h^{m+1}, $\hat{\Gamma}_{h,*}^m$, $\hat{\Gamma}_{h,*}^{m+1}$ and $\Gamma_{h,*}^{m+1}$.** In this subsection, we show the equivalence of L^p and $W^{1,p}$ norms on the curves $\Gamma_h^m, \Gamma_h^{m+1}, \hat{\Gamma}_{h,*}^m, \hat{\Gamma}_{h,*}^{m+1}$ and $\Gamma_{h,*}^{m+1}$ by using the velocity estimates established in the previous subsection. In view of the norm equivalence results in Lemma 3.1, it suffices to show that the distance between these curves are small in the $W^{1,\infty}$ norm.

From the velocity estimate (4.72) we can derive the following result by using the stepsize condition $\tau \le ch^{k+1}$:

$$\|e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim h^2. \tag{4.80}$$

Then, using the triangle inequality and (4.73)–(4.76), we get

$$\|e_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim \|e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} + \|\hat{e}_h^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} + \tau \lesssim h^{1.75}, \tag{4.81}$$

$$\|\hat{e}_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim \|e_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim h^{1.75}. \tag{4.82}$$

By utilizing relation (3.38) and the two estimates above, we have

$$
\begin{aligned}
&\|\hat{X}_{h,*}^{m+1} - \hat{X}_{h,*}^{m}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \\
&\le \|\hat{X}_{h,*}^{m+1} - X_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|X_h^{m+1} - X_h^{m}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|X_h^{m} - \hat{X}_{h,*}^{m}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \\
&= \|\hat{e}_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1} - \hat{e}_h^m - \tau I_h(H^m n^m - g^m)\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|\hat{e}_h^{m}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \\
&\lesssim \|\hat{e}_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|\hat{e}_h^{m}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \tau \\
&\lesssim h^{1.75}.
\end{aligned} \tag{4.83}
$$

From (3.35) we see that

$$
\|X_{h,*}^{m+1} - \hat{X}_{h,*}^{m}\|_{W^{1,\infty}(\hat{\Gamma}_{h,*}^m)} \lesssim \tau. \tag{4.84}
$$

Lemma 4.8 essentially helps us to bound the L^p norm of the tangential part of the error displacement, i.e. $e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m$, by the L^p norm of its normal part and the $W^{1,p}$ semi-norm of the tangential part which can be furthermore controlled by (4.70) and (4.71) respectively. This lemma is needed to estimate $\|\hat{X}_{h,*}^{m+1} - \hat{X}_{h,*}^{m}\|_{W^{1,\infty}(\hat{\Gamma}_{h,*}^m)}$.

Lemma 4.8. *The following estimate for the error displacement $e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m$ holds:*

$$
\begin{aligned}
&\|I_h T_*^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)} \\
&\lesssim h^{k-1} \|I_h \bar{N}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{L^p(\hat{\Gamma}_{h,*}^m)} \\
&\quad + \|\nabla_{\hat{\Gamma}_{h,*}^m} I_h \bar{T}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{L^p(\hat{\Gamma}_{h,*}^m)}, \quad \forall\, p \in [2,\infty].
\end{aligned} \tag{4.85}
$$

Proof. Using the triangle inequality, we have

$$
\begin{aligned}
&\|I_h T_*^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)} \\
&\le \|I_h \bar{T}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)} \\
&\quad + \|I_h (\bar{T}_{h,*}^m - I_h T_*^m)(e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)} \\
&\lesssim \|I_h \bar{T}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)} \\
&\quad + h^{-1} \|\bar{T}_{h,*}^m - I_h T_*^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \|e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m\|_{L^p(\hat{\Gamma}_{h,*}^m)} \quad \text{(here (3.14) is used)} \\
&\lesssim \|I_h \bar{T}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)} \\
&\quad + h^{k-1} \|e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m\|_{L^p(\hat{\Gamma}_{h,*}^m)} \quad \text{(Lemma 3.8 is used)} \\
&\lesssim \|I_h \bar{T}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{W^{1,p}(\hat{\Gamma}_{h,*}^m)} \\
&\quad + h^{k-1} \|I_h \bar{N}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{L^p(\hat{\Gamma}_{h,*}^m)} \\
&\lesssim \|\nabla_{\hat{\Gamma}_{h,*}^m} I_h \bar{T}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{L^p(\hat{\Gamma}_{h,*}^m)} \\
&\quad + h^{k-1} \|I_h \bar{N}_{h,*}^m (e_h^{m+1} - \hat{e}_h^m - \tau I_h T_*^m v^m)\|_{L^p(\hat{\Gamma}_{h,*}^m)},
\end{aligned} \tag{4.86}
$$

where, in the last inequality, we have applied Poincaré inequality (Lemma 3.10). □

The following identities have been proved in [2, Eqs. (A.15) and (A.17)]:

$$N_*^m(\hat X_{h,*}^{m+1}-\hat X_{h,*}^m)=(X^{m+1}-\mathrm{id})\circ a^m+\rho_h \quad \text{at the nodes,} \tag{4.87}$$

$$\text{where } |\rho_h|\le C_0\tau^2+C_0|T_*^m(\hat X_{h,*}^{m+1}-\hat X_{h,*}^m)|^2 \quad \text{at the nodes,} \tag{4.88}$$

$$T_*^m(\hat X_{h,*}^{m+1}-\hat X_{h,*}^m)=T_*^m(X_h^{m+1}-X_h^m)+T_*^m(N_*^{m+1}-N_*^m)\hat e_h^{m+1} \quad \text{at the nodes,} \tag{4.89}$$

where C_0 is a constant that is independent of κ_l.

Note that n_*^{m+1} is a smooth extension of $n(\cdot,t_{m+1})$ from Γ^{m+1} to a neighborhood of Γ^{m+1} which contains Γ^m for sufficiently small τ, and the gradient of n_*^{m+1} is bounded uniformly with respect to m and τ. By considering both $n_*^{m+1}=n(\cdot,t_{m+1})\circ\hat X_{h,*}^{m+1}$ and $n_*^m=n(\cdot,t_m)\circ\hat X_{h,*}^m$ as functions defined on $\hat\Gamma_{h,*}^m$, and using estimates in (4.92) and (4.84), we have

$$\begin{aligned}
|n_*^{m+1}-n_*^m| &= |n(\hat X_{h,*}^{m+1},t_{m+1})-n(\hat X_{h,*}^m,t_m)|\\
&= |n_*^{m+1}(\hat X_{h,*}^{m+1})-n_*^{m+1}(\hat X_{h,*}^m)+n_*^{m+1}(\hat X_{h,*}^m)-n_*^{m+1}(X_{h,*}^{m+1})\\
&\quad +n(X_{h,*}^{m+1},t_{m+1})-n(\hat X_{h,*}^m,t_m)|\\
&\lesssim |\hat X_{h,*}^{m+1}-\hat X_{h,*}^m|+|\hat X_{h,*}^m-X_{h,*}^{m+1}|+\tau \quad \text{at the nodes}\\
&\lesssim |\hat X_{h,*}^{m+1}-\hat X_{h,*}^m|+\tau \quad \text{at the nodes,}
\end{aligned} \tag{4.90}$$

where the second to last inequality uses the smoothness of n_*^{m+1} in a neighborhood of Γ^{m+1}, and the last inequality uses (4.84).

Combining (4.87)–(4.89) with the velocity estimates, we derive

$$\begin{aligned}
&\|\hat X_{h,*}^{m+1}-\hat X_{h,*}^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}\\
&\le \|I_hN_*^m(\hat X_{h,*}^{m+1}-\hat X_{h,*}^m)\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}+\|I_hT_*^m(\hat X_{h,*}^{m+1}-\hat X_{h,*}^m)\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}\\
&\lesssim \tau+h^{-1}(\tau^2+\|I_hT_*^m(\hat X_{h,*}^{m+1}-\hat X_{h,*}^m)\|_{L^\infty(\hat\Gamma_{h,*}^m)}^2)+\|I_hT_*^m(\hat X_{h,*}^{m+1}-\hat X_{h,*}^m)\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}\\
&\qquad\qquad \text{(inverse inequality and (4.87)–(4.88) are used)}\\
&\lesssim \tau+\|I_hT_*^m(\hat X_{h,*}^{m+1}-\hat X_{h,*}^m)\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)} \quad \text{((4.83) is used)}\\
&\le \tau+\|I_hT_*^m(X_h^{m+1}-X_h^m)\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}+\|I_hT_*^m((N_*^{m+1}-N_*^m)\hat e_h^{m+1})\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}\\
&\qquad\qquad \text{((4.89) is used)}\\
&\lesssim \tau+\|I_hT_*^m(e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m)\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)} \quad \text{((4.40) is used)}\\
&\quad +h^{-1}(\tau+\|\hat X_{h,*}^{m+1}-\hat X_{h,*}^m\|_{L^\infty(\hat\Gamma_{h,*}^m)})\|\hat e_h^{m+1}\|_{L^\infty(\hat\Gamma_{h,*}^m)} \quad \text{((4.90) is used)}\\
&\lesssim \tau+\|\nabla_{\hat\Gamma_{h,*}^m}I_h\bar T_{h,*}^m(e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m)\|_{L^\infty(\hat\Gamma_{h,*}^m)}\\
&\quad +h^{k-1}\|I_h\bar N_{h,*}^m(e_h^{m+1}-\hat e_h^m-\tau I_hT_*^mv^m)\|_{L^\infty(\hat\Gamma_{h,*}^m)} \quad \text{(Lemma 4.8 is used)}\\
&\quad +h^{-1}(\tau+\|\hat X_{h,*}^{m+1}-\hat X_{h,*}^m\|_{L^\infty(\hat\Gamma_{h,*}^m)})\|\hat e_h^{m+1}\|_{L^\infty(\hat\Gamma_{h,*}^m)}\\
&\lesssim \tau+h^{-3/2}\tau(\tau+h^{k+1})+h^{-3/2}\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}+h^{-7/2}\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2\\
&\qquad\qquad \text{(inverse inequality, (4.70) and (4.71) are used)}\\
&\quad +h^{-1}(\tau+\|\hat X_{h,*}^{m+1}-\hat X_{h,*}^m\|_{L^\infty(\hat\Gamma_{h,*}^m)})\|\hat e_h^{m+1}\|_{L^\infty(\hat\Gamma_{h,*}^m)}
\end{aligned}$$

$$\lesssim \tau + h^{0.75}\|\hat X_{h,*}^{m+1} - \hat X_{h,*}^m\|_{L^\infty(\hat\Gamma_{h,*}^m)}, \tag{4.91}$$

where the last inequality follows from the induction assumption $\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \lesssim h^{1.75}$ and the estimate $\|\hat e_h^{m+1}\|_{L^\infty(\hat\Gamma_{h,*}^m)} \lesssim h^{1.75}$ in (4.82). By absorbing $h^{0.75}\|\hat X_{h,*}^{m+1} - \hat X_{h,*}^m\|_{L^\infty(\hat\Gamma_{h,*}^m)}$ into the left-hand side, we get

$$\|\hat X_{h,*}^{m+1} - \hat X_{h,*}^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)} \lesssim \tau. \tag{4.92}$$

This implies the norm equivalence between $\hat\Gamma_{h,*}^m$ and $\hat\Gamma_{h,*}^{m+1}$ according to Lemma 3.1. Moreover,

$$\begin{aligned}
&\|X_h^{m+1} - X_h^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)} \\
&= \|e_h^{m+1} - \hat e_h^m - \tau I_h(H^m n^m - g^m)\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)} \quad \text{(relation (3.38) is used)} \\
&\lesssim \tau + h^{-1/2}\|e_h^{m+1}\|_{H^1(\hat\Gamma_{h,*}^m)} + h^{-1/2}\|\hat e_h^m\|_{H^1(\hat\Gamma_{h,*}^m)} \\
&\lesssim h^{1.25},
\end{aligned} \tag{4.93}$$

where the last inequality uses (3.10) and (4.81). This implies the norm equivalence between Γ_h^m and Γ_h^{m+1} according to Lemma 3.1.

The norm equivalence between Γ_h^m and $\hat\Gamma_{h,*}^m$ is a consequence of the induction assumption $\|\hat e_h^m\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)} \lesssim h^{1.25}$ in (3.10), and the norm equivalence between Γ_h^m and $\Gamma_{h,*}^m$ follows from (4.84). Therefore, the norms of finite element functions on $\Gamma_h^m, \Gamma_h^{m+1}, \hat\Gamma_{h,*}^m, \hat\Gamma_{h,*}^{m+1}$ and $\Gamma_{h,*}^{m+1}$ with a common nodal vector are all equivalent.

To distinguish the domain of definition more clearly, we temporarily denote by $\hat X_{h,*}^{m+1} : \Gamma_{h,\mathrm{f}}^0 \to \hat\Gamma_{h,*}^{m+1}$ and $\hat Y_{h,*}^{m+1} : \hat\Gamma_{h,*}^m \to \hat\Gamma_{h,*}^{m+1}$ the finite element functions with the same nodal vector but defined on $\Gamma_{h,\mathrm{f}}^0$ and $\hat\Gamma_{h,*}^m$, respectively. Then (4.92) can be written as

$$\|\hat Y_{h,*}^{m+1} - \mathrm{id}\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)} \lesssim \tau.$$

As a result, for sufficiently small τ, the map $\hat Y_{h,*}^{m+1} = \mathrm{id} + (\hat Y_{h,*}^{m+1} - \mathrm{id})$ is invertible and satisfies that $\|(\hat Y_{h,*}^{m+1})^{-1}\|_{W^{1,\infty}(\hat\Gamma_{h,*}^{m+1})} \lesssim 1$. From (4.92) we conclude that, by using the triangle inequality and the inverse inequality,

$$\|\hat X_{h,*}^{m+1}\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \lesssim \|\hat X_{h,*}^m\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} + h^{-k+1}\|\hat X_{h,*}^{m+1} - \hat X_{h,*}^m\|_{W^{1,\infty}(\Gamma_{h,\mathrm{f}}^0)} \lesssim 1. \tag{4.94}$$

Since $\hat X_{h,*}^{m+1} = \hat Y_{h,*}^{m+1} \circ \hat X_{h,*}^m$, it follows that

$$\begin{aligned}
\|(\hat X_{h,*}^{m+1})^{-1}\|_{W^{1,\infty}(\hat\Gamma_{h,*}^{m+1})} &= \|(\hat X_{h,*}^m)^{-1} \circ (\hat Y_{h,*}^{m+1})^{-1}\|_{W^{1,\infty}(\hat\Gamma_{h,*}^{m+1})} \\
&\le \|(\hat X_{h,*}^m)^{-1}\|_{W^{1,\infty}(\hat\Gamma_{h,*}^m)}\|(\hat Y_{h,*}^{m+1})^{-1}\|_{W^{1,\infty}(\hat\Gamma_{h,*}^{m+1})} \lesssim 1.
\end{aligned} \tag{4.95}$$

The estimates in (4.94)–(4.95) imply that the constant κ_l defined in (3.1) satisfies that

$$\kappa_{l+1} \le C_{\kappa_l}. \tag{4.96}$$

As a result, all the estimates in Section 3 proved for $\hat\Gamma_{h,*}^m$ also hold for $\hat\Gamma_{h,*}^{m+1}$ (with some constants depending only on κ_l). In particular, (3.7) and Lemma 3.8 hold at time level $m+1$, and therefore

$$\|\bar{n}_{h,*}^{m+1} - \hat{n}_{h,*}^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^{m+1})} \lesssim h^k. \tag{4.97}$$

4.7. **Stability of orthogonal projection on the error.** In this subsection we establish the stability of converting $\|e_h^{m+1}\cdot\bar{n}_{h,*}^m\|^2_{L_h^2(\hat{\Gamma}_{h,*}^m)}$ to $\|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m+1}\|^2_{L_h^2(\hat{\Gamma}_{h,*}^{m+1})}$ at each time level. We decompose their difference into the following five parts:

$$\begin{aligned}
&\|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m+1}\|^2_{L_h^2(\hat{\Gamma}_{h,*}^{m+1})} - \|e_h^{m+1}\cdot\bar{n}_{h,*}^m\|^2_{L_h^2(\hat{\Gamma}_{h,*}^m)}\\
&= \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m+1}\|^2_{L_h^2(\hat{\Gamma}_{h,*}^{m+1})} - \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m+1}\|^2_{L_h^2(\hat{\Gamma}_{h,*}^m)} \quad (\text{change of } \hat{\Gamma}_{h,*}^{m+1} \text{ to } \hat{\Gamma}_{h,*}^m)\\
&\quad + \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m+1}\|^2_{L_h^2(\hat{\Gamma}_{h,*}^m)} - \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^m\|^2_{L_h^2(\hat{\Gamma}_{h,*}^m)} \quad (\text{change of } \bar{n}_{h,*}^{m+1} \text{ to } \bar{n}_{h,*}^m)\\
&\quad + \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^m\|^2_{L_h^2(\hat{\Gamma}_{h,*}^m)} - \|e_h^{m+1}\cdot\bar{n}_{h,*}^m\|^2_{L_h^2(\hat{\Gamma}_{h,*}^m)} \quad (\text{change of } \hat{e}_h^{m+1} \text{ to } e_h^{m+1})\\
&=: M_1^m + M_2^m + M_3^m.
\end{aligned} \tag{4.98}$$

By the fundamental theorem of calculus, (4.92) and the norm equivalence of curves $\hat{\Gamma}_{h,*}^m$ and $\hat{\Gamma}_{h,*}^{m+1}$ in Section 4.6, we know

$$\begin{aligned}
M_1^m &= \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m+1}\|^2_{L_h^2(\hat{\Gamma}_{h,*}^{m+1})} - \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m+1}\|^2_{L_h^2(\hat{\Gamma}_{h,*}^m)}\\
&\lesssim \|\nabla_{\hat{\Gamma}_{h,*}^m}(\hat{X}_h^{m+1} - \hat{X}_h^m)\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}\\
&\lesssim \tau\|\hat{e}_h^{m+1}\|^2_{L^2(\hat{\Gamma}_{h,*}^m)} \quad (\text{here (4.92) is used}).
\end{aligned} \tag{4.99}$$

The estimation of M_2^m and M_3^m requires Lemma 4.9 which tells us that the L^∞ norms of the quantities n_*^m, $\hat{n}_{h,*}^m$ and $\bar{n}_{h,*}^m$ at adjacent time levels differ at most $O(\tau)$ from each other. This additional $O(\tau)$ will help us to eliminate the factor $\frac{1}{\tau}$ in the fourth line of (4.115).

Lemma 4.9. *We have the following estimates for the difference between normal vectors at two adjacent time levels:*

$$|n_*^{m+1} - n_*^m| \lesssim \tau \quad \textit{at the nodes}, \tag{4.100}$$

$$\|\hat{n}_{h,*}^{m+1} - \hat{n}_{h,*}^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \lesssim \tau, \tag{4.101}$$

$$\|\bar{n}_{h,*}^{m+1} - \bar{n}_{h,*}^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \lesssim \tau. \tag{4.102}$$

Proof. Note that n_*^{m+1} is a smooth extension of $n(\cdot, t_{m+1})$ from Γ^{m+1} to a neighborhood of Γ^{m+1} which contains Γ^m when τ is sufficiently small, and the gradient of n_*^{m+1} is bounded uniformly with respect to m and τ. From (4.90), (4.92) and (4.84), it follows that

$$\begin{aligned}
|n_*^{m+1} - n_*^m| &\lesssim |\hat{X}_{h,*}^{m+1} - \hat{X}_{h,*}^m| + |\hat{X}_{h,*}^m - X_{h,*}^{m+1}| + \tau \quad \text{at the nodes}\\
&\lesssim \tau \quad \text{at the nodes.}
\end{aligned} \tag{4.103}$$

The second and the third results in Lemma 4.9 follow from Lemma 3.14 (item 7), (4.30)–(4.31) and the norm equivalences in Section 4.6, i.e.,

$$\begin{aligned}
&\|\hat{n}_{h,*}^{m+1} - \hat{n}_{h,*}^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|\bar{n}_{h,*}^{m+1} - \bar{n}_{h,*}^m\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}\\
&\lesssim \|\nabla_{\hat{\Gamma}_{h,*}^m}(\hat{X}_{h,*}^{m+1} - \hat{X}_{h,*}^m)\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} \lesssim \tau,
\end{aligned} \tag{4.104}$$

where the last inequality follows from (4.92). □

The third result in Lemma 4.9 implies that

$$M_2^m = \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m+1}\|^2_{L^2_h(\hat{\Gamma}^m_{h,*})} - \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m}\|^2_{L^2_h(\hat{\Gamma}^m_{h,*})} \lesssim \tau\|\hat{e}_h^{m+1}\|^2_{L^2(\hat{\Gamma}^m_{h,*})}. \tag{4.105}$$

We decompose M_3^m into several parts as follows:

$$\begin{aligned}
M_3^m &= \|\hat{e}_h^{m+1}\cdot\bar{n}_{h,*}^{m}\|^2_{L^2_h(\hat{\Gamma}^m_{h,*})} - \|e_h^{m+1}\cdot\bar{n}_{h,*}^{m}\|^2_{L^2_h(\hat{\Gamma}^m_{h,*})} \\
&= \int_{\hat{\Gamma}^m_{h,*}}^h (\hat{e}_h^{m+1} - e_h^{m+1})\cdot\bar{n}_{h,*}^m(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&= \int_{\hat{\Gamma}^m_{h,*}}^h \Big(I_hT_*^{m+1}(\hat{e}_h^{m+1} - e_h^{m+1}) + f_h\Big)\cdot\hat{n}_{h,*}^m(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&= -\int_{\hat{\Gamma}^m_{h,*}}^h I_hT_*^{m+1}e_h^{m+1}\cdot(\hat{n}_{h,*}^m - n_*^m)(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&\quad + \int_{\hat{\Gamma}^m_{h,*}}^h I_hT_*^{m+1}e_h^{m+1}\cdot(n_*^{m+1} - n_*^m)(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&\quad + \int_{\hat{\Gamma}^m_{h,*}}^h f_h\cdot\hat{n}_{h,*}^m(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&= -\int_{\hat{\Gamma}^m_{h,*}}^h I_h(T_*^{m+1}-T_*^m)e_h^{m+1}\cdot(\hat{n}_{h,*}^m - n_*^m)(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&\quad - \int_{\hat{\Gamma}^m_{h,*}}^h I_hT_*^m(e_h^{m+1} - \hat{e}_h^m - \tau I_hT_*^m v^m)\cdot(\hat{n}_{h,*}^m - n_*^m)(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&\quad - \Big(\int_{\hat{\Gamma}^m_{h,*}}^h - \int_{\hat{\Gamma}^m_{h,*}}\Big)\tau I_hT_*^m v^m\cdot(\hat{n}_{h,*}^m - n_*^m)(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&\quad - \int_{\hat{\Gamma}^m_{h,*}} \tau I_hT_*^m v^m\cdot(\hat{n}_{h,*}^m - n_*^m)(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&\quad + \int_{\hat{\Gamma}^m_{h,*}}^h I_hT_*^{m+1}e_h^{m+1}\cdot(n_*^{m+1} - n_*^m)(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&\quad + \int_{\hat{\Gamma}^m_{h,*}}^h f_h\cdot\hat{n}_{h,*}^m(\hat{e}_h^{m+1}+e_h^{m+1})\cdot\bar{n}_{h,*}^m \\
&=: \sum_{i=1}^6 M_{3i}^m,
\end{aligned} \tag{4.106}$$

where we have applied (3.33) and (3.16) in the third equality, and have used the nodal orthogonality relation in the fourth equality.

Lemma 4.9 directly implies

$$M_{31}^m \lesssim \tau(\|\hat{e}_h^{m+1}\|^2_{L^2(\hat{\Gamma}^m_{h,*})} + \|e_h^{m+1}\|^2_{L^2(\hat{\Gamma}^m_{h,*})}), \tag{4.107}$$

$$M_{35}^m \lesssim \tau(\|\hat{e}_h^{m+1}\|^2_{L^2(\hat{\Gamma}^m_{h,*})} + \|e_h^{m+1}\|^2_{L^2(\hat{\Gamma}^m_{h,*})}). \tag{4.108}$$

By the velocity estimates in (4.70)–(4.71), we derive that

$$\begin{aligned}
M_{32}^m &\lesssim \tau h^k \Big\| \frac{e_h^{m+1} - \hat{e}_h^m}{\tau} - I_h T_*^m v^m \Big\|_{L^2(\hat{\Gamma}_{h,*}^m)} (\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\lesssim \tau h^{k-1}(\tau + h^{k+1})(\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\quad + \tau h^k \big(h^{-1}\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} \\
&\quad + h^{-3}\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}^2\big)(\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\lesssim \tau h^{k-1}(\tau + h^{k+1})(\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\quad + \tau(h^{k-1}\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + h^{k-3}\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}^2) \\
&\qquad\qquad (\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\lesssim \tau h^{k-1}(\tau + h^{k+1}) \\
&\qquad\qquad (\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\quad + \tau h^{k-1}\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}(\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\qquad\qquad + \tau h^{k-0.25}\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}^2,
\end{aligned} \tag{4.109}$$

where we have used the estimate $\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} \lesssim h^{2.75}$ which follows from (4.76), (4.78) and the induction assumption in (3.9). The superconvergence estimate in Lemma 3.5 leads to

$$M_{33}^m \lesssim \tau h^{k+1}(\|\hat{e}_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)}), \tag{4.110}$$

and, applying Lemma 4.1, we obtain

$$M_{34}^m \lesssim \tau h^{k+1}(\|\hat{e}_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{H^1(\hat{\Gamma}_{h,*}^m)}). \tag{4.111}$$

Finally, by using the estimates in (3.34) and Lemma 4.9, as well as the relation $(1 - n_*^m (n_*^m)^\top)\hat{e}_h^m = 0$ at the nodes, we have

$$\begin{aligned}
M_{36}^m &= \int_{\hat{\Gamma}_{h,*}^m}^h f_h \cdot \hat{n}_{h,*}^m (\hat{e}_h^{m+1} + e_h^{m+1}) \cdot \bar{n}_{h,*}^m \\
&\lesssim \|(1 - n_*^{m+1}(n_*^{m+1})^\top) e_h^{m+1}\|_{L_h^2(\hat{\Gamma}_{h,*}^m)}^2 (\|\hat{e}_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}) \\
&\lesssim \|(1 - n_*^m(n_*^m)^\top)(e_h^{m+1} - \hat{e}_h^m)\|_{L_h^2(\hat{\Gamma}_{h,*}^m)}^2 (\|\hat{e}_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}) \\
&\quad + \|(n_*^{m+1}(n_*^{m+1})^\top - n_*^m(n_*^m)^\top) e_h^{m+1}\|_{L_h^2(\hat{\Gamma}_{h,*}^m)}^2 (\|\hat{e}_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}) \\
&\lesssim \tau^2 \Big(\Big\| \frac{e_h^{m+1} - \hat{e}_h^{m+1}}{\tau} - I_h T_*^m v^m \Big\|_{L_h^2(\hat{\Gamma}_{h,*}^m)}^2 + 1 \Big) (\|\hat{e}_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^\infty(\hat{\Gamma}_{h,*}^m)}) \\
&\lesssim \tau^2 (\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\quad + \tau^2 (\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|\nabla_{\hat{\Gamma}_{h,*}^m} e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}),
\end{aligned}$$

where the boundedness of $\big\|\frac{e_h^{m+1} - \hat{e}_h^m}{\tau} - I_h T_*^m v^m\big\|_{L^2(\hat{\Gamma}_{h,*}^m)}$ comes from a combination of the velocity estimates (4.62) and (4.70) as well as the induction assumption $\|\hat{e}_h^m\|_{H^1(\hat{\Gamma}_{h,*}^m)} \lesssim h^{1.75}$. Therefore, by using Young's inequality, we have

$$\begin{aligned}
M_{36}^m &\lesssim \tau^2 (\|\hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)} + \|e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}) \\
&\quad + \epsilon^{-1}\tau^3 + \epsilon\tau (\|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}^2 + \|\nabla_{\hat{\Gamma}_{h,*}^m} e_h^{m+1}\|_{L^2(\hat{\Gamma}_{h,*}^m)}^2).
\end{aligned} \tag{4.112}$$

By collecting the estimates of M_{3j}^m, $j = 1, \ldots, 6$, we obtain the following estimate:

$$
\begin{aligned}
M_3^m \lesssim\ & \epsilon^{-1}\tau^3 + \tau(\|\hat e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|\hat e_h^{m}\|^2_{L^2(\hat\Gamma^m_{h,*})}) \\
& + (\epsilon + h^{k-0.25} + h^{2k-2})\tau \\
& \quad (\|\nabla_{\hat\Gamma^m_{h,*}} \hat e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|\nabla_{\hat\Gamma^m_{h,*}} e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e_h^{m}\|^2_{L^2(\hat\Gamma^m_{h,*})}) \\
& + \tau h^{k+1}(\|\hat e_h^{m+1}\|_{H^1(\hat\Gamma^m_{h,*})} + \|e_h^{m+1}\|_{H^1(\hat\Gamma^m_{h,*})}).
\end{aligned}
\tag{4.113}
$$

4.8. **Error estimates.** Note that

$$
\begin{aligned}
& \frac{1}{\tau}(\|\hat e_h^{m+1}\cdot \bar n_{h,*}^{m+1}\|^2_{L^2_h(\hat\Gamma^{m+1}_{h,*})} - \|\hat e_h^{m}\cdot \bar n_{h,*}^{m}\|^2_{L^2_h(\hat\Gamma^{m}_{h,*})}) + A_{h,*}(e_h^{m+1}, e_h^{m+1}) \\
& = \frac{1}{\tau}(\|e_h^{m+1}\cdot \bar n_{h,*}^{m}\|^2_{L^2_h(\hat\Gamma^{m}_{h,*})} - \|\hat e_h^{m}\cdot \bar n_{h,*}^{m}\|^2_{L^2_h(\hat\Gamma^{m}_{h,*})}) + A_{h,*}(e_h^{m+1}, e_h^{m+1}) \\
& \quad + \frac{1}{\tau}(\|\hat e_h^{m+1}\cdot \bar n_{h,*}^{m+1}\|^2_{L^2_h(\hat\Gamma^{m+1}_{h,*})} - \|e_h^{m+1}\cdot \bar n_{h,*}^{m}\|^2_{L^2_h(\hat\Gamma^{m}_{h,*})}),
\end{aligned}
\tag{4.114}
$$

where the first line on the right-hand side above can be estimated by choosing $\phi_h = e_h^{m+1}$ in the error equation (4.21) and using the estimates of the linear and bilinear forms developed in Sections 4.1 and 4.2. The second line on the right-hand side above can be estimated by using (4.98) and the estimates of M_j^m, $j = 1, \ldots, 3$ in Section 4.7. This leads to the following result:

$$
\begin{aligned}
& \frac{1}{\tau}(\|\hat e_h^{m+1}\cdot \bar n_{h,*}^{m+1}\|^2_{L^2_h(\hat\Gamma^{m+1}_{h,*})} - \|\hat e_h^{m}\cdot \bar n_{h,*}^{m}\|^2_{L^2_h(\hat\Gamma^{m}_{h,*})}) + A_{h,*}(e_h^{m+1}, e_h^{m+1}) \\
& \lesssim A^T_{h,*}(\hat e_h^m, \hat e_h^m) - B^m(\hat e_h^m, e_h^{m+1}) - J^m(e_h^{m+1}) - K^m(e_h^{m+1}) - d^m(e_h^{m+1}) \\
& \quad - \sum_{i=1}^3 F_i^m(e_h^{m+1}) + A^N_{h,*}(\hat e_h^m, I_h\bar T^m_{h,*}e_h^{m+1}) + B^m(\hat e_h^m, I_h\bar T^m_{h,*}e_h^{m+1}) + Q^m(I_h\bar T^m_{h,*}e_h^{m+1}) \\
& \quad + \frac{1}{\tau}\sum_{i=1}^3 M_i^m \quad \text{(here (4.98) is used)} \\
& \lesssim \epsilon^{-1}(\tau + h^{k+1})^2 + \epsilon^{-1}(\|\hat e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|\hat e_h^{m}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|e_h^{m}\|^2_{L^2(\hat\Gamma^m_{h,*})}) \\
& \quad + \epsilon(\|\nabla_{\hat\Gamma^m_{h,*}} e_h^{m}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e_h^{m}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|\nabla_{\hat\Gamma^m_{h,*}} e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \|\nabla_{\hat\Gamma^m_{h,*}} \hat e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})}).
\end{aligned}
\tag{4.115}
$$

Then, by using the results in (3.44), (4.73) and (4.77), we can simplify (4.115) to the following inequality:

$$
\begin{aligned}
& \frac{\|\hat e_h^{m+1}\cdot \bar n_{h,*}^{m+1}\|^2_{L^2_h(\hat\Gamma^{m+1}_{h,*})} - \|\hat e_h^{m}\cdot \bar n_{h,*}^{m}\|^2_{L^2_h(\hat\Gamma^{m}_{h,*})}}{2\tau} + C^{-1}\|\nabla_{\hat\Gamma^m_{h,*}} \hat e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} \\
& \lesssim \epsilon^{-1}(\tau + h^{k+1})^2 + \epsilon^{-1}\|\hat e_h^{m}\cdot \bar n_{h,*}^{m}\|^2_{L^2_h(\hat\Gamma^{m}_{h,*})} + \epsilon\|\nabla_{\hat\Gamma^m_{h,*}} \hat e_h^{m}\|^2_{L^2(\hat\Gamma^m_{h,*})},
\end{aligned}
\tag{4.116}
$$

where ϵ is an arbitrary small constant. The last term in (4.116) can be absorbed by its left-hand side. Then, by applying the discrete Grönwall's inequality, the norm equivalence in Section 4.6 and (3.44), we obtain the following error estimate:

$$
\max_{0\le m\le l} \|\hat e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} + \sum_{m=0}^{l} \tau\|\nabla_{\hat\Gamma^m_{h,*}} \hat e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})} \le C(\tau + h^{k+1})^2. \tag{4.117}
$$

For $h^{2k} \lesssim \tau \lesssim h^{k+1}$ and sufficiently small h, from (4.117) we can recover the induction hypothesis (3.9) at time level t_{m+1}. In view of (4.78)–(4.79), we also obtain the following result:

$$\max_{0\le m\le l}\|e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})}+\sum_{m=0}^{l}\tau\|\nabla_{\hat\Gamma^m_{h,*}}e_h^{m+1}\|^2_{L^2(\hat\Gamma^m_{h,*})}\le C(\tau+h^{k+1})^2. \tag{4.118}$$

Note that the constants C on the right-hand side of (4.117) and (4.118) depend on the κ_l defined in (3.1), and the condition on the mesh size under which the error estimates are established is $h\le h_{\kappa_l}$ (for some constant h_{κ_l} which may depend on κ_l). In order to conclude Theorem 2.1, it remains to show that the constant κ_l defined in (3.1) is independent of τ and l (though possibly depending on T). This is presented in the next subsection.

4.9. **Uniform boundedness of κ_l.** For any $j=0,1,\dots,k$, we can prove that

$$\max_{0\le m\le l}\|\hat X^m_{h,*}\|_{W_h^{j,\infty}(\Gamma^0_{h,\mathrm{f}})}\le C_0'\quad\text{if}\quad\max_{0\le m\le l}\|\hat X^m_{h,*}\|_{W_h^{j-1,\infty}(\Gamma^0_{h,\mathrm{f}})}\le C_0,$$

where C_0 and C_0' are constants which are independent of τ, h and κ_l (with C_0' depending on C_0). For illustration, however, we only prove $\|\hat X^m_{h,*}\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}\le C_0'$ under the condition $\|\hat X^m_{h,*}\|_{W_h^{k-1,\infty}(\Gamma^0_{h,\mathrm{f}})}\le C_0$. The case $j\ne k$ can be proved similarly; see [2, Appendix].

In this subsection, we regard $\hat X^m_{h,*}$ and X^m_h as the maps from the piecewise flat curve $\Gamma^0_{h,\mathrm{f}}$ to $\hat\Gamma^m_{h,*}$ and Γ^m_h, respectively. Let $v^m_{\mathrm{f}}=v^m\circ a^m\circ\hat X^m_{h,*}$ and $g^m_{\mathrm{f}}=g^m\circ a^m\circ\hat X^m_{h,*}$, which are functions defined on the piecewise flat curve $\Gamma^0_{h,\mathrm{f}}$. By using relations (4.87)–(4.89), we have

$$\begin{aligned}
&\|\hat X^{m+1}_{h,*}-\hat X^m_{h,*}\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}\\
&\le\|I_h[(X^{m+1}-\mathrm{id})\circ a^m\circ\hat X^m_{h,*}]\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}+\|\rho_h\circ\hat X^m_{h,*}\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}\\
&\quad+\|I_h[I_h(T^m_*\circ\hat X^m_{h,*})I_h(N^{m+1}_*\circ\hat X^{m+1}_{h,*}-N^m_*\circ\hat X^m_{h,*})(\hat e^{m+1}_h\circ\hat X^m_{h,*})]\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}\\
&\quad+\|I_h[(T^m_*\circ\hat X^m_{h,*})(X^{m+1}_h-X^m_h)]\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}\\
&=:E^m_1+E^m_2+E^m_3.
\end{aligned}$$

By using the stability of I_h on $C^0(\Gamma^0_{h,\mathrm{f}})\cap W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})$, chain rule, the inverse inequality and (4.88), we have

$$\begin{aligned}
E^m_1&\le C_0\|(X^{m+1}-\mathrm{id})\circ a^m\circ\hat X^m_{h,*}\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}+\|\rho_h\circ\hat X^m_{h,*}\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}\\
&\le C_0\|\nabla_{\hat\Gamma^m_{h,*}}[(X^{m+1}-\mathrm{id})\circ a^m]\circ\hat X^m_{h,*}\nabla^k_{\Gamma^0_{h,\mathrm{f}}}\hat X^m_{h,*}\|_{L_h^\infty(\Gamma^0_{h,\mathrm{f}})}\\
&\quad+C_0\sum_{j=2}^{k-1}\|X^{m+1}-\mathrm{id}\|_{W^{j,\infty}(\Gamma^m)}+C_0h^{-k}\|\rho_h\circ\hat X^m_{h,*}\|_{L^\infty(\Gamma^0_{h,\mathrm{f}})}\\
&\le C_0\tau\|\hat X^m_{h,*}\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}+C_0\tau+C_0h^{-k}\big(\tau^2+\|I_hT^m_*(\hat X^{m+1}_{h,*}-\hat X^m_{h,*})\|^2_{L^\infty(\hat\Gamma^m_{h,*})}\big)\\
&\le C_0\tau\|\hat X^m_{h,*}\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}+C_0\tau+C_0h^{-k}\|I_hT^m_*(\hat X^{m+1}_{h,*}-\hat X^m_{h,*}-\tau I_hT^m_*v^m)\|^2_{H^1(\hat\Gamma^m_{h,*})}\\
&\le C_0\tau\|\hat X^m_{h,*}\|_{W_h^{k,\infty}(\Gamma^0_{h,\mathrm{f}})}+C_0\tau,
\end{aligned}$$

where the last inequality follows from the velocity estimates (4.70), (4.71) and Lemma 4.8. Furthermore, using the inverse inequality, we have

$$\begin{aligned} E_2^m &\le C_0 h^{-k-\frac12} \| I_h [I_h (T_*^m \circ \hat X_{h,*}^m) I_h (N_*^{m+1} \circ \hat X_{h,*}^{m+1} - N_*^m \circ \hat X_{h,*}^m)(\hat e_h^{m+1} \circ \hat X_{h,*}^m)] \|_{L_h^2(\Gamma_{h,\mathrm f}^0)} \\ &\le C_0 h^{-k-\frac12} C_{\kappa_l} \tau C_{\kappa_l} (\tau + h^{k+1}), \end{aligned}$$

where we have used the estimate $\| N_*^{m+1} \circ \hat X_{h,*}^{m+1} - N_*^m \circ \hat X_{h,*}^m \|_{L_h^\infty(\Gamma_{h,\mathrm f}^0)} \le C_{\kappa_l} \tau$ which follows from (4.90) and (4.92), and the estimate $\| \hat e_h^{m+1} \circ \hat X_{h,*}^m \|_{L_h^2(\Gamma_{h,\mathrm f}^0)} \le C_{\kappa_l}(\tau + h^{k+1})$ which follows from the error estimate (4.117). Under the condition $\tau \le c h^{k+1}$ we obtain

$$E_2^m \le C_{\kappa_l} h^{\frac12} \tau \le C_0 \tau \quad \text{under the condition } C_{\kappa_l} h^{\frac12} \le 1.$$

With the above estimates of E_1^m and E_2^m, we have

$$\begin{aligned} &\| \hat X_{h,*}^{m+1} - \hat X_{h,*}^m \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \\ &\le C_0 \tau (1 + \| \hat X_{h,*}^m \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)}) + C_0 \| I_h [(T_*^m \circ \hat X_{h,*}^m)(X_h^{m+1} - X_h^m)] \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)}. \end{aligned}$$

Using relation (4.40) we can estimate the last term in the above inequality as follows:

$$\begin{aligned} &\| I_h [(T_*^m \circ \hat X_{h,*}^m)(X_h^{m+1} - X_h^m)] \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \\ &\le \| I_h [(T_*^m \circ \hat X_{h,*}^m)(X_h^{m+1} - X_h^m - \tau I_h v_{\mathrm f}^m)] \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \\ &\quad + \tau \| I_h [(T_*^m \circ \hat X_{h,*}^m) v_{\mathrm f}^m] \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \\ &= \| I_h [(T_*^m \circ \hat X_{h,*}^m)(e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v_{\mathrm f}^m + \tau I_h g_{\mathrm f}^m)] \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \\ &\quad + \tau \| I_h [(T_*^m \circ \hat X_{h,*}^m) v_{\mathrm f}^m] \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \\ &\le \| I_h [(T_*^m \circ \hat X_{h,*}^m)(e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v_{\mathrm f}^m)] \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \\ &\quad + C_0 \tau (C_{\kappa_l} h^{-k+1} \tau + 1 + \| \hat X_h^m \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)}), \end{aligned}$$

where the last inequality follows from the following estimates:

$$\begin{aligned} &\| I_h g_{\mathrm f}^m \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \le h^{-k+1} \| I_h g_{\mathrm f}^m \|_{W^{1,\infty}(\Gamma_{h,\mathrm f}^0)} \le C_{\kappa_l} h^{-k+1} \tau \quad \text{(in view of (3.37))}, \\ &\| (T_*^m \circ \hat X_{h,*}^m) v_{\mathrm f}^m \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \le C_0 (1 + \| \hat X_{h,*}^m \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)}) \qquad \text{(chain rule of differentation)}. \end{aligned}$$

Therefore, by using the inverse inequality, under the condition $\tau \le c h^{k+1}$ and $C_{\kappa_l} h^{\frac12} \le 1$, we have

$$\begin{aligned} &\| I_h [(T_*^m \circ \hat X_{h,*}^m)(X_h^{m+1} - X_h^m)] \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)} \\ &\le C_0 h^{-k+1/2} \| I_h [(T_*^m \circ \hat X_{h,*}^m)(e_h^{m+1} - \hat e_h^m - \tau I_h T_*^m v_{\mathrm f}^m)] \|_{H^1(\Gamma_{h,\mathrm f}^0)} \\ &\quad + C_0 \tau (1 + \| \hat X_h^m \|_{W_h^{k,\infty}(\Gamma_{h,\mathrm f}^0)}) \\ &\le C_{\kappa_l} h^{-k-1/2} \tau (\tau + h^{k+1}) + C_{\kappa_l} h^{-k-1/2} \tau \| \nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m \|_{L^2(\hat\Gamma_{h,*}^m)} \\ &\quad + C_{\kappa_l} h^{-k-5/2} \tau \| \nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m \|_{L^2(\hat\Gamma_{h,*}^m)}^2 + C_0 \tau (1 + \| \hat X_h^m \|_{W^{k,\infty}(\Gamma_{h,\mathrm f}^0)}), \end{aligned}$$

where Lemma 4.8 and (4.70)–(4.71) are used in the last inequality. Then, using the error estimate in (4.117) and the stepsize condition $\tau \le c h^{k+1}$, and $\| \nabla_{\hat\Gamma_{h,*}^m} \hat e_h^m \|_{L^2(\hat\Gamma_{h,*}^m)}$

$\le C_{\kappa_l} h^{1.75}$, we have

$$\begin{aligned}
&\|I_h[(T_*^m \circ \hat{X}_{h,*}^m)(X_h^{m+1} - X_h^m)]\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \\
&\le C_{\kappa_l} h^{-k-1/2}\tau \|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + C_{\kappa_l} h^{-k-5/2}\tau \|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}^2 \\
&\quad + C_0 \tau (1 + \|\hat{X}_h^m\|_{W^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)}),
\end{aligned} \tag{4.119}$$

where we have used the mesh size condition $C_{\kappa_l} h^{\frac{1}{2}} \le 1$ again. In view of the estimates above, we have proved the following result:

$$\begin{aligned}
&\|\hat{X}_{h,*}^{m+1} - \hat{X}_{h,*}^m\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \\
&\le C_{\kappa_l} h^{-k-1/2}\tau \|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + C_{\kappa_l} h^{-k-5/2}\tau \|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}^2 \\
&\quad + C_0 \tau (1 + \|\hat{X}_h^m\|_{W^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)}).
\end{aligned} \tag{4.120}$$

Therefore, by using the triangle inequality,

$$\begin{aligned}
&\|\hat{X}_{h,*}^{m+1}\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} - \|\hat{X}_{h,*}^0\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \\
&\le \sum_{j=0}^m \|\hat{X}_{h,*}^{j+1} - \hat{X}_{h,*}^j\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \\
&\le C_{\kappa_l} h^{-k-1/2} \sum_{j=0}^m \tau \|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)} + C_{\kappa_l} h^{-k-5/2} \sum_{j=0}^m \tau \|\nabla_{\hat{\Gamma}_{h,*}^m} \hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}^2 \\
&\quad + C_0 + \sum_{j=0}^m C_0 \tau \|\hat{X}_{h,*}^j\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \\
&\le C_{\kappa_l} h^{-k-1/2}(\tau + h^{k+1}) + C_{\kappa_l} h^{-k-5/2}(\tau + h^{k+1})^2 \quad \text{(here (4.117) is used)} \\
&\quad + C_0 + \sum_{j=0}^m C_0 \tau \|\hat{X}_{h,*}^j\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \\
&\le C_0 + \sum_{j=0}^m C_0 \tau \|\hat{X}_{h,*}^j\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \quad \text{(under condition } C_{\kappa_l} h^{\frac{1}{2}} \le 1).
\end{aligned} \tag{4.121}$$

By applying the discrete Grönwall's inequality, we obtain the following result under the condition $C_{\kappa_l} h^{\frac{1}{2}} \le 1$:

$$\max_{0\le m\le l} \|\hat{X}_{h,*}^{m+1}\|_{W_h^{k,\infty}(\Gamma_{h,\mathrm{f}}^0)} \le C_0. \tag{4.122}$$

The proof of $\|(\hat{X}_{h,*}^{m+1})^{-1}\|_{W^{1,\infty}(\hat{\Gamma}_{h,*}^{m+1})} \le C_0$ is simpler, i.e., the same as [2, Appendix], and therefore omitted. This proves that

$$\kappa_{l+1} \le C_0,$$

with a constant C_0 which is independent of τ and l. This proves the boundedness of the quantity κ_l defined in (3.1). Moreover, the condition $C_{\kappa_l} h^{\frac{1}{2}} \le 1$ is essentially requiring $h \le h_0$ for some constant h_0 independent of l. This completes the proof of Theorem 2.1. ∎

5. Characterization of particle trajectories (Proof of Theorem 2.2)

Let $\{x_{j,\#}(t) : t \in [0,T]\}$ be the trajectory of the particle under the flow determined by (1.10), with initial position $x_j^0 \in \Gamma_h^0$. Let $X_{h,\#}^m$ be the finite element function with nodal vector $(x_{1,\#}(t_m),\dots,x_{J,\#}(t_m))^\top$. Thus $X_{h,\#}^m$ maps the initial curve Γ_h^0 to some finite element curve $\Gamma_{h,\#}^m$ which interpolates the smooth curve Γ^m at the nodes $x_{j,\#}(t_m)$, $j=1,\dots,J$.

Let I_h be the interpolation operator onto the initial approximate curve Γ_h^0. Then the following identity holds at the nodes of Γ_h^0:

$$X_{h,\#}^{m+1} = X_{h,\#}^m + \tau I_h[v^m \circ X_{h,\#}^m] + O(\tau^2),$$

which is simply the Taylor expansion of the flow in (1.10) at the nodes. Therefore, the error $e_{h,\#}^m = X_{h,\#}^m - X_h^m$ satisfies the following relation:

$$\begin{aligned} e_{h,\#}^{m+1} &= e_{h,\#}^m - (X_h^{m+1} - X_h^m - \tau I_h[v^m \circ \hat X_{h,*}^m]) + \tau I_h[v^m \circ X_{h,\#}^m - v^m \circ \hat X_{h,*}^m] + O(\tau^2) \\ &= e_{h,\#}^m - (e_h^{m+1} - \hat e_h^m - \tau I_h[(T_*^m v^m)\circ \hat X_{h,*}^m] + \tau I_h g^m) \quad \text{(here (4.40) is used)} \\ &\quad + \tau I_h[v^m \circ X_{h,\#}^m - v^m \circ \hat X_{h,*}^m] + O(\tau^2). \end{aligned}$$

By using the smoothness of v^m on Γ^m, we have $|v^m \circ X_{h,\#}^m - v^m \circ \hat X_{h,*}^m| \le C(|e_{h,\#}^m| + |\hat e_h^m|)$ and therefore the following inequality holds at the nodes of Γ_h^0:

$$|e_{h,\#}^{m+1}| \le (1+C\tau)|e_{h,\#}^m| + |e_h^{m+1} - \hat e_h^m - \tau I_h[(T_*^m v^m)\circ \hat X_{h,*}^m]| + C\tau^2 + C\tau|\hat e_h^m|.$$

By taking the discrete L^2 norm on Γ_h^0 and using the equivalence between discrete and continuous L^2 norms on Γ_h^0, we have

$$\begin{aligned} \|e_{h,\#}^{m+1}\|_{L_h^2(\Gamma_h^0)} &\le (1+C\tau)\|e_{h,\#}^m\|_{L_h^2(\Gamma_h^0)} + C\|e_h^{m+1} - \hat e_h^m - \tau I_h[(T_*^m v^m)\circ \hat X_{h,*}^m]\|_{L^2(\Gamma_h^0)} \\ &\quad + C\tau^2 + C\tau\|\hat e_h^m\|_{L_h^2(\Gamma_h^0)} \\ &\le (1+C\tau)\|e_{h,\#}^m\|_{L_h^2(\Gamma_h^0)} + Ch^{-1}\tau(\tau + h^{k+1}) \\ &\quad + Ch^{-1}\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} + Ch^{-3}\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2, \end{aligned}$$

where $\|\hat e_h^m\|_{L_h^2(\Gamma_h^0)}$ is estimated by using (4.117). By iterating the inequality above with respect to m (equivalently, using the discrete Grönwall's inequality), we obtain

$$\begin{aligned} \|e_{h,\#}^{m+1}\|_{L_h^2(\Gamma_h^0)} &\le Ch^{-1}(\tau + h^{k+1}) + Ch^{-1}\sum_{m=0}^{l}\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)} \\ &\quad + Ch^{-3}\sum_{m=0}^{l}\tau\|\nabla_{\hat\Gamma_{h,*}^m}\hat e_h^m\|_{L^2(\hat\Gamma_{h,*}^m)}^2 \\ &\le Ch^k + Ch^k + Ch^{2k-1} \le Ch^k, \end{aligned}$$

where we have used $\tau \le ch^{k+1}$ and the error estimate in (4.117). This proves that the particle trajectory produced by the stabilized BGN method converges to the particle trajectory determined by (1.10). The latter minimizes the rate of deformation while maintaining the shape of the curve under curve shortening flow at every time $t \in [0,T]$. This fully characterizes the particle trajectory produced by the stabilized BGN method and gives the first rigorous explanation to why parametric FEMs of the BGN type could maintain mesh quality of the surfaces/curves evolving under curvature flows.

TABLE 1. Rate of convergence of the error with $\tau \sim h^{k+1}$

N	$\max_{1\le m\le N_t} \|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}$		
	$k=1$	$k=2$	$k=3$
2^4	1.11e-1	4.55e-2	4.46e-2
2^5	3.30e-2	7.11e-3	3.60e-3
2^6	8.90e-3	9.18e-4	2.25e-4
Convergence rate	1.89	2.95	4.00

6. NUMERICAL EXPERIMENTS

We test the convergence of the proposed stabilized BGN method in (1.5) for approximating curve shortening flow with the following benchmark example (see [6, Section 4]) of dumbbell shape curve as the initial data

$$\begin{pmatrix} x(\xi) \\ y(\xi) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\xi) \\ 0.9(\cos^2(2\pi\xi)+0.1)\sin(2\pi\xi) \end{pmatrix}, \quad \xi \in [0,1]. \tag{6.1}$$

We solve the problem numerically by the stabilized BGN method on the time interval $[0,T]$ with $T = 0.15$. Since there is no closed expression for the solution with initial data (6.1), we instead compute the reference solution with very fine time and space grids, i.e. $N = 2^{11}, N_t = 2^{22}$ and $k = 1$. Although our proof of Theorem 2.1 only guarantees the convergence of numerical solutions for finite elements of degree $k \ge 2$, we perform numerical experiments for finite elements of degree $k = 1, 2, 3$.

The time stepsize condition $\tau = O(h^{k+1})$ is imposed by choosing the number of mesh points N and the number of time levels N_t in a consistent way. Namely, for $N = 2^4, 2^5, 2^6$ we choose $N_t = 2^5, 2^7, 2^9$ for $k = 1$, $N_t = 2^5, 2^8, 2^{11}$ for $k = 2$, and $N_t = 2^5, 2^9, 2^{13}$ for $k = 3$, respectively. The discrete $L^\infty(0,T;L^2)$ errors of the numerical solutions, i.e.,

$$\max_{0\le m\le N_t} \|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}$$

are presented in Table 1, where the convergence rates for finite elements of degree $k = 2, 3$ are consistent with the theoretical result proved in Theorem 2.1. The numerical results show that the stabilized BGN method has optimal-order convergence also for piecewise linear finite elements. The proof of this result is different from the current paper and therefore needs to be studied in future work.

It is also desirable to test the sharpness and necessity of the CFL condition $\tau \le ch^{k+1}$. To this end, we compute the errors and rate of convergence in the regime of $\tau \sim h$ (in the experiment we simply take $N = N_t$). The results are shown in Table 2, which indicates a linear rate of convergence for all cases. This means the convergence might hold for a larger regime of weaker CFL condition.

Besides, we examine the convergence of the stabilization term. Since the stabilization term is in the weak form, we denote by $Stab$ the Riesz representation of the stabilization term, defined as follows:

$$\int_{\Gamma_h^m} Stab \cdot \phi_h = \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} \mathrm{id} \cdot \nabla_{\Gamma_h^m} I_h[\phi_h - (\phi_h \cdot \bar{n}_h^m)\bar{n}_h^m] \quad \forall \phi_h \in S_h(\Gamma_h^m).$$

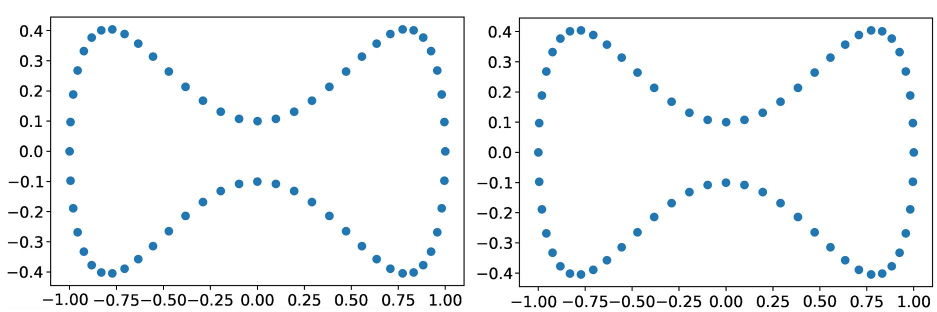
TABLE 2. Rate of convergence of the error with $\tau \sim h$

N	$\max_{1\le m\le N_t} \|\hat{e}_h^m\|_{L^2(\hat{\Gamma}_{h,*}^m)}$		
	$k=1$	$k=2$	$k=3$
2^7	1.47e-2	1.34e-2	1.34e-2
2^8	7.34e-3	7.01e-3	7.01e-3
2^9	3.68e-3	3.59e-3	3.59e-5
Convergence rate	1.00	0.96	0.96

TABLE 3. Rate of convergence of the stabilization term

N	$\max_{1\le m\le N_t} \|Stab\|_{L^\infty(\Gamma_h^m)}$		
	$k=1$	$k=2$	$k=3$
2^7	7.57e-2	4.18e-2	1.62e-3
2^8	1.86e-2	1.18e-2	1.96e-4
2^9	4.67e-3	3.06e-3	2.41e-5
Convergence rate	1.99	1.95	3.02

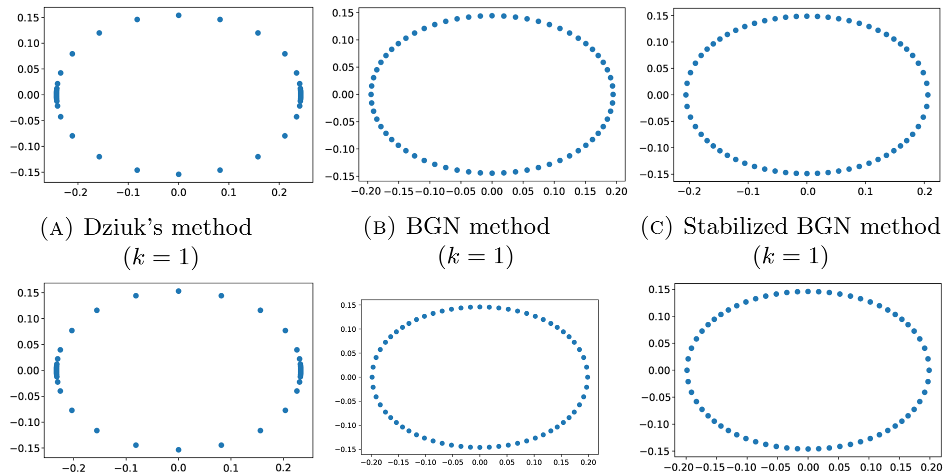
The $L_t^\infty L_x^\infty$ norm of $Stab$ are presented in Table 3 with fixed $N_t = 2^5, 2^5, 2^6$ for $k = 1, 2, 3$, respectively. The numerical results in Table 3 show that the stabilization term is $O(h^2)$ for $k = 1$ and $O(h^k)$ for $k = 2, 3$.



(A) $k = 1$, $N = 2^6$

(B) $k = 2$, $N = 2^5$

FIGURE 1. Initial nodal distribution



(A) Dziuk's method ($k = 1$) (B) BGN method ($k = 1$) (C) Stabilized BGN method ($k = 1$)

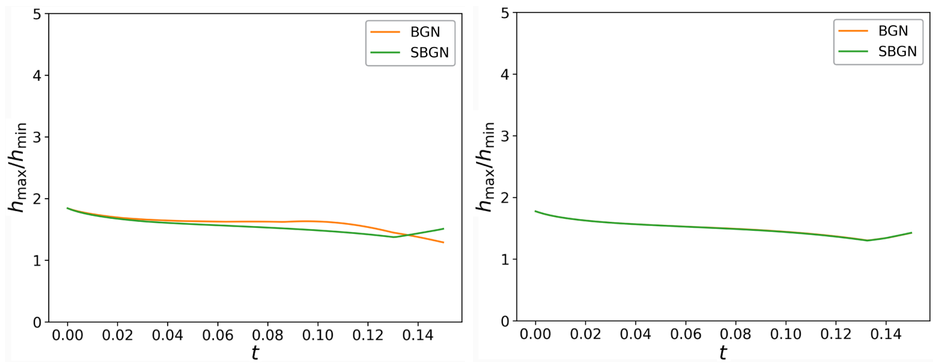
(D) Dziuk's method ($k = 2$) (E) BGN method ($k = 2$) (F) Stabilized BGN method ($k = 2$)

FIGURE 2. Mesh distributions of different methods at $T = 0.15$, with $N = 2^6$ for finite elements of degree $k = 1$, and $N = 2^5$ for finite elements of degree $k = 2$

In addition to testing the convergence rates of the proposed method, we test the performance of the stabilized BGN method in improving the distribution of mesh points of curve shortening flow with initial condition (6.1). For the initial distribution of mesh points shown in Figure 1, we test the performance of Dziuk's method, the BGN method and the stabilized BGN method proposed in this paper. The distribution of mesh points at $T = 0.15$, with number of time levels $N_t = 2^7$, is presented in Figures 2 for finite elements of degree $k = 1, 2$, where N denotes the total number of finite elements. The numerical results in Figure 2 show that, while Dziuk's method leads to clustering of mesh points, the stabilized BGN method can keep the mesh quality (distribution of mesh points) good similarly as the BGN method. To be more quantitative on the mesh quality, we present the mesh ratio $h_{\max}/h_{\min}$ in Figure 3, which shows that the stabilized BGN method has similar mesh quality as the BGN method.

7. CONCLUSIONS

We have proposed a stabilized BGN method with possibly arbitrary high-order finite elements based on mass lumping techniques using Gauss–Lobatto points, and proved the optimal-order convergence of the method in the L^2 norm under the stepsize condition $\tau \leq ch^{k+1}$. The stabilized BGN method differs from the classical BGN method from a stabilization term, with the same effect as the BGN method in improving the mesh quality, with an additional stabilization term helping to establish stability estimates for the artificial tangential velocity. We have found the underlying geometric PDEs to which the stabilized BGN method converges, i.e., the system of equations in (1.10), which is used to establish stability estimates for the artificial tangential velocity and to characterize the limit of particle trajectories produced by the stabilized BGN method. The convergence of the method is



(A) $k = 1$, $N = 2^6$ (B) $k = 2$, $N = 2^5$

FIGURE 3. Mesh ratio $h_{\max}/h_{\min}$

supported by the numerical results, which also show that the proposed stabilized BGN method has the same effect as the original BGN method in maintaining good mesh quality of the evolving curve.

Our analysis requires the projected normal vector $\bar{n}_h^m$ to be defined as a continuous finite element function, which is essential for applying integration by parts in many places throughout this article. Additionally, the quadrature points must coincide with the nodes used to define $\bar{n}_h^m$ to ensure that the terms $L_1(I_h\bar{T}_h^m\phi_h)$ and $L_2(I_h\bar{T}_h^m\phi_h)$ vanish on the right-hand side of (4.41); see the text below (4.41). These requirements necessitate that the quadrature points include the endpoints of each finite element, thereby excluding Gauss quadrature. Instead, the Gauss–Lobatto quadrature satisfies all these requirements. The underlying framework and techniques established in this paper may be applied/extended to other geometric flows and parametric finite element approximations which contain artificial tangential motions of the BGN type.

Acknowledgments

We would like to thank Yupei Xie (PhD student at The Hong Kong Polytechnic University) and the anonymous referees for carefully reading the manuscript and providing valuable comments.

References

[1] G. Bai and B. Li, *Erratum: Convergence of Dziuk's semidiscrete finite element method for mean curvature flow of closed surfaces with high-order finite elements*, SIAM J. Numer. Anal. **61** (2023), no. 3, 1609–1612, DOI 10.1137/22M1521791. MR4605925

[2] G. Bai and B. Li. A new approach to the analysis of parametric finite element approximations to mean curvature flow. *Found. Comput. Math.*, 2023. doi: 10.1007/s10208-023-09622-x.

[3] E. Bänsch, P. Morin, and R. H. Nochetto, *A finite element method for surface diffusion: the parametric case*, J. Comput. Phys. **203** (2005), no. 1, 321–343, DOI 10.1016/j.jcp.2004.08.022. MR2104399

[4] W. Bao, H. Garcke, R. Nürnberg, and Q. Zhao, *Volume-preserving parametric finite element methods for axisymmetric geometric evolution equations*, J. Comput. Phys. **460** (2022), Paper No. 111180, 23, DOI 10.1016/j.jcp.2022.111180. MR4404537

[5] J. W. Barrett, H. Garcke, and R. Nürnberg, *Parametric Finite Element Approximations of Curvature-Driven Interface Evolutions*, Geometric partial differential equations. Part I, Handb. Numer. Anal., vol. 21, Elsevier/North-Holland, Amsterdam, [2020] ©2020, pp. 275–423. MR4378429

[6] J. W. Barrett, K. Deckelnick, and V. Styles, *Numerical analysis for a system coupling curve evolution to reaction diffusion on the curve*, SIAM J. Numer. Anal. **55** (2017), no. 2, 1080–1100, DOI 10.1137/16M1083682. MR3639582

[7] J. W. Barrett, H. Garcke, and R. Nürnberg, *A parametric finite element method for fourth order geometric evolution equations*, J. Comput. Phys. **222** (2007), no. 1, 441–462, DOI 10.1016/j.jcp.2006.07.026. MR2298053

[8] J. W. Barrett, H. Garcke, and R. Nürnberg, *A variational formulation of anisotropic geometric evolution equations in higher dimensions*, Numer. Math. **109** (2008), no. 1, 1–44, DOI 10.1007/s00211-007-0135-5. MR2377611

[9] J. W. Barrett, H. Garcke, and R. Nürnberg, *On the parametric finite element approximation of evolving hypersurfaces in* $\mathbb{R}^3$, J. Comput. Phys. **227** (2008), no. 9, 4281–4307, DOI 10.1016/j.jcp.2007.11.023. MR2406538

[10] J. W. Barrett, H. Garcke, and R. Nürnberg, *Parametric approximation of Willmore flow and related geometric evolution equations*, SIAM J. Sci. Comput. **31** (2008), no. 1, 225–253, DOI 10.1137/070700231. MR2460777

[11] S. Bartels, *A simple scheme for the approximation of the elastic flow of inextensible curves*, IMA J. Numer. Anal. **33** (2013), no. 4, 1115–1125, DOI 10.1093/imanum/drs041. MR3119710

[12] A. Bonito, R. H. Nochetto, and M. S. Pauletti, *Parametric FEM for geometric biomembranes*, J. Comput. Phys. **229** (2010), no. 9, 3171–3188, DOI 10.1016/j.jcp.2009.12.036. MR2601095

[13] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008, DOI 10.1007/978-0-387-75934-0. MR2373954

[14] K. Deckelnick and G. Dziuk, *On the Approximation of the Curve Shortening Flow*, Calculus of variations, applications and computations (Pont-à-Mousson, 1994), Pitman Res. Notes Math. Ser., vol. 326, Longman Sci. Tech., Harlow, 1995, pp. 100–108. MR1419337

[15] K. Deckelnick and G. Dziuk, *Error analysis for the elastic flow of parametrized curves*, Math. Comp. **78** (2009), no. 266, 645–671, DOI 10.1090/S0025-5718-08-02176-5. MR2476555

[16] K. Deckelnick, G. Dziuk, and C. M. Elliott, *Computation of geometric partial differential equations and mean curvature flow*, Acta Numer. **14** (2005), 139–232, DOI 10.1017/S0962492904000224. MR2168343

[17] A. Demlow, *Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces*, SIAM J. Numer. Anal. **47** (2009), no. 2, 805–827, DOI 10.1137/070708135. MR2485433

[18] D. M. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. **18** (1983), no. 1, 157–162. MR697987

[19] G. Dziuk, *An algorithm for evolutionary surfaces*, Numer. Math. **58** (1991), no. 6, 603–611, DOI 10.1007/BF01385643. MR1083523

[20] G. Dziuk, *Convergence of a semi-discrete scheme for the curve shortening flow*, Math. Models Methods Appl. Sci. **4** (1994), no. 4, 589–606, DOI 10.1142/S0218202594000339. MR1291140

[21] G. Dziuk, *Computational parametric Willmore flow*, Numer. Math. **111** (2008), no. 1, 55–80, DOI 10.1007/s00211-008-0179-1. MR2448203

[22] G. Dziuk and C. M. Elliott, *Finite elements on evolving surfaces*, IMA J. Numer. Anal. **27** (2007), no. 2, 262–292, DOI 10.1093/imanum/drl023. MR2317005

[23] G. Dziuk and C. M. Elliott, *A fully discrete evolving surface finite element method*, SIAM J. Numer. Anal. **50** (2012), no. 5, 2677–2694, DOI 10.1137/110828642. MR3022237

[24] G. Dziuk and C. M. Elliott, *Finite element methods for surface PDEs*, Acta Numer. **22** (2013), 289–396, DOI 10.1017/S0962492913000056. MR3038698

[25] G. Dziuk, D. Kröner, and T. Müller, *Scalar conservation laws on moving hypersurfaces*, Interfaces Free Bound. **15** (2013), no. 2, 203–236, DOI 10.4171/IFB/301. MR3105772

[26] C. M. Elliott, H. Garcke, and B. Kovács, *Numerical analysis for the interaction of mean curvature flow and diffusion on closed surfaces*, Numer. Math. **151** (2022), no. 4, 873–925, DOI 10.1007/s00211-022-01301-3. MR4453294

[27] C. M. Elliott and H. Fritz, *On approximations of the curve shortening flow and of the mean curvature flow based on the DeTurck trick*, IMA J. Numer. Anal. **37** (2017), no. 2, 543–603, DOI 10.1093/imanum/drw020. MR3649420

[28] H. Fritz, *Finite elemente approximation der Ricci-Krümmung und simulation des Ricci-Deturck-Flusses*, PhD thesis, 2013.

[29] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364

[30] X. Gui, B. Li, and J. Wang, *Convergence of renormalized finite element methods for heat flow of harmonic maps*, SIAM J. Numer. Anal. **60** (2022), no. 1, 312–338, DOI 10.1137/21M1402212. MR4377027

[31] J. Hu and B. Li, *Evolving finite element methods with an artificial tangential velocity for mean curvature flow and Willmore flow*, Numer. Math. **152** (2022), no. 1, 127–181, DOI 10.1007/s00211-022-01309-9. MR4474058

[32] B. Kovács, *High-order evolving surface finite element method for parabolic problems on evolving surfaces*, IMA J. Numer. Anal. **38** (2018), no. 1, 430–459, DOI 10.1093/imanum/drx013. MR3800028

[33] B. Kovács, B. Li, and C. Lubich, *A convergent evolving finite element algorithm for mean curvature flow of closed surfaces*, Numer. Math. **143** (2019), no. 4, 797–853, DOI 10.1007/s00211-019-01074-2. MR4026373

[34] B. Kovács, B. Li, and C. Lubich, *A convergent evolving finite element algorithm for Willmore flow of closed surfaces*, Numer. Math. **149** (2021), no. 3, 595–643, DOI 10.1007/s00211-021-01238-z. MR4344594

[35] B. Kovács, B. Li, C. Lubich, and C. A. P. Guerra, *Convergence of finite elements on an evolving surface driven by diffusion on the surface*, Numer. Math. **137** (2017), no. 3, 643–689, DOI 10.1007/s00211-017-0888-4. MR3712288

[36] J. M. Lee, *Introduction to Riemannian Manifolds*, Graduate Texts in Mathematics, vol. 176, Springer, Cham, 2018. Second edition of [MR1468735]. MR3887684

[37] B. Li, *Convergence of Dziuk's linearly implicit parametric finite element method for curve shortening flow*, SIAM J. Numer. Anal. **58** (2020), no. 4, 2315–2333, DOI 10.1137/19M1305483. MR4134364

[38] B. Li, *Convergence of Dziuk's semidiscrete finite element method for mean curvature flow of closed surfaces with high-order finite elements*, SIAM J. Numer. Anal. **59** (2021), no. 3, 1592–1617, DOI 10.1137/20M136935X. MR4269969

[39] C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Progress in Mathematics, vol. 290, Birkhäuser/Springer Basel AG, Basel, 2011, DOI 10.1007/978-3-0348-0145-4. MR2815949

[40] A. Mierswa, *Error estimates for a finite difference approximation of mean curvature flow for surfaces of torus type*, PhD Thesis, Otto-von-Guericke-Universität, Magdeburg, 2020.

[41] K. Mikula and D. Ševčovič, *Evolution of plane curves driven by a nonlinear function of curvature and anisotropy*, SIAM J. Appl. Math. **61** (2001), no. 5, 1473–1501, DOI 10.1137/S0036139999359288. MR1824511

[42] C. Ye and J. Cui, *Convergence of Dziuk's fully discrete linearly implicit scheme for curve shortening flow*, SIAM J. Numer. Anal. **59** (2021), no. 6, 2823–2842, DOI 10.1137/21M1391626. MR4333668

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