WEAK DISCRETE MAXIMUM PRINCIPLE OF ISOPARAMETRIC FINITE ELEMENT METHODS IN CURVILINEAR POLYHEDRA

BUYANG LI, WEIFENG QIU, YUPEI XIE, AND WENSHAN YU

ABSTRACT. The weak maximum principle of the isoparametric finite element method is proved for the Poisson equation under the Dirichlet boundary condition in a (possibly concave) curvilinear polyhedral domain with edge openings smaller than π, which include smooth domains and smooth deformations of convex polyhedra. The proof relies on the analysis of a dual elliptic problem with a discontinuous coefficient matrix arising from the isoparametric finite elements. Therefore, the standard H^2 elliptic regularity which is required in the proof of the weak maximum principle in the literature does not hold for this dual problem. To overcome this difficulty, we have decomposed the solution into a smooth part and a nonsmooth part, and estimated the two parts by H^2 and $W^{1,p}$ estimates, respectively.

As an application of the weak maximum principle, we have proved a maximum-norm best approximation property of the isoparametric finite element method for the Poisson equation in a curvilinear polyhedron. The proof contains non-trivial modifications of Schatz's argument due to the non-conformity of the iso-parametric finite elements, which requires us to construct a globally smooth flow map which maps the curvilinear polyhedron to a perturbed larger domain on which we can establish the $W^{1,\infty}$ regularity estimate of the Poisson equation uniformly with respect to the perturbation.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \in \{2,3\}$ and consider a quasi-uniform triangulation of the domain Ω with mesh size h, denoted by $\mathscr{K}_h$. Hence, $\Omega_h = (\bigcup_{K \in \mathscr{K}_h} K)^\circ$ is an approximation of Ω. Let $S_h(\Omega_h)$ be a finite element space subject to the triangulation $\mathscr{K}_h$, and denote by $\mathring{S}_h(\Omega_h) = \{v_h \in S_h(\Omega_h) : v_h = 0 \text{ on } \partial\Omega_h\}$ the finite element subspace under the homogeneous boundary condition. A function $u_h \in S_h(\Omega_h)$ is called discrete harmonic if it satisfies

$$\int_{\Omega_h} \nabla u_h \cdot \nabla \chi_h = 0 \quad \forall \chi_h \in \mathring{S}_h(\Omega_h). \tag{1.1}$$

For a given mesh and finite element space, if all the discrete harmonic functions satisfy the following inequality:

$$\|u_h\|_{L^\infty(\Omega_h)} \lesssim \|u_h\|_{L^\infty(\partial\Omega_h)}, \tag{1.2}$$

then it is said that the *discrete maximum principle* holds.

The discrete maximum principle of finite element methods (FEMs) has attracted much attention from numerical analysts due to its importance for the stability and accuracy of numerical solutions; for example, see [10, 13, 38, 44, 47]. However, strong restrictions on the geometry of the mesh are required for the discrete maximum principle to hold. For example, for piecewise linear finite elements on a two-dimensional triangular mesh, the discrete maximum principle

generally requires the angles of the triangles to be less than $\pi/2$; see [47, §5]. In three dimensions, it is hard to have such discrete maximum principe even for piecewise linear finite elements; see [6, 27, 28, 48].

Schatz considered a different approach in [39] by proving the weak maximum principle (also called the Agmon–Miranda maximum principle) ,

$$\|u_h\|_{L^\infty(\Omega_h)} \leqslant C \|u_h\|_{L^\infty(\partial\Omega_h)}, \tag{1.3}$$

for some constant C which is independent of u_h and h, for a wide class of H^1-conforming finite elements on a general quasi-uniform triangulation of a two-dimensional polygonal domain. It was shown in [39] that the weak maximum principle can be used to prove the maximum-norm stability and best approximation results of FEMs in a plane polygonal domain, i.e.,

$$\|u - R_h u\|_{L^\infty(\Omega)} \leqslant C\ell_h \inf_{v_h \in \mathring{S}_h} \|u - v_h\|_{L^\infty(\Omega)} \quad \forall\, u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{1.4}$$

where $R_h : H_0^1(\Omega) \to \mathring{S}_h$ is the Ritz projection operator, and

$$\ell_h = \begin{cases} \ln(2 + 1/h) & \text{for piecewise linear elements,} \\ 1 & \text{for higher-order finite elements.} \end{cases}$$

Such maximum-norm stability and best approximation results have a number of applications in the error estimates of finite element solutions for parabolic problems [25, 31, 33, 34], Stokes systems [5], nonlinear problems [15, 18, 36], optimal control problems [1, 2], and so on.

In three dimensions, the weak maximum principle was extended to convex polyhedral domains in [32] and used to prove the L^∞-norm stability and best approximation results of FEMs on convex polyhedral domains, removing an extra logarithmic factor $\ln(2 + 1/h)$ in the stability constant for quadratic and higher-order elements obtained in other approaches (for example, see [30]). When Ω is a smooth domain and $\Omega_h = \Omega$ (the triangulation is assumed to match the curved boundary exactly), the weak maximum principle of quadratic or higher-order FEMs is a result of the maximum-norm stability result in [40, 42], and the weak maximum principle of linear finite elements can be proved similarly as in [32]. In all these articles, the triangulation is assumed to match the boundary of the domain exactly, with $\Omega_h = \Omega$.

In the practical computation, the curved boundary of a bounded smooth domain, or more generally a curvilinear polygon or polyhedron (which may contain both curved faces, curved edges, and corners), is generally approximated by isoparametric finite elements instead of being matched exactly by the triangulation. In this case, the weak maximum principle of FEMs has not been proved yet. Correspondingly, the best approximation results such as (1.4) are not known for isoparametric FEMs in a curved domain.

Some related results have been proved in the case $\Omega_h \neq \Omega$. For the Poisson equation with Dirichlet boundary conditions in convex smooth domains, the piecewise linear finite element space with a zero extension in $\Omega \backslash \Omega_h$ is conforming, i.e., $S_h(\Omega_h) \subset H_0^1(\Omega)$. In this case, pointwise error estimates of FEMs have been established in [4, 42]. For general bounded smooth domains which may be concave, thus the finite element space may be non-conforming, Kashiwabara & Kemmochi [26] have obtained the following error estimate for piecewise linear finite elements for the Poisson equation under the Neumann boundary condition:

$$\|\tilde{u} - u_h\|_{L^\infty(\Omega_h)} \leqslant Ch|\log h| \inf_{v_h \in S_h} \|\tilde{u} - v_h\|_{W^{1,\infty}(\Omega_h)} + Ch^2|\log h|\|u\|_{W^{2,\infty}(\Omega)}, \tag{1.5}$$

where $\tilde{u}$ is any extension of u in $W^{2,\infty}(\Omega_\delta)$ and Ω_δ is a neighborhood of $\overline{\Omega}$. In the case $u \in W^{2,\infty}(\Omega)$, this error estimate is a consequence of the best approximation result in (1.4). More recently, the $W^{1,\infty}$ stability of the Ritz projection was proved in [16] for isoparametric FEMs on $C^{r+1,1}$-smooth domains based on weighted-norm estimates, where r denotes the degree of

finite elements. For curvilinear polyhedra or smooth domains which may be concave, the weak maximum principle and the best approximation results in the L^∞ norm have not been proved.

In this article, we close the gap mentioned above by proving the weak maximum principle in (1.1) for isoparametric finite elements of degree $r \geqslant 1$ in a bounded smooth domain or a curvilinear polyhedron (possibly concave) with edge openings smaller than π. As an application of the weak maximum principle, we prove that the finite element solution $u_h \in \mathring{S}_h(\Omega_h)$ of the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

using isoparametric finite elements of degree $r \geqslant 1$ has the following optimal-order error bound (for any $p > N$):

$$\|u - u_h\|_{L^\infty(\Omega)} \leqslant C\ell_h \|u - \check{I}_h u\|_{L^\infty(\Omega)} + Ch^{r+1}\ell_h \|f\|_{L^p(\Omega)}, \tag{1.7}$$

where u_h is extended to be zero in $\Omega \backslash \Omega_h$, and $\check{I}_h u$ denotes a Lagrange interpolation operator (which will be defined in the next section). Inequality (1.7) can be viewed as a variant of the best approximation result in (1.4) by taking account of the geometry change of the domain, which produces an additional optimal-order term $Ch^{r+1}\|f\|_{L^p(\Omega)}$ independent of the higher regularity of f. In particular, inequality (1.7) implies the following error estimate:

$$\|u - u_h\|_{L^\infty(\Omega)} \leqslant C\ell_h h^s \|u\|_{C^s(\overline{\Omega})} + Ch^{r+1}\ell_h \|f\|_{L^p(\Omega)} \quad \text{for } u \in C^s(\overline{\Omega}), \ 0 \leqslant s \leqslant r+1, \tag{1.8}$$

which adapts to the regularity of u.

The weak maximum principle is proved by converting the finite element weak form on Ω_h to a weak form on Ω by using a bijective transformation $\Phi_h : \Omega_h \to \Omega$ which is piecewisely defined on the triangles/tetrahedra. This yields a bilinear form with a discontinuous coefficient matrix. The main technical difficulty is that the elliptic partial differential equation associated to this coefficient matrix does not have the H^2 regularity estimate, which is required in the proof of weak maximum principle in the literature; see [32]. We overcome this difficulty by decomposing the finite element solution v_h (of a duality problem) into two parts, $v_h = v_{h,1} + v_{h,2}$, with $v_{h,1}$ corresponding to the Poisson equation with H^2 regularity, and $v_{h,2}$ corresponding to an elliptic equation with discontinuous coefficients but with a small source term arising from the geometry perturbation, and then estimate the two parts separately by using the H^2 and $W^{1,p}$ regularity of the respective problems.

The maximum-norm error estimate is proved by using Schatz argument through estimating the difference between the solutions of the Poisson equations in Ω_h and Ω. However, in order to avoid using the partial derivatives of f in the proof of (1.7), we have to estimate the error between the solutions of the Poisson equation in the two domains Ω_h and Ω under the Dirichlet boundary conditions, respectively. This is accomplished by perturbing the curvilinear polyhedron through a globally smooth flow map pointing outward the domain and establishing the $W^{1,\infty}$ regularity estimate of the Poisson equation in a slightly larger perturbed domain Ω^t (uniformly with respect to the perturbation), which contains both Ω_h and Ω and satisfies that $\text{dist}(x, \partial\Omega) \sim h^{r+1}$ for $x \in \partial\Omega^t$.

The rest of this article is organized as follows. In Section **??**, we present the main results to be proved in this article, including the weak maximum principle of the isoparametric FEM in a curvilinear polyhedron, and the best approximation result of finite element solutions in the maximum norm. The proofs of the two main results are presented in Sections 2 and 3, respectively. The conclusions are presented in Section 4.

In this article, we assume that $\Omega \subset \mathbb{R}^N$, with $N \in \{2, 3\}$, is either a bounded smooth domain or a curvilinear polyhedron (possibly concave) with edge openings smaller than π. More specifically,

in the three-dimensional space, this means that for every $x \in \partial\Omega$ there is a neighborhood U_x and a smooth diffeomorphism $\varphi_x : U_x \to B_0(\varepsilon_x)$ mapping x to 0 such that one of the following three conditions holds:

(1) x is a smooth point, i.e., $\varphi_x(U_x \cap \Omega) = B_0(\varepsilon_x) \cap \mathbb{R}_+^3$, where $\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x_3 > 0\}$ is a half space in $\mathbb{R}^3$.

(2) x is an edge point, i.e., $\varphi_x(U_x \cap \Omega) = B_0(\varepsilon_x) \cap K_x$, where $K_x = \mathbb{R} \times \Sigma$, where $\Sigma \subseteq \mathbb{R}^2$ is a sector with angle less than π.

(3) x is a vertex point, i.e., $\varphi_x(U_x \cap \Omega) = B_0(\varepsilon_x) \cap K_x$, where K_x is a convex polyhedral cone with a vertex at 0. Therefore, the boundary of K_x consists of several smooth faces intersecting at some edges which pass through the vertex 0.

We refer to [3, Definition 2.1] for the definition of general curvilinear polyhedron.

Let $\mathscr{K}$ be the set of closed simplices in a quasi-uniform triangulation of the domain Ω with isoparametric finite elements of degree $r \geqslant 1$ approximating the boundary $\partial\Omega$, as described in [29], with flat interior simplices which have at most one vertex on $\partial\Omega$ and possibly curved boundary simplices. Each boundary simplex contains a possibly curved face or edge interpolating $\partial\Omega$ with an accuracy of $O(h^{r+1})$, where h denotes the mesh size of the triangulation. Hence, $\Omega_h = (\bigcup_{K \in \mathscr{K}} K)^\circ$ is an approximation to Ω such that $\text{dist}(x, \partial\Omega) = O(h^{r+1})$ for $x \in \partial\Omega_h$.

We prove the following weak maximum principle of the isoparametric FEM.

Theorem 1.1. *For the isoparametric FEM of degree $r \geqslant 1$ on a quasi-uniform triangulation of Ω, all the discrete harmonic functions $u_h \in S_h(\Omega_h)$ satisfying (1.1) have the following estimate:*

$$\|u_h\|_{L^\infty(\Omega_h)} \leqslant C \|u_h\|_{L^\infty(\partial\Omega_h)}, \tag{1.1}$$

where the constant C is independent of u_h and the mesh size h.

In the isoparametric finite elements described in [29], each curved simplex $K \in \mathscr{K}$ is the image of a map $F_K : \hat{K} \to K$ defined on the reference simplex $\hat{K}$, which is a polynomial of degree no larger than r and transforms the finite element structure of $\hat{K}$ to K. There is a homeomorphism $\Phi_h : \Omega_h \to \Omega$, which is piecewise smooth on each simplex and globally Lipschitz continuous. If we denote $\Phi_{h,K} := \Phi_h|_K$ and $\check{K} := \Phi_h(K)$, then $\Phi_{h,K} : K \to \check{K}$ is a diffeomorphism which transforms the finite element structure of K to $\check{K}$. Therefore, $\check{\mathscr{K}} = \{\check{K} : K \in \mathscr{K}\}$ is a triangulation of the curved domain Ω, with

$$\Omega_h = \bigcup_{K \in \mathscr{K}} K \quad \text{and} \quad \Omega = \bigcup_{K \in \mathscr{K}} \check{K}.$$

One can define isoparametric finite element space $S_h(\Omega_h)$ as

$$S_h(\Omega_h) = \{v_h \in H^1(\Omega_h) : v_h|_K \circ F_K \text{ is a polynomial on } \hat{K} \text{ of degree} \leqslant r \text{ for } K \in \mathscr{K}\}. \tag{1.2}$$

The finite element spaces on Ω can be defined as

$$S_h(\Omega) = \{\check{v}_h \in H^1(\Omega) : \check{v}_h \circ \Phi_h \in S_h(\Omega_h)\} \quad \text{and} \quad \mathring{S}_h(\Omega) = \{\check{v}_h \in S_h(\Omega) : \check{v}_h = 0 \text{ on } \partial\Omega\}. \tag{1.3}$$

For a finite element function $v_h \in S_h(\Omega_h)$, we can associate it with a finite element function $\check{v}_h \in S_h(\Omega)$ defined by $v_h \circ \Phi_h^{-1} := \check{v}_h$.

Remark 1.1. *By using the notation which link $v_h \in S_h(\Omega_h)$ and $\check{v}_h \in S_h(\Omega)$, the weak maximum principle in (1.1) can be equivalently written as*

$$\|\check{u}_h\|_{L^\infty(\Omega)} \leqslant C \|\check{u}_h\|_{L^\infty(\partial\Omega)}. \tag{1.4}$$

For a function $f \in C^0(\overline{\Omega}_h)$, one can define its local interpolation $I_{h,K}f$ on a simplex $K \in \mathscr{K}$ as the function satisfying

$$(I_{h,K}f) \circ F_K := I_{\hat{K}}(f \circ F_K),$$

where $I_{\hat{K}}$ is the standard Lagrange interpolation on the reference simplex $\hat{K}$ (onto the space of polynomials of degree $\leqslant r$). The global interpolation $I_h f \in S_h(\Omega_h)$ is defined as

$$I_h f|_K := I_{h,K} f \quad \forall\, K \in \mathcal{K}.$$

For the analysis of the isoparametric FEM, we also define an interpolation operator $\check{I}_h : C(\overline{\Omega}) \to S_h(\Omega)$ by

$$(\check{I}_h v) \circ \Phi_h = I_h(v \circ \Phi_h) \quad \forall\, v \in C(\overline{\Omega}).$$

As an application of the weak maximum principle, we establish an L^∞-norm best approximation result of isoparametric FEM for the Poisson equation in a curvilinear polyehdron. We assume that the triangulation can be extended to a bigger domain which contains $\overline{\Omega}$, as stated below.

Assumption 1.1. The curvilinear polyhedral domain Ω can be extended to a larger convex polyhedron Ω_* with piecewise flat boundaries such that $\overline{\Omega} \subset \Omega_*$ and the triangulation $\mathcal{K}$ can be extended to a quasi-uniform triangulation $\mathcal{K}_*$ on Ω_* (thus the triangulation in $\Omega_* \backslash \overline{\Omega}$ is also isoparametric on its boundary $\partial\Omega$).

Remark 1.2. Here Ω_* can be chosen as a large cube whose interior contains $\overline{\Omega}$. Note that the triangulation $\mathcal{K}$ is obtained from some triangulation $\widetilde{\mathcal{K}}$ consisting of flat simplexes by the method in Lenoir's paper [29]. We can first extend $\widetilde{K}$ to a quasi-uniform flat triangulation $\widetilde{\mathcal{K}_*}$ of Ω_*, and then modify those flat simplexes $\widetilde{\mathcal{K}}$ with one of whose edges/faces attaches to the boundary $\partial\Omega$, to isoparametric elements by the method in Lenoir's paper [29]. This leads to a quasi-uniform triangulation $\mathcal{K}_*$ on Ω_* which extends $\mathcal{K}$. By our construction, the triangulation on $\Omega_* \backslash \overline{\Omega}$ is also isoparametric on its boundary $\partial\Omega$.

Theorem 1.2. *For $f \in L^p(\Omega)$ with some $p > N$, we consider the Poisson equation*

$$\begin{cases} -\Delta u = f & in \ \Omega \\ \quad\ u = 0 & on \ \partial\Omega \end{cases} \tag{1.5}$$

and the isoparametric FEM of degree $r \geqslant 1$ for (1.5): Find $u_h \in \mathring{S}_h(\Omega_h)$ such that

$$\int_{\Omega_h} \nabla u_h \cdot \nabla \chi_h \, \mathrm{d}x = \int_{\Omega_h} \tilde{f} \chi_h \, \mathrm{d}x \quad \forall\, \chi_h \in \mathring{S}_h(\Omega_h), \tag{1.6}$$

where $\tilde{f} \in L^p(\Omega \cup \Omega_h)$ is any extension of $f \in L^p(\Omega)$ satisfying $\|\tilde{f}\|_{L^p(\Omega \cup \Omega_h)} \leqslant C\|f\|_{L^p(\Omega)}$. Assuming that the triangulation satisfies Assumption 1.1, there exist positive constants h_0 and C (independent of f, u and h) such that the solutions of (1.5) and (1.6) satisfy the following inequality for $h \leqslant h_0$:

$$\|u - u_h\|_{L^\infty(\Omega)} \leqslant C\ell_h \|u - \check{I}_h u\|_{L^\infty(\Omega)} + C h^{r+1} \ell_h \|f\|_{L^p(\Omega)}, \tag{1.7}$$

where u_h is extended to be zero on $\Omega \backslash \Omega_h$, and ℓ_h is defined as

$$\ell_h = \begin{cases} \ln(2 + 1/h) & \textit{for piecewise linear elements,} \\ 1 & \textit{for higher-order finite elements.} \end{cases}$$

The proofs of Theorems 1.1 and 1.2 are presented in the next two sections, respectively. For the simplicity of notation, we denote by C a generic positive constant which may be different at different occurrences, possibly depending on the specific domain Ω and the shape-regularity and quasi-uniformity of the triangulation, and the polynomial degree $r \geqslant 1$, but is independent of the mesh size h.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into six parts, presented in the following six subsections.

2.1. Properties of the isomparametric FEM

In this subsection, we summarize the basic properties of the isoparametric FEM to be used in the proof of Theorem 1.1.

Lemma 2.1 ([29, Theorem 1, Theorem 2, Proposition 2, Proposition 3, Proposition 4]). *Let $\mathscr{K}$ be the triangulation of Ω by isoparametric finite elements of degree $r \geqslant 1$, with the maps $F_K : \hat{K} \to K$ and $\Phi_{h,K} : K \to \check{K}$ described in Section **??**. Let D^s denote the Fréchet derivative of order s. Then the following results hold:*

1. *$F_K : \hat{K} \to K$ is a diffeomorphism such that*
$$\begin{aligned} \|D^s F_K\|_{L^\infty(\hat{K})} &\leqslant Ch^s \qquad \forall s \in [1, r+1] \\ \|D^s F_K^{-1}\|_{L^\infty(K)} &\leqslant Ch^{-s} \quad \forall s \in [1, r+1] \end{aligned} \tag{2.1}$$

2. *$\Phi_{h,K} : K \to \check{K}$ is a diffeomorphism such that*
$$\begin{aligned} \|D^s(\Phi_{h,K} - \mathrm{Id})\|_{L^\infty(K)} &\leqslant Ch^{r+1-s} \quad \forall s \in [1, r+1] \\ \|D^s(\Phi_{h,K}^{-1} - \mathrm{Id})\|_{L^\infty(\check{K})} &\leqslant Ch^{r+1-s} \quad \forall s \in [1, r+1] \end{aligned} \tag{2.2}$$

3. *For $v \in H^m(K)$ and integer $m \in [0, r+1]$, the norms $\|v\|_{H^m(K)}$ and $\|v \circ \Phi_{h,K}^{-1}\|_{H^m(\check{K})}$ are uniformly equivalent with respect to h.*

4. *Each curved simplex $K \in \mathscr{K}$ corresponds to a flat simplex $\widetilde{K}$ (which has the same vertices as K), and there is a unique linear bijection $F_{\widetilde{K}} : \hat{K} \to \widetilde{K}$ which maps the reference simplex $\hat{K}$ onto $\widetilde{K}$. The map $\widetilde{\Psi}_K := F_K \circ F_{\widetilde{K}}^{-1} : \widetilde{K} \to K$ is a diffeomorphism satisfying the following estimates:*
$$\begin{aligned} \|D(\widetilde{\Psi}_K - \mathrm{Id})\|_{L^\infty(\widetilde{K})} &\leqslant Ch, \quad \|D(\widetilde{\Psi}_K^{-1} - \mathrm{Id})\|_{L^\infty(K)} \leqslant Ch \\ \|D^s\widetilde{\Psi}_K\|_{L^\infty(\widetilde{K})} &\leqslant C, \quad \|D^s\widetilde{\Psi}_K^{-1}\|_{L^\infty(K)} \leqslant C \quad \forall s \in [1, r+1]. \end{aligned} \tag{2.3}$$

5. *For $v \in H^m(K)$ and integer $m \in [0, r+1]$, the norms $\|v\|_{H^m(K)}$ and $\|v \circ \widetilde{\Psi}_K\|_{H^m(\widetilde{K})}$ are uniformly equivalent with respect to h.*

Let $W_h^{k,p}(\Omega)$ be the space of functions on Ω whose restriction on each $\check{K} \in \mathscr{K}$ lies in $W^{k,p}(\check{K})$, equipped with the following norm:

$$\|v\|_{W_h^{k,p}(\Omega)} := \begin{cases} \left(\sum_{K \in \mathscr{K}} \|v\|_{W^{k,p}(K)}^p \right)^{\frac{1}{p}} & \text{for } 1 \leqslant p < \infty, \\ \sup_{K \in \mathscr{K}} \|v\|_{W^{k,p}(K)} & \text{for } p = \infty. \end{cases}$$

In the case $p = 2$ we write $H^{l,h}(\Omega) = W_h^{l,2}(\Omega)$. The following local interpolation error estimate was proved in [29, Lemma 7]; also see [11, Theorem 4.3.4]. Although it was proved only for $p = 2$ in [29, Lemma 7], the proof can be extended to $1 \leqslant p \leqslant \infty$ straightforwardly.

Lemma 2.2 (Lagrange interpolation). *Let $\check{I}_{h,K} : C(\overline{\Omega}) \to S_h(\Omega)$ be the interpolation operator defined by*

$$\check{I}_{h,K} f \circ \Phi_h := I_{h,K} f \quad \forall f \in C(\overline{\Omega}).$$

Then, for $1 \leqslant k \leqslant r + 1$ and $1 \leqslant p \leqslant \infty$ such that $W_h^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ (e.g., $kp > N$ when $p > 1$ or $k \geqslant N$ when $p = 1$), the following error estimate holds:

$$|u - \check{I}_{h,K}u|_{W^{i,p}(\check{K})} \leqslant Ch^{k-i}\|u\|_{W^{k,p}(\check{K})} \quad \forall\, 0 \leqslant i \leqslant k, \ \forall\, \check{K} \in \mathscr{K}, \ \forall\, u \in C(\overline{\Omega}) \cap W_h^{k,p}(\Omega).$$

Since the Lagrange interpolation is defined by using the pointwise values of a function at the Lagrange nodes, its stability in the $W^{k,p}$ norm is valid only when $W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, i.e., in the case "$kp > N$ and $p > 1$" or "$k \geqslant N$ and $p = 1$". One can remove this restriction by using the Scott–Zhang interpolation, which can be constructed first in the flat triangulation $\widetilde{\mathscr{K}} = \{\widetilde{K} : K \in \mathscr{K}\}$ as in [7, Section 4.8] and then be transformed to $\mathscr{K}$ via the maps $\widetilde{\Psi}_K$. Namely, by denoting $\widetilde{\Omega}_h = \bigcup_{K \in \mathscr{K}} \widetilde{K}$ and $\widetilde{\Psi}_h : \widetilde{\Omega}_h \to \Omega_h$, we can define

$$(\mathcal{I}^h v) \circ \widetilde{\Psi}_h := \widetilde{\mathcal{I}}^h(v \circ \widetilde{\Psi}_h) \quad \forall\, v \in L^1(\Omega_h),$$

where $\widetilde{\mathcal{I}}^h$ denotes the Scott–Zhang interpolation on the flat triangulation $\widetilde{\mathscr{K}}$. Since the maps $\widetilde{\Psi}_h$ induces norm equivalence on every simplex, as a result of (2.3), we have the following result.

Lemma 2.3 (Scott–Zhang interpolation). *There is a global interpolation operator*

$$\mathcal{I}^h : L^1(\Omega_h) \to S_h(\Omega_h)$$

such that

$$|u - \mathcal{I}^h u|_{W_h^{i,p}(\Omega_h)} \leqslant Ch^{k-i}\|u\|_{W_h^{k,p}(\Omega_h)} \quad \forall\, 0 \leqslant i \leqslant k, \ \forall\, 1 \leqslant k \leqslant r + 1, \ \forall\, u \in W_h^{k,p}(\Omega_h).$$

The inverse estimate for isoparametric finite elements follows from Lemma 2.1, Part 1. This is presented in the following lemma.

Lemma 2.4 (Inverse estimate). *For $1 \leqslant k \leqslant l \leqslant r + 1$ and $1 \leqslant p, q \leqslant \infty$ the following estimate holds:*

$$\|\check{u}_h\|_{W^{l,p}(\check{K})} \leqslant Ch^{k-l+N/p-N/q}\|\check{u}_h\|_{W^{k,q}(\check{K})} \quad \forall\, \check{u}_h \in S_h(\Omega), \ \forall\, \check{K} \in \mathscr{K}. \tag{2.4}$$

The following lemma says that the $(r+1)$th-order derivative of a finite element function in $S_h(\Omega)$ can be bounded by its lower-order derivatives. This result is often used to prove a super-approximation property which is stated in Lemma 2.6 for iso-parametric finite elements.

Lemma 2.5. *The following result holds for iso-parametric finite element functions in $S_h(\Omega)$:*

$$|D^{r+1}\check{v}_h|(x) \leqslant C \sum_{i=1}^{r} |D^i \check{v}_h|(x) \quad \forall\, x \in \check{K}, \ \forall\, \check{K} \in \mathscr{K}, \ \forall\, \check{v}_h \in S_h(\Omega). \tag{2.5}$$

Proof. Let $M_K := \Phi_{h,K} \circ \widetilde{\Psi}_K$, which is a diffeomorpshism between the flat simplex $\widetilde{K}$ and the curved simplex $\check{K}$ (according to Lemma 2.1), satisfying the following estimates:

$$\|D^s M_K\|_{L^\infty(\widetilde{K})} \leqslant C \quad \text{and} \quad \|D^s M_K^{-1}\|_{L^\infty(\check{K})} \leqslant C \quad \forall\, 1 \leqslant s \leqslant r + 1.$$

According to the definition of $S_h(\Omega)$, a function $\check{v}_h$ is in $S_h(\Omega)$ if and only if the pull-back function $\check{v}_h \circ M_K$ is a polynomial degree $\leqslant r$ on the flat simplex $\widetilde{K}$. Therefore, from the estimate on higher order derivatives of composed functions (see [13, Lemma 3]), we have

$$|D^{r+1}\check{v}_h|(x)$$
$$= |D^{r+1}((\check{v}_h \circ M_K) \circ M_K^{-1})|(x)$$
$$\leqslant C \sum_{l=1}^{r+1} |D^l(\check{v}_h \circ M_K)(M_K^{-1}(x))| \sum_{i \in I(l,r+1)} |DM_K^{-1}(x)|^{i_1} |D^2 M_K^{-1}(x)|^{i_2} \dots |D^{r+1} M_K^{-1}(x)|^{i_{r+1}}$$

$$\leqslant C \sum_{l=1}^{r+1} |D^l(\check{v}_h \circ M_K)|(M_K^{-1}(x))$$

$$= C \sum_{l=1}^{r} |D^l(\check{v}_h \circ M_K)|(M_K^{-1}(x)),$$

where

$$I(l, r+1) := \{i = (i_1, i_2, ..., i_{r+1}) \in \mathbb{Z}^{r+1} : i_k \geqslant 0, \sum_{k=1}^{r+1} i_k = l; \sum_{k=1}^{r+1} k i_k = r+1\}.$$

We can estimate $|D^l(\check{v}_h \circ M_K)|(M_K^{-1}(x))$ using the same estimate on higher order derivatives of composed function

$$|D^l(\check{v}_h \circ M_K)|(M_K^{-1}(x))$$

$$\leqslant C \sum_{k=1}^{l} |D^k \check{v}_h|(x) \sum_{i \in I(k,l)} |DM_K(M_K^{-1}(x))|^{i_1} |D^2 M_K(M_K^{-1}(x))|^{i_2} ... |D^l M_K(M_K^{-1}(x))|^{i_l}$$

$$\leqslant C \sum_{k=1}^{l} |D^k \check{v}_h|(x)$$

The result of Lemma 2.5 is obtained by combining the two estimates above. $\qquad\square$

The result above is the key to the superapproximation results for the isoparametric case. For the standard elements the $r + 1$ derivative just vanishes.

Lemma 2.6 (Super-approximation). *Let $\omega \in C_0^\infty(\mathbb{R}^N)$ be a smooth cut-off function such that $0 \leqslant \omega \leqslant 1$ and $\operatorname{supp}(\omega) \cap \Omega \subset \Omega_0 \subset \Omega$, with $\Omega_0(d) := \{x \in \Omega : \operatorname{dist}(x, \Omega_0) \leqslant d\} \subset \Omega_1$ for some $d > h$. Then the following estimate holds for $\check{v}_h \in \mathring{S}_h(\Omega)$:*

$$\|\omega \check{v}_h - \check{I}_h(\omega \check{v}_h)\|_{H^1(\Omega_1)} \leqslant Ch\Big(\sum_{j=1}^{r} h^{j-1} \|\omega\|_{W^{j,\infty}(\mathbb{R}^N)}\Big) \|\check{v}_h\|_{H^1(\Omega_1)} + Ch^r \|\omega\|_{r+1,\infty} \|\check{v}_h\|_{L^2(\Omega_1)},$$

$$\|\omega \check{v}_h - \check{I}_h(\omega \check{v}_h)\|_{H^1(\Omega_1)} \leqslant C\Big(\sum_{j=1}^{r+1} h^{j-1} \|\omega\|_{W^{j,\infty}(\mathbb{R}^N)}\Big) \|\check{v}_h\|_{L^2(\Omega_1)}.$$

Proof. Since $\operatorname{supp}(\omega \check{v}_h) \subset \Omega_0$, it follows that $\check{I}_h(\omega \check{v}_h)$ vanishes on all $\check{K}$ such that $\check{K} \cap \Omega_0 = \varnothing$. Since $\Omega_0(d) \subset \Omega_1$, all the simplices $\check{K}$ such that $\check{K} \cap \Omega_0 \neq \varnothing$ are contained in Ω_1. Therefore, we have

$$\|\omega \check{v}_h - \check{I}_h(\omega \check{v}_h)\|_{H^1(\Omega_1)}^2 = \sum_{\check{K} \cap \Omega_0 \neq \varnothing} \|\omega \check{v}_h - \check{I}_h(\omega \check{v}_h)\|_{H^1(\check{K})}^2$$

$$\leqslant \sum_{\check{K} \cap \Omega_0 \neq \varnothing} Ch^{2r} \|\omega \check{v}_h\|_{H^{r+1}(\check{K})}^2 \tag{2.6}$$

$$\leqslant \sum_{\check{K} \cap \Omega_0 \neq \varnothing} Ch^{2r} \Big(|\check{v}_h|_{H^{r+1}(\check{K})}^2 + \sum_{i=0}^{r} \|\omega\|_{W^{r+1-i,\infty}(\mathbb{R}^N)}^2 \|\check{v}_h\|_{H^i(\check{K})}^2\Big).$$

The term $|\check{v}_h|_{H^{r+1}(\check{K})}^2$ can be estimated by using Lemma 2.5, i.e.,

$$|\check{v}_h|_{H^{r+1}(\check{K})}^2 \leqslant C \sum_{i=1}^{r} |\check{v}_h|_{H^i(\check{K})}^2. \tag{2.7}$$

For $0 \leqslant i \leqslant r$, the term $|\check{v}_h|_{H^i(\check{K})}$ can be estimated by using the inverse estimate for isoparametric finite element functions (see Lemma 2.4). This yields the first result of Lemma 2.6. The second result can be proved similarly. $\qquad\square$

2.2. The perturbed bilinear form associated to the isoparametric FEM

By using the notation $\check{u}_h \circ \Phi_h = u_h$ and $\check{v}_h \circ \Phi_h = v_h$ for $u_h, v_h \in S_h(\Omega_h)$, the following identity holds:

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h \, \mathrm{d}x = \int_{\Omega_h} \nabla(\check{u}_h \circ \Phi_h) \cdot \nabla(\check{u}_h \circ \Phi_h) \, \mathrm{d}x$$
$$= \int_{\Omega} A_h \nabla \check{u}_h \cdot \nabla \check{v}_h \, \mathrm{d}x \quad \forall \, \check{v}_h \in \mathring{S}_h(\Omega), \tag{2.8}$$

where

$$A_h = (\nabla \Phi_h (\nabla \Phi_h)^\top J^{-1}) \circ \Phi_h^{-1}$$

is a piecewise smooth (globally discontinuous) and symmetric matrix-valued function, and $J = \det(\nabla \Phi_h) \in L^\infty(\Omega_h)$ is the Jacobian of the mapping $\Phi_h : \Omega_h \to \Omega$, piecewisely defined on every simplex $K \in \mathscr{K}$. Therefore, a function $u_h \in S_h(\Omega_h)$ is discrete harmonic if and only if

$$\int_{\Omega} A_h \nabla \check{u}_h \cdot \nabla \check{v}_h \, \mathrm{d}x = 0 \quad \forall \, \check{v}_h \in \mathring{S}_h(\Omega), \tag{2.9}$$

Identity (2.9) will be used frequently in the following proof.

Since the map $\Phi_h : \Omega_h \to \Omega$ is close to the identity map $\mathrm{Id} : \mathbb{R}^N \to \mathbb{R}^N$ (which satisfies $\mathrm{Id}(x) \equiv x$), it follows that the matrix A_h is close to the identity matrix. In particular, the following results are corollaries of the second statement of Lemma 2.1:

$$\|\nabla^j (\Phi_h - \mathrm{Id})\|_{L^\infty(\Omega_h)} \leqslant Ch^{r+1-j} \quad \text{and} \quad \|A_h - I\|_{L^\infty(\Omega)} \leqslant Ch^r, \quad \text{for } j = 0, 1. \tag{2.10}$$

Therefore, for sufficiently small mesh size h, the perturbed bilinear form $\check{B}_h : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$\check{B}_h(v, \chi) = \int_{\Omega} A_h \nabla v \cdot \nabla \chi \, \mathrm{d}x \tag{2.11}$$

is continuous and coercive on $H_0^1(\Omega)$, i.e.,

$$\check{B}_h(v, \chi) \leqslant C \|\nabla v\|_{L^2(\Omega)} \|\nabla \chi\|_{L^2(\Omega)} \qquad \forall \, v, \chi \in H^1(\Omega),$$
$$\check{B}_h(v, v) \geqslant C^{-1} \|\nabla v\|_{L^2(\Omega)}^2 \sim \|v\|_{H^1(\Omega)}^2 \quad \forall \, v \in H_0^1(\Omega). \tag{2.12}$$

More precisely, the difference between $\check{B}_h(u, v)$ and $B(u, v)$ is estimated in the following lemma.

Lemma 2.7. *There exists a positive constant $h_1 > 0$ such that for $h \leqslant h_1$ the following result holds: If $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$, then*

$$\left| \check{B}_h(u, v) - B(u, v) \right| \leqslant Ch^r \|\nabla u\|_{L^p(\Lambda_h)} \|\nabla v\|_{L^q(\Lambda_h)}$$

where $\Lambda_h := \{x \in \Omega : \mathrm{dist}(x, \partial \Omega) \leqslant 2h\}$.

Proof. Since $\Phi_h = \mathrm{Id}$ at all interior simplices, it follows that $A_h \circ \Phi_h = I$ outside the subdomain $D_h = \{x \in \Omega_h : \mathrm{dist}(x, \partial \Omega_h) \leqslant h\}$. Correspondingly, $A_h = I$ outside the subdomain $\Phi_h(D_h)$ and therefore,

$$\left| \check{B}_h(u, v) - B(u, v) \right| \leqslant \|A_h - I\|_{L^\infty(\Phi_h(D_h))} \|\nabla u\|_{L^p(\Phi_h(D_h))} \|\nabla v\|_{L^q(\Phi_h(D_h))}$$
$$\leqslant Ch^r \|\nabla u\|_{L^p(\Phi_h(D_h))} \|\nabla v\|_{L^q(\Phi_h(D_h))}.$$

If $x \in D_h$, then there exists $x' \in \partial\Omega_h$ such that $|x - x'| = \mathrm{dist}(x, \partial\Omega_h) \leqslant h$ and

$$|\Phi_h(x) - \Phi_h(x')| \leqslant |\Phi_h(x) - x| + |x - x'| + |x' - \Phi_h(x')| \leqslant Ch^{r+1} + h + Ch^{r+1},$$

which implies that

$$\mathrm{dist}(\Phi_h(x), \partial\Omega) \leqslant Ch^{r+1} + h.$$

For sufficiently small h we obtain $\mathrm{dist}(\Phi_h(x), \partial\Omega) \leqslant 2h$ and therefore $\Phi_h(D_h) \subset \Lambda_h$. $\qquad\square$

2.3. Reduction of the problem

Let $x_0 \in \overline{\Omega}$ be a point satisfying

$$|\check{u}_h(x_0)| = \|\check{u}_h\|_{L^\infty(\Omega)} \quad \text{with} \quad d = \mathrm{dist}(x_0, \partial\Omega).$$

If $d \geqslant 2kh$ for some fixed $k \geqslant 1$, i.e., x_0 is relatively far away from the boundary $\partial\Omega$, then we can choose $\Omega_1 = \{x_0\}$ and $\Omega_2 = S_{d/2}(x_0)$ and use the interior L^∞ estimate established in [40, Corollary 5.1]. This yields the following result:

$$|\check{u}_h(x_0)| \leqslant Cd^{-\frac{N}{2}}\|\check{u}_h\|_{L^2(S_d(x_0))}.$$

Otherwise, we have $d < 2kh$. In this case, assuming that $x_0 \in \check{K}$ for some curved simplex $\check{K} \in \check{\mathcal{K}}$, by the inverse estimate in Lemma 2.4 we have

$$|\check{u}_h(x_0)| = \|\check{u}_h\|_{L^\infty(\check{K})} \leqslant Ch^{-\frac{N}{2}}\|\check{u}_h\|_{L^2(\check{K})} \leqslant Ch^{-\frac{N}{2}}\|\check{u}_h\|_{L^2(S_{2kh}(x_0))}.$$

Overall, for either $d \geqslant 2kh$ or $d < 2kh$, the following estimate holds:

$$|\check{u}_h(x_0)| \leqslant C\rho^{-\frac{N}{2}}\|\check{u}_h\|_{L^2(S_\rho(x_0))}, \quad \text{with} \quad \rho = d + 2kh. \tag{2.13}$$

To estimate the term $\|\check{u}_h\|_{L^2(S_\rho(x_0))}$ on the right-hand side of (2.13), we use the following duality property:

$$\|\check{u}_h\|_{L^2(S_\rho(x_0))} = \sup_{\substack{\mathrm{supp}(\varphi)\subset S_\rho(x_0) \\ \|\varphi\|_{L^2(S_\rho(x_0))}\leqslant 1}} |(\check{u}_h, \varphi)|,$$

where $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$ (or $L^2(\Omega)^N$ for vector-valued functions), i.e.,

$$(u, v) := \int_\Omega u \cdot v \, \mathrm{d}x.$$

Hence, there exists a function $\varphi \in C_0^\infty(\Omega)$ with the following properties:

$$\mathrm{supp}(\varphi) \subset S_\rho(x_0), \quad \|\varphi\|_{L^2(S_\rho(x_0))} \leqslant 1, \tag{2.14}$$

$$\|\check{u}_h\|_{L^2(S_\rho(x_0))} \leqslant 2|(\check{u}_h, \varphi)|. \tag{2.15}$$

For this function φ, we define $v \in H_0^1(\Omega)$ and $u \in H^1(\Omega)$ to be the solutions of the following elliptic equations (in the weak form):

$$\begin{cases} (A_h\nabla v, \nabla\chi) = (\varphi, \chi) & \forall \chi \in H_0^1(\Omega), \\ \quad\quad\quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

and

$$\begin{cases} (A_h\nabla u, \nabla\chi) = 0 & \forall \chi \in H_0^1(\Omega), \\ \quad\quad\quad u = \check{u}_h & \text{on } \partial\Omega, \end{cases} \tag{2.17}$$

respectively. The maximum principle of the continuous problem (2.17) implies that

$$\|u\|_{L^\infty(\Omega)} \leqslant \|\check{u}_h\|_{L^\infty(\partial\Omega)}. \tag{2.18}$$

Therefore, we have

$$
\begin{aligned}
\|\check{u}_h\|_{L^2(S_\rho(x_0))} &\leqslant 2|(\check{u}_h, \varphi)| &&\text{(here we have used (2.15))}\\
&= 2|(\check{u}_h - u, \varphi) + (u, \varphi)|\\
&= 2|(A_h \nabla(\check{u}_h - u), \nabla v) + (u, \varphi)| &&\text{(here we have used (2.16))}\\
&= 2|(A_h \nabla \check{u}_h, \nabla v) + (u, \varphi)| &&\text{(here we have used (2.17))}\\
&\leqslant 2|(A_h \nabla \check{u}_h, \nabla v)| + 2\|u\|_{L^\infty(\Omega)}\|\varphi\|_{L^1(S_\rho(x_0))} &&\text{(since } \mathrm{supp}(\varphi) \subset S_\rho(x_0))\\
&\leqslant 2|(A_h \nabla \check{u}_h, \nabla v)| + C\rho^{\frac{N}{2}}\|\check{u}_h\|_{L^\infty(\partial\Omega)}\|\varphi\|_{L^2(S_\rho(x_0))}, &&(2.19)
\end{aligned}
$$

where we have used (2.18) and the Hölder inequality in deriving the last inequality. Combing inequalities (2.13) and (2.19), we have

$$
\|\check{u}_h\|_{L^\infty(\Omega)} = |\check{u}_h(x_0)| \leqslant C\rho^{-\frac{N}{2}}|(A_h \nabla \check{u}_h, \nabla v)| + C\|\check{u}_h\|_{L^\infty(\partial\Omega)} \tag{2.20}
$$

where we have used the fact that $\|\varphi\|_{L^2(S_\rho(x_0))} \leqslant 1$.

It remains to estimate $\rho^{-\frac{N}{2}}|(A_h \nabla \check{u}_h, \nabla v)|$. To this end, we define $R_h : H_0^1(\Omega) \to \mathring{S}_h(\Omega)$ to be the Ritz projection associated with the perturbed bilinear form defined in (2.11), i.e.,

$$
(A_h \nabla(v - R_h v), \nabla \check{\chi}_h) = 0 \quad \forall \check{\chi}_h \in \mathring{S}_h(\Omega), \tag{2.21}
$$

which is well defined in view of the coercivity of the bilinear form; see (2.12). By using identity (2.9) for the discrete harmonic function u_h and the definition of the Ritz projection R_h in (2.21), we have

$$
\begin{aligned}
(A_h \nabla \check{u}_h, \nabla v) &= (A_h \nabla \check{u}_h, \nabla(v - R_h v))\\
&= (A_h \nabla(\check{u}_h - \check{\chi}_h), \nabla(v - R_h v)) \quad \forall \check{\chi}_h \in \mathring{S}_h(\Omega). &&(2.22)
\end{aligned}
$$

In particular, we can choose $\check{\chi}_h = \chi_h \circ \Phi_h^{-1} \in \mathring{S}_h(\Omega)$ to satisfy $\chi_h = u_h$ on all interior Lagrange nodes while $\chi_h = 0$ on all the boundary nodes (which implies $\chi_h = 0$ on $\partial\Omega_h$ and therefore $\check{\chi}_h \equiv 0$ on $\partial\Omega$). Then

$$
\|\check{\chi}_h - \check{u}_h\|_{L^\infty(\Omega)} \leqslant C\|\check{u}_h\|_{L^\infty(\partial\Omega)}. \tag{2.23}
$$

Let $\Lambda_h = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leqslant 2h\}$ be a neighborhood of the boundary $\partial\Omega$, when h sufficiently small, $\check{u}_h - \check{\chi}_h = 0$ outside Λ_h. Then

$$
\begin{aligned}
|(A_h \nabla(\check{u}_h - \check{\chi}_h), \nabla(v - R_h v))| &\leqslant C\|\nabla(\check{\chi}_h - \check{u}_h)\|_{L^\infty(\Omega)}\|\nabla(v - R_h v)\|_{L^1(\Lambda_h)}\\
&\leqslant Ch^{-1}\|\check{u}_h\|_{L^\infty(\partial\Omega)}\|\nabla(v - R_h v)\|_{L^1(\Lambda_h)}, &&(2.24)
\end{aligned}
$$

where we have used (2.23) and the inverse estimate for finite element functions. Substituting (2.22) and (2.24) into (2.20), we obtain

$$
\|\check{u}_h\|_{L^\infty(\Omega)} \leqslant C\big(\rho^{-\frac{N}{2}}h^{-1}\|\nabla(v - R_h v)\|_{L^1(\Lambda_h)} + 1\big)\|u_h\|_{L^\infty(\partial\Omega)}. \tag{2.25}
$$

The proof of Theorem 1.1 will be completed if the following result holds:

$$
\rho^{-\frac{N}{2}}h^{-1}\|\nabla(v - R_h v)\|_{L^1(\Lambda_h)} \leqslant C, \tag{2.26}
$$

which will be proved in the following subsections.

2.4. Regularity decomposition

In order to estimate the left-hand side of (2.26), we need to use a local energy estimate and a duality argument, which is based on the regularity result of the following elliptic equation (in the weak form): Find $v \in H_0^1(\Omega)$ such that

$$(A_h \nabla v, \nabla \chi) = (f, \chi) \quad \forall \chi \in H_0^1(\Omega), \tag{2.27}$$

where A_h is a globally discontinuous matrix-valued function defined in Section 2.2.

Due to the discontinuity of the coefficient matrix A_h, the standard H^2 regularity does not hold for the elliptic equation (2.27). We decompose the solution $v \in H_0^1(\Omega)$ of equation (2.27) into the following two parts:

$$v = v_1 + v_2, \tag{2.28}$$

where $v_1 \in H_0^1(\Omega)$ and $v_2 \in H_0^1(\Omega)$ are the weak solutions of the equations

$$(\nabla v_1, \nabla \chi) = (f, \chi) \qquad\qquad \forall \chi \in H_0^1(\Omega), \tag{2.29}$$

$$(A_h \nabla v_2, \nabla \chi) = ((I - A_h) \nabla v_1, \nabla \chi) \qquad\qquad \forall \chi \in H_0^1(\Omega). \tag{2.30}$$

Equation (2.29) has a constant coefficient and therefore the classical $W^{2,q}$ regularity estimate holds for $1 < q < 2 + \varepsilon$, for some $\varepsilon > 0$ which depends on the interior angles at the edges and corners of the domain Ω (see [14, Corollaries 3.7, 3.9 and 3.12]), i.e.,

$$\|v_1\|_{W^{2,q}(\Omega)} \leqslant C_q \|f\|_{L^q(\Omega)} \quad \forall 1 < q < 2 + \varepsilon. \tag{2.31}$$

Since equation (2.30) has discontinuous coefficients, the $W^{2,q}$ regularity estimate does not hold. We have to estimate v_2 by using the $W^{1,p}$ estimate in the following lemma.

Lemma 2.8. *For every $1 < p < \infty$ there exists $h_p > 0$ (which depends on p), such that for $h \leqslant h_p$, the solution $w \in H_0^1(\Omega)$ of the equation*

$$(A_h \nabla w, \nabla \chi) = (\vec{g}, \nabla \chi) \quad \forall \chi \in H_0^1(\Omega) \quad with \quad \vec{g} \in L^p(\Omega)^N \cap L^2(\Omega)^N, \tag{2.32}$$

satisfies $w \in W^{1,p}(\Omega)$ and

$$\|w\|_{W^{1,p}(\Omega)} \leqslant C_p \|\vec{g}\|_{L^p(\Omega)}, \tag{2.33}$$

where C_p is a constant which is independent of h (possibly depending on p).

Proof. We can rewrite equation (2.32) into the following form:

$$(\nabla w, \nabla \chi) = (\vec{g}, \nabla \chi) + ((I - A_h) \nabla w, \nabla \chi) \quad \forall \chi \in H_0^1(\Omega),$$

and apply the $W^{1,p}$ regularity estimate for the Poisson equation (which holds in a smooth domain or curvilinear polyhedron with edge openings smaller than π; see [14, Corollaries 3.7, 3.9 and 3.12]). This yields the following inequality:

$$\|w\|_{W^{1,p}(\Omega)} \leqslant C_p \|\vec{g}\|_{L^p(\Omega)} + C_p \|I - A_h\|_{L^\infty(\Omega)} \|w\|_{W^{1,p}(\Omega)}.$$

Since $\|A_h - I\|_{L^\infty} \leqslant Ch$, for sufficiently small h (depending on p) the last term on the right-hand side can be absorbed by the left-hand side. This yields the result of Lemma 2.8. $\qquad\square$

By combining the $W^{2,q}$ regularity estimate in (2.31) and the $W^{1,p}$ regularity estimate in Lemma 2.8, we can prove the following result.

Lemma 2.9. *Let $1 < p, q < \infty$ be numbers such that $1/q \leqslant 1/n + 1/p$, and assume that $h \leqslant h_p$, where h_p is given in Lemma 2.8. Let $w \in H_0^1(\Omega)$ be the weak solution of the equation*

$$(A_h \nabla w, \nabla \chi) = (f, \chi) + (\vec{g}, \nabla \chi) \quad \forall \chi \in H_0^1(\Omega) \tag{2.34}$$

for some $f \in L^q(\Omega) \cap L^2(\Omega)$ and $\vec{g} \in L^p(\Omega)^N \cap L^2(\Omega)^N$. Then $w \in W^{1,p}(\Omega)$ and

$$\|w\|_{W^{1,p}(\Omega)} \leqslant C_p\|f\|_{L^q(\Omega)} + C_p\|\vec{g}\|_{L^p(\Omega)}. \qquad (2.35)$$

Proof. We consider the decomposition $w = w_1 + w_2$ with $w_1, w_2 \in H_0^1(\Omega)$ weakly solving

$$(\nabla w_1, \nabla \chi) = (f, \chi) \qquad\qquad \forall \chi \in H_0^1(\Omega),$$
$$(A_h \nabla w_2, \chi) = ((I - A_h)\nabla w_1 + \vec{g}, \nabla \chi) \quad \forall \chi \in H_0^1(\Omega).$$

Note that for $\chi \in W_0^{1,p'}(\Omega)$ where $1/p + 1/p' = 1$

$$|(f, \chi)| \leqslant \|f\|_{L^q(\Omega)}\|\chi\|_{L^{q'}(\Omega)} \quad (1/q + 1/q' = 1)$$
$$\leqslant C\|f\|_{L^q(\Omega)}\|\chi\|_{W^{1,p'}(\Omega)} \quad \text{(embedding } W^{1,p'} \hookrightarrow L^{q'} \text{ used)},$$

therefore we have $\|f\|_{W^{-1,p}(\Omega)} \leqslant C\|f\|_{L^q(\Omega)}$. By the $W^{1,p}$ regularity estimate for the Poisson equation on curvilinear polyhedron (see [14, Corollaries 3.7, 3.9 and 3.12]), there holds

$$\|w_1\|_{W^{1,p}(\Omega)} \leqslant C_p\|f\|_{W^{-1,p}(\Omega)} \leqslant C_p\|f\|_{L^q(\Omega)}$$

Then we apply the $W^{1,p}$ estimate in Lemma 2.8 to the equation of w_2. This yields the following result:

$$\|w_2\|_{W^{1,p}(\Omega)} \leqslant C_p\|\vec{g} + (I - A_h)\nabla w_1\|_{L^p(\Omega)} \leqslant C_p\|\vec{g}\|_{L^p(\Omega)} + C_p\|f\|_{L^q(\Omega)}.$$

The result of Lemma 2.9 follows from combining the estimates for w_1 and w_2. $\qquad\square$

The following lemma was proved in [32, Lemma 2.2] for polyhedral domains. The proof of this result for smooth domains and curvilinear polyhedron is the same.

Lemma 2.10. *If $\chi \in W_0^{1,p}(\Omega)$ for some $1 < p < \infty$ and $x^* \in \partial\Omega$, then*

$$\|\chi\|_{L^p(S_{d_*}(x^*))} \leqslant Cd_*\|\nabla\chi\|_{L^p(\Omega)},$$

where $S_{d_*}(x^*) := \{x \in \Omega : |x - x^*| < d_*\}$.

Lemma 2.11. *Let $1 < p < \infty$ and $h \leqslant h_p$, where h_p is given in Lemma 2.8. For*

$$f \in L^p(\Omega) \cap L^2(\Omega) \text{ with } \mathrm{supp}(f) \subset S_{d_*}(x_0), \text{ where } x_0 \in \overline{\Omega} \text{ and } \mathrm{dist}(x_0, \partial\Omega) \leqslant d_*,$$

the solution $v \in H_0^1(\Omega)$ of equation (2.27) satisfies

$$\|v\|_{W^{1,p}(\Omega)} \leqslant C_p d_*\|f\|_{L^p(\Omega)} \qquad (2.36)$$

Proof. We consider the decomposition $v = v_1 + v_2$ in (2.28)–(2.30). If $\mathrm{dist}(x_0, \partial\Omega) \leqslant d_*$, then $S_{d_*}(x_0) \subset S_{2d_*}(\bar{x}_0)$ for some $\bar{x}_0 \in \partial\Omega$. Note that for $\chi \in W_0^{1,p'}(\Omega)$ where $1/p + 1/p' = 1$, we have

$$|(f, \chi)| \leqslant \|f\|_{L^p(S_{d_*}(x_0))}\|\chi\|_{L^{p'}(S_{d_*}(x_0))}$$
$$\leqslant \|f\|_{L^p(S_{d_*}(x_0))}\|\chi\|_{L^{p'}(S_{2d_*}(\bar{x}_0))}$$
$$\leqslant Cd_*\|f\|_{L^p(\Omega)}\|\nabla\chi\|_{L^{p'}(\Omega)}, \quad \text{(Lemma 2.10 used)}$$

which implies that $\|f\|_{W^{-1,p}(\Omega)} \leqslant Cd_*\|f\|_{L^p(\Omega)}$. Thus by the $W^{1,p}$ regularity estimate for the Poisson equation on curvilinear polyhedron (see [14, Corollaries 3.7, 3.9 and 3.12]), there holds:

$$\|v_1\|_{W^{1,p}(\Omega)} \leqslant C_p\|f\|_{W^{-1,p}(\Omega)} \leqslant C_p d_*\|f\|_{L^p(\Omega)}.$$

By applying Lemma 2.8 to equation (2.30), we obtain

$$\|v_2\|_{W^{1,p}(\Omega)} \leqslant C_p\|(I - A_h)\nabla v_1\|_{L^p(\Omega)} \leqslant C_p h\|v_1\|_{W^{1,p}(\Omega)} \leqslant C_p h d_*\|f\|_{L^p(\Omega)}.$$

The last two inequalities imply the result of Lemma 2.11. $\qquad\square$

The next lemma is about the Cacciopoli inequality for harmonic functions which is the same as in [41, Lemma 8.3]. The result holds for smooth domains and curvilinear polyhedra on which the elliptic H^2 regularity result holds for the Poisson equation.

Lemma 2.12. *Let D and D_d be two subdomains of Ω satisfying $D \subset D_d \subset \Omega$, with*

$$D_d = \{x \in \Omega : dist(x, D) \leqslant d\},$$

where d is a positive constant. If $v \in H_0^1(\Omega)$ and v is harmonic on D_d, i.e.

$$(\nabla v, \nabla w) = 0 \quad \forall w \in H_0^1(D_d),$$

then the following estimates hold:

$$|v|_{H^2(D)} \leqslant Cd^{-1}\|v\|_{H^1(D_d)}, \tag{2.37a}$$

$$\|v\|_{H^1(D)} \leqslant Cd^{-1}\|v\|_{L_2(D_d)}. \tag{2.37b}$$

We also need the following interior estimate in the estimation of v_2.

Lemma 2.13. *Let $1 < p, q < \infty$ be numbers such that $1/q \leqslant 1/n + 1/p$ and assume that $h \leqslant h_p$, where h_p is given in Lemma 2.8. Let $D \subset D_d \subset \Omega$ be subdomains, with $D_d = \{x \in \Omega : dist(x, D) \leqslant d\}$. If $v \in W_0^{1,p}(\Omega) \cap H_0^1(\Omega)$ satisfies equation*

$$(A_h \nabla v, \nabla \chi) = 0 \quad \forall \chi \in H_0^1(D_d), \tag{2.38}$$

or

$$(\nabla v, \nabla \chi) = 0 \quad \forall \chi \in H_0^1(D_d). \tag{2.39}$$

Then

$$\|v\|_{W^{1,p}(D)} \leqslant \frac{C_p}{d}(\|v\|_{L^p(D_d)} + \|v\|_{W^{1,q}(D_d)}). \tag{2.40}$$

Proof. We focus on the first case: v satisfies equation (2.38). The proof for the second case is the same and therefore omitted.

First, we choose a cut-off function $\omega \in C_0^\infty(\mathbb{R}^N)$, $\omega \equiv 1$ on D, $\text{supp}(\omega) \cap \Omega \subset D_d$, with $\|\omega\|_{W^{1,\infty}(\mathbb{R}^N)} \leqslant Cd^{-1}$. Then $\omega v \in H_0^1(\Omega)$ satisfies the following equation:

$$\begin{aligned}
(A_h \nabla(\omega v), \nabla \chi) &= (\omega A_h \nabla v, \nabla \chi) + (A_h \nabla \omega, v \nabla \chi) \\
&= (A_h \nabla v, \nabla(\omega \chi)) - (A_h \nabla v, \chi \nabla \omega) + (A_h \nabla \omega, v \nabla \chi) \\
&= (A_h v \nabla \omega, \nabla \chi) - (A_h \nabla v \cdot \nabla \omega, \chi) \quad \forall \chi \in H_0^1(\Omega)
\end{aligned}$$

where we have used the identity $(A_h \nabla v, \nabla(\omega \chi)) = 0$ in the derivation of the last equality, which is a consequence of (2.38) and $\omega \chi \in H_0^1(D_d)$. Then we can apply Lemma 2.9 to the above equation satisfied by ωv. This yields the following result:

$$\begin{aligned}
\|\omega v\|_{W^{1,p}(\Omega)} &\leqslant C_p\|A_h v \nabla \omega\|_{L^p(\Omega)} + C_p\|A_h \nabla v \cdot \nabla \omega\|_{L^q(\Omega)} \\
&\leqslant \frac{C_p}{d}\|v\|_{L^p(D_d)} + \frac{C_p}{d}\|v\|_{W^{1,q}(D_d)}.
\end{aligned}$$

Since $\omega = 1$ on D, the last inequality implies the result of Lemma 2.13. $\qquad\square$

Lemma 2.14. *Let $1 < p, q < \infty$ be numbers such that $1/q \leqslant 1/n + 1/p$ and assume that $h \leqslant \min\{h_p, h_q\}$, where h_p, h_q are given in Lemma 2.8. Let $D \subset D_d \subset \Omega$ be subdomains, with $D_d = \{x \in \Omega : dist(x, D) \leqslant d\}$. If the source function f has $\text{supp}(f) \cap D_d = \varnothing$, then the solution v_2 of equation (2.30) satisfies the following estimate:*

$$\|v_2\|_{W^{1,p}(D)} \leqslant \frac{C_{p,q}}{d}h\|v_1\|_{W^{1,q}(\Omega)}. \tag{2.41}$$

Proof. We consider a cut-off function ω such that $\omega \equiv 1$ in D and $\text{supp}(\omega) \subset D_{d/2}$, with $\|\omega\|_{W^{1,\infty}(\mathbb{R}^N)} \leqslant Cd^{-1}$. Then the following equation can be written down similarly as in the proof of Lemma 2.13:

$$(A_h \nabla(\omega v_2), \nabla \chi) = (\omega(I - A_h)\nabla v_1, \nabla \chi) + ((I - A_h)\nabla v_1 \cdot \nabla \omega, \chi)$$
$$+ (v_2 A_h \nabla \omega, \nabla \chi) - (A_h \nabla v_2 \cdot \nabla \omega, \chi) \quad \forall \chi \in H_0^1(\Omega).$$

By applying Lemma 2.9 to the equation above, we obtain

$$\|v_2\|_{W^{1,p}(D)} \leqslant C_p h \|v_1\|_{W^{1,p}(D_{d/2})} + \frac{C_p h}{d}\|v_1\|_{W^{1,q}(\Omega)} + \frac{C_p}{d}\|v_2\|_{L^p(\Omega)} + \frac{C_p}{d}\|v_2\|_{W^{1,q}(\Omega)}$$
$$\leqslant C_p h \|v_1\|_{W^{1,p}(D_{d/2})} + \frac{C_p h}{d}\|v_1\|_{W^{1,q}(\Omega)} + \frac{C_p}{d}\|v_2\|_{W^{1,q}(\Omega)}, \tag{2.42}$$

where we have used Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$.

Since $\text{supp}(f) \cap D_d = \varnothing$, it follows that the solution v_1 of (2.29) satisfies equation (2.39). Therefore, Lemma 2.13 implies that

$$\|v_1\|_{W^{1,p}(D_{d/2})} \leqslant \frac{C_p}{d}(\|v\|_{L^p(D_d)} + \|v\|_{W^{1,q}(D_d)}) \leqslant \frac{C_p}{d}\|v_1\|_{W^{1,q}(\Omega)}.$$

By applying Lemma 2.8 to equation (2.30), we also obtain

$$\|v_2\|_{W^{1,q}(\Omega)} \leqslant C_q \|I - A_h\|_{L^\infty(\Omega)}\|v_1\|_{W^{1,q}(\Omega)} \leqslant C_q h \|v_1\|_{W^{1,q}(\Omega)}.$$

Then, substituting the last two inequalities into (2.42), we obtain the result of Lemma 2.14. □

2.5. $W^{1,p}$ stablity of the Ritz projection (with discontinuous coefficients)

In [22] the $W^{1,\infty}$ stability of the Ritz projection is proved for the Poisson equation in convex polyhedral domains. The proof is based on the following properties of the domain and finite elements:

(P1) Hölder estimates of the Green function for the Poisson equation, i.e.,

$$\frac{|\partial_{x_i} G(x, \xi) - \partial_{y_i} G(y, \xi)|}{|x - y|^\sigma} \leqslant C\left(|x - \xi|^{-2-\sigma} + |y - \xi|^{-2-\sigma}\right)$$
$$\frac{|\partial_{x_i}\partial_{\xi_j} G(x, \xi) - \partial_{y_i}\partial_{\xi_j} G(y, \xi)|}{|x - y|^\sigma} \leqslant C\left(|x - \xi|^{-3-\sigma} + |y - \xi|^{-3-\sigma}\right) \tag{2.43}$$

for $i, j = 1, 2, 3$.
(P2) Elliptic H^2 regularity result for the Poisson equation.
(P3) Exact triangulation which matches the boundary $\partial\Omega$.
(P4) Error estimates for the Lagrange interpolation holds as in Lemma 2.2.

Note that the Hölder estimates for the Green function in (2.43) was proved in [22] for general curvilinear polyhedral domains with edge opening smaller than π, instead of merely classical polyhedral domains. If we define a modified Ritz projection R_h^* associated to the Poisson equation (without the discontinuous coefficient A_h), i.e.,

$$\int_\Omega \nabla(v - R_h^* v) \cdot \nabla \check{\chi}_h \, dx = 0 \quad \forall \check{\chi}_h \in \mathring{\check{S}}_h(\Omega), \tag{2.44}$$

then all the properties in (P1)–(P4) are possessed by the curvilinear polyhedral domain Ω and the finite element space $\mathring{\check{S}}_h(\Omega)$. The latter is based on the triangulation $\check{K}$ which matches the boundary $\partial\Omega$ exactly. Therefore, the $W^{1,\infty}$ stability still holds for the modified Ritz projection defined in (2.44). The result is stated in the following lemma.

Lemma 2.15.

$$\|R_h^* v\|_{W^{1,\infty}(\Omega)} \leqslant C\|v\|_{W^{1,\infty}(\Omega)} \quad \forall\, v \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega). \tag{2.45}$$

By real interpolation between the H^1 and $W^{1,\infty}$ stability estimates (see [8, result in (5.1)]), we obtain the $W^{1,p}$ stability of the modified Ritz projection for $2 \leqslant p \leqslant \infty$. The result can also be extended to $1 < p \leqslant 2$ by a duality argument as in [7, Section 8.5], which requires Poisson equation to have the $W^{1,p'}$ regularity (this is true for a curvilinear polyhedron with edge opening smaller than π). The result is summarized below.

Lemma 2.16 ($W^{1,p}$ *stability of the modified Ritz projection* R_h^*). *For any* $1 < p \leqslant \infty$, *there exists a positive constant* h_p *such that for* $h \leqslant h_p$ *the following result holds:*

$$\|R_h^* u\|_{W^{1,p}(\Omega)} \leqslant C_p\|u\|_{W^{1,p}(\Omega)} \quad \forall\, u \in W^{1,p}(\Omega) \cap H_0^1(\Omega). \tag{2.46}$$

By a "perturbation" argument, similar as [7, Section 8.6], one can obtain the $W^{1,p}$ stability of the Ritz projection R_h. This is stated in the following proposition.

Proposition 2.17 ($W^{1,p}$ *stability of the Ritz projection* R_h). *For any* $1 < p < \infty$, *there exists a positive constant* h_p *such that for* $h \leqslant h_p$ *the following result holds:*

$$\|R_h u\|_{W^{1,p}(\Omega)} \leqslant C_p\|u\|_{W^{1,p}(\Omega)} \quad \forall\, u \in W^{1,p}(\Omega) \cap H_0^1(\Omega). \tag{2.47}$$

Proof. For $v \in H_0^1(\Omega)$, its Ritz projection $R_h v \in \mathring{S}_h(\Omega)$ satisfies the following equation:

$$\int_\Omega \nabla(v - R_h v) \cdot \nabla \check{\chi}_h \,\mathrm{d}x = \int_\Omega (I - A_h)\nabla(v - R_h v) \cdot \nabla \check{\chi}_h \,\mathrm{d}x \quad \forall\, \check{\chi}_h \in \mathring{S}_h(\Omega).$$

If we define w to be the solution of the following elliptic equation (in the weak form):

$$\int_\Omega \nabla w \cdot \nabla \check{\chi} \,\mathrm{d}x = -\int_\Omega (I - A_h)\nabla(v - R_h v) \cdot \nabla \check{\chi} \,\mathrm{d}x \quad \forall\, \check{\chi} \in H_0^1(\Omega),$$

then

$$\int_\Omega \nabla(w + v - R_h v) \cdot \nabla \check{\chi}_h \,\mathrm{d}x = 0 \quad \forall\, \check{\chi}_h \in \mathring{S}_h(\Omega),$$

which means that $R_h v = R_h^*(w + v)$. Lemma 2.16 implies that

$$
\begin{aligned}
\|R_h v\|_{W^{1,p}(\Omega)} = \|R_h^*(w + v)\|_{W^{1,p}(\Omega)} &\leqslant C_p\|w + v\|_{W^{1,p}(\Omega)} \\
&\leqslant C_p\|I - A_h\|_{L^\infty(\Omega)}\|v - R_h v\|_{W^{1,p}(\Omega)} + C_p\|v\|_{W^{1,p}(\Omega)} \\
&\leqslant C_p h\|R_h v\|_{W^{1,p}(\Omega)} + C_p\|v\|_{W^{1,p}(\Omega)}.
\end{aligned}
$$

There exists a constant h_p such that for $h \leqslant h_p$ the first term on the right-hand side can be absorbed by the left-hand side. In this case we obtain the result of Proposition 2.17. $\square$

As a result of Proposition 2.17, we obtain the following $W^{1,p}$ error estimate for the Ritz projection.

Lemma 2.18. *For any* $1 < q < 2 + \varepsilon$, *there exists a positive constant* h_q *such that for* $h \leqslant h_q$ *the solution of equation* (2.27) *has the following error bound:*

$$\|v - R_h v\|_{W^{1,q}(\Omega)} \leqslant C_q h\|f\|_{L^q(\Omega)} \quad \forall\, f \in L^q(\Omega) \cap L^2(\Omega).$$

Proof. We consider the decomposition $v = v_1 + v_2$ in (2.28)–(2.30). The $W^{2,q}$ estimate in (2.31) and the $W^{1,p}$ estimate in Lemma 2.8 imply that v_1 and v_2 satisfy the following estimates:

$$
\begin{aligned}
\|v_1\|_{W^{2,q}(\Omega)} &\leqslant C_q\|f\|_{L^q(\Omega)} \quad \forall\, 1 < q < 2 + \varepsilon, \\
\|v_2\|_{W^{1,q}(\Omega)} &\leqslant C_q h\|v_1\|_{W^{1,q}(\Omega)} \leqslant C_q h\|f\|_{L^q(\Omega)}.
\end{aligned}
$$

Applying the $W^{1,q}$ stability of the Ritz projection, we obtain the following estimates:

$$\|v_1 - R_h v_1\|_{W^{1,q}(\Omega)} \leqslant C_q \inf_{\check{\chi}_h \in \mathring{S}_h(\Omega)} \|v_1 - \check{\chi}_h\|_{W^{1,q}(\Omega)} \leqslant C_q h \|v_1\|_{W^{2,q}(\Omega)} \leqslant C_q h \|f\|_{L^q(\Omega)},$$

$$\|v_2 - R_h v_2\|_{W^{1,q}(\Omega)} \leqslant C_q \|v_2\|_{W^{1,q}(\Omega)} \leqslant C_q h \|f\|_{L^q(\Omega)}.$$

The result of Lemma 2.18 is obtained by combining the two estimates above. $\qquad\square$

Finally, the L^p error estimate for the Ritz projection follows from a standard duality argument, again by using the regularity decomposition as in (2.28)–(2.30) for the dual problem.

Lemma 2.19. *For any $1 < q < 2 + \varepsilon$, there exists a positive constant h_q such that for $h \leqslant h_q$ the following error estimate holds:*

$$\|u - R_h u\|_{L^{q'}(\Omega)} \leqslant C_q h \|u - R_h u\|_{W^{1,q'}(\Omega)} \quad \forall\, u \in H_0^1(\Omega) \cap W^{1,q'}(\Omega), \tag{2.48}$$

where $1/q + 1/q' = 1$.

Proof. By using the duality between $L^q(\Omega)$ and $L^{q'}(\Omega)$, we can express the $L^{q'}$ error of the Ritz projection as

$$\|R_h u - u\|_{L^{q'}(\Omega)} = \sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ \|\varphi\|_{L^q(\Omega)} \leqslant 1}} (R_h u - u, \varphi),$$

In particular, there exists $\varphi \in C_0^\infty(\Omega)$ with $\|\varphi\|_{L^q(\Omega)} \leqslant 1$ such that

$$\|R_h u - u\|_{L^{q'}(\Omega)} \leqslant 2(R_h u - u, \varphi).$$

Let $v \in H_0^1(\Omega)$ be the weak solution of the following elliptic equation (in the weak form):

$$(A_h \nabla v, \nabla \chi) = (\varphi, \chi) \quad \forall\, \chi \in H_0^1(\Omega).$$

Then

$$\begin{aligned}
(R_h u - u, \varphi) &= (A_h \nabla v, \nabla(R_h u - u)) \\
&= (A_h \nabla(R_h u - u), \nabla v) \\
&= (A_h \nabla(R_h u - u), \nabla(v - R_h v)) \\
&\leqslant C \|R_h u - u\|_{W^{1,q'}(\Omega)} \|R_h v - v\|_{W^{1,q}(\Omega)} \\
&\leqslant C_q h \|R_h u - u\|_{W^{1,q'}(\Omega)} \|\varphi\|_{L^q(\Omega)} \quad \text{(Lemma 2.18 is used here)} \\
&\leqslant C_q h \|R_h u - u\|_{W^{1,q'}(\Omega)}.
\end{aligned}$$

This proves the result of Lemma 2.19 . $\qquad\square$

2.6. Estimation of $\rho^{-\frac{N}{2}} h^{-1} \|\nabla(v - R_h v)\|_{L^1(\Lambda_h)}$

In this subsection, we prove (2.26) by utilizing the results established in Sections 2.4–2.5, where v is the solution of (2.16). This would complete the proof of Theorem 1.1. To this end, we consider a dyadic decomposition of the domain as in the literature; see [22, 32, 39].

Let $R_0 = \text{diam}(\Omega)$ and $d_j = R_0 2^{-j}$ for $j \geqslant 0$. We define a sequence of subdomains

$$D_j = \{x \in \Omega : d_{j+1} \leqslant |x - x_0| \leqslant d_j\} \quad \text{for } j \geqslant 0.$$

For each j we denote by D_j^l a subdomain slightly larger than D_j, defined by

$$D_j^l = D_{j-l} \cup \cdots \cup D_j \cup D_{j+1} \cup \cdots \cup D_{j+l} \quad (D_i := \varnothing \text{ for } i < 0.)$$

Let $J = [\ln_2(R_0/2\kappa\rho)] + 1$, where $[\ln_2(R_0/2\kappa\rho)]$ denotes the biggest integer not exceeding $\ln_2(R_0/2\kappa\rho)$. The constant $\kappa > 32$ will be determined below, and the generic constant C

will be independent on κ until it is determined (unless it contains a subscript κ). The definition above implies that

$$\frac{1}{2}\kappa\rho \leqslant d_{J+1} \leqslant \kappa\rho$$

and

$$\text{measure}(D_j \cap \Lambda_h) \leqslant Chd_j^{N-1}. \tag{2.49}$$

Note that v is the solution of (2.16), where $\varphi = 0$ outside $S_\rho(x_0)$. Therefore, $\varphi = 0$ in D_j^3 for $1 \leqslant j \leqslant J$. This result will be used below.

By using the subdomains defined above, we have

$$\rho^{-\frac{N}{2}}h^{-1}\|\nabla(v - R_h v)\|_{L^1(\Lambda_h)}$$

$$\leqslant \rho^{-\frac{N}{2}}h^{-1}\left(\sum_{j=0}^J \|\nabla(v - R_h v)\|_{L^1(\Lambda_h \cap D_j)} + \|\nabla(v - R_h v)\|_{L^1(\Lambda_h \cap S_{\kappa\rho}(x_0))}\right)$$

$$\leqslant C\rho^{-\frac{N}{2}}h^{-1}\sum_{j=0}^J h^{\frac{1}{2}}d_j^{\frac{N-1}{2}}\|\nabla(v - R_h v)\|_{L^2(\Lambda_h \cap D_j)}$$

$$+ C\kappa^{\frac{N-1}{2}}\rho^{-\frac{1}{2}}h^{-\frac{1}{2}}\|\nabla(v - R_h v)\|_{L^2(\Lambda_h \cap S_{\kappa\rho}(x_0))}, \tag{2.50}$$

where the Hölder inequality and (2.49) are used in the derivation of the last inequality. By choosing $q = 2$ in Lemma 2.18 we have

$$\|\nabla(v - R_h v)\|_{L^2(\Omega)} \leqslant Ch\|\varphi\|_{L^2(\Omega)} \leqslant Ch. \tag{2.51}$$

Then, substituting (2.51) into the last term on the right-hand side of (2.50) and using the fact that $\rho \geqslant h$ (which follows from the definition of ρ in (2.13)), we obtain

$$\rho^{-\frac{N}{2}}h^{-1}\|\nabla(v - R_h v)\|_{L^1(\Lambda_h)} \leqslant C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\sum_{j=0}^J d_j^{\frac{N-1}{2}}\|\nabla(v - R_h v)\|_{L^2(D_j)} + C_\kappa, \tag{2.52}$$

where C_κ denotes a constant which depends on the parameter κ.

It remains to estimate $\|\nabla(v - R_h v)\|_{L^2(D_j)}$. To this end, we use the following interior energy estimate for the solution of (2.16):

$$\|v - R_h v\|_{H^1(D_j)} \leqslant C\|v - \check{I}_h v\|_{H^1(D_j^1)} + Cd_j^{-1}\|v - \check{I}_h v\|_{L^2(D_j^1)} + Cd_j^{-1}\|v - R_h v\|_{L^2(D_j^1)}. \tag{2.53}$$

The proof of such interior energy estimate is omitted as it only requires the coefficient matrix A_h to be L^∞ in the perturbed bilinear form in (2.11), without additional smoothness, and therefore is the same as the proof for standard finite elements for the Poisson equation.

We use the decomposition $v = v_1 + v_2$ in (2.28)–(2.30) with $f = \varphi$ supported in $S_\rho(x_0)$, and consider interpolation error of v_1 and v_2, respectively. First, by applying the result of Lemma 2.2 and using the fact that $d_j > h$, we have

$$\|v_1 - \check{I}_h v_1\|_{H^1(D_j^1)} + d_j^{-1}\|v_1 - \check{I}_h v_1\|_{L^2(D_j^1)} \leqslant Ch\|v_1\|_{H^2(D_j^2)} \leqslant Chd_j^{-1+\frac{N}{2}-\frac{N}{p}}\|v_1\|_{W^{1,p}(\Omega)}$$

$$\text{for } \frac{2N}{N+2} < p < 2, \tag{2.54}$$

where we have used the following inequality in deriving the last inequality:

$$\|v_1\|_{H^2(D_j^2)} \leqslant Cd_j^{\frac{1}{2}-\frac{3}{p}}\|v_1\|_{W^{1,p}(\Omega)} \quad \text{for } \frac{2N}{N+2} < p < 2. \tag{2.55}$$

The inequality above follows from Lemma 2.12 (because v_1 is the solution of (2.29) with $f = \varphi = 0$ in D_j^3), the Hölder inequality and the Sobolev embedding inequality, i.e.,

$$\|v_1\|_{H^2(D_j^2)} \leqslant Cd_j^{-2}\|v_1\|_{L^2(D_j^3)}$$

$$\leqslant Cd_j^{-2+\frac{N}{2}-\frac{N}{p_*}}\|v_1\|_{L^{p*}(D_j^3)} \quad \text{if } p_* > 2$$

$$\leqslant Cd_j^{-1+\frac{N}{2}-\frac{N}{p}}\|v_1\|_{W^{1,p}(\Omega)} \quad \text{for } \frac{N}{p_*} = \frac{N}{p} - 1 \text{ and } \frac{2N}{N+2} < p < 2$$
$$\text{so that } p_* > 2 \text{ and } W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega).$$

Here we require $\kappa > 32$ to guarantee that $d_{J+5} > \rho$, which is required in the use Lemma 2.12. This proves the last inequality in (2.54).

Next, we consider the interpolation error of v_2 by using Lemma 2.2 and Hölder inequality, i.e.,

$$\|v_2 - \check{I}_h v_2\|_{H^1(D_j^1)} + d_j^{-1}\|v_2 - \check{I}_h v_2\|_{L^2(D_j^1)} \leqslant Cd_j^{\frac{N}{2}-\frac{N}{p_1}}\|v_2\|_{W^{1,p_1}(D_j^2)}$$

$$\leqslant Cd_j^{\frac{N}{2}-\frac{N}{q_1}}h\|v_1\|_{W^{1,q_1}(\Omega)}$$

$$\text{for some } p_1 > N \text{ and } \frac{N}{q_1} = \frac{N}{p_1} + 1, \qquad (2.56)$$

where we have applied Corollary 2.14 in deriving the last inequality. (Here we only need p_1 to be slightly bigger than N, and therefore the corresponding q_1 here can be smaller than 2, so that we can use Hölder inequality to estimate $\|\varphi\|_{L^{q_1}(S_\rho(x_0))}$ below.)

By combining (2.54) and (2.56), we obtain

$$C\|v - \check{I}_h v\|_{H^1(D_j^1)} + Cd_j^{-1}\|v - \check{I}_h v\|_{L^2(D_j^1)}$$

$$\leqslant Chd_j^{-1+\frac{N}{2}-\frac{N}{p}}\|v_1\|_{W^{1,p}(\Omega)} + Cd_j^{\frac{N}{2}-\frac{N}{q_1}}h\|v_1\|_{W^{1,q_1}(\Omega)}$$

$$\leqslant Chd_j^{-1+\frac{N}{2}-\frac{N}{p}}\rho\|\varphi\|_{L^p(S_\rho(x_0))} + Chd_j^{\frac{N}{2}-\frac{N}{q_1}}\rho\|\varphi\|_{L^{q_1}(S_\rho(x_0))}$$

$$\leqslant Chd_j^{-1+\frac{N}{2}-\frac{N}{p}}\rho^{1-\frac{N}{2}+\frac{N}{p}} + Chd_j^{\frac{N}{2}-\frac{N}{q_1}}\rho^{1-\frac{N}{2}+\frac{N}{q_1}}, \qquad (2.57)$$

where we have applied Lemma 2.11 to equation (2.29) in the derivation of the second inequality, and used Hölder inequality in the derivation of the last inequality.

Finally, substituting (2.57) into (2.53), we obtain

$$d_j^{\frac{N-1}{2}}\|\nabla(v - R_h v)\|_{L^2(D_j)}$$

$$\leqslant Chd_j^{N-\frac{3}{2}-\frac{N}{p}}\rho^{1-\frac{N}{2}+\frac{N}{p}} + Chd_j^{N-\frac{1}{2}-\frac{N}{q_1}}\rho^{1-\frac{N}{2}+\frac{N}{q_1}} + Cd_j^{\frac{N-3}{2}}\|v - R_h v\|_{L^2(D_j^1)}$$

$$\leqslant Chd_j^{N-\frac{3}{2}-\frac{N}{p}}\rho^{1-\frac{N}{2}+\frac{N}{p}} + Cd_j^{\frac{N-3}{2}}\|v - R_h v\|_{L^2(D_j^1)}, \qquad (2.58)$$

where we have chosen $p = q_1 < 2$ and used $d_j \leqslant C$ in the derivation of the last inequality. Here we can make p as close to 2 as possible so that $p = q'$ satisfies the condition in Lemma 2.19 (which will be used in the subsequent analysis).

Now we substitute (2.58) into (2.52) and use the result $\sum_{j=0}^{J} d_j^{N-\frac{3}{2}-\frac{N}{p}} \leqslant C_\kappa \rho^{N-\frac{3}{2}-\frac{N}{p}}$, we obtain

$$\sum_{j=0}^{J} d_j^{\frac{N-1}{2}} \|\nabla(v - R_h v)\|_{L^2(D_j)} \leqslant C_\kappa h \rho^{\frac{N-1}{2}} + \sum_{j=0}^{J} C d_j^{\frac{N-3}{2}} \|v - R_h v\|_{L^2(D_j^1)}, \qquad (2.59)$$

and therefore

$$\rho^{-\frac{N}{2}} h^{-1} \|\nabla(v - R_h v)\|_{L^1(\Lambda_h)} \leqslant C \rho^{-\frac{N}{2}} h^{-\frac{1}{2}} \sum_{j=0}^{J} d_j^{\frac{N-1}{2}} \|\nabla(v - R_h v)\|_{L^2(D_j)} + C_\kappa$$

$$\leqslant C_\kappa + C \rho^{-\frac{N}{2}} h^{-\frac{1}{2}} \sum_{j=0}^{J} d_j^{\frac{N-3}{2}} \|v - R_h v\|_{L^2(D_j^1)}. \qquad (2.60)$$

It remains to estimate $\sum_{j=0}^{J} d_j^{\frac{N-3}{2}} \|v - R_h v\|_{L^2(D_j^1)}$. To this end, we let χ be a smooth cut-off function satisfying

$$\chi = 1 \text{ on } D_j^1, \quad \chi = 0 \text{ outside } D_j^2 \quad \text{and} \quad |\nabla \chi| \leqslant C d_j^{-1}.$$

For $N = 2, 3$ the following Sobolev interpolation inequality holds:

$$\|\chi(v - R_h v)\|_{L^2(\Omega)} \leqslant \|\chi(v - R_h v)\|_{L^p(\Omega)}^{1-\theta} \|\chi(v - R_h v)\|_{H^1(\Omega)}^{\theta} \quad \text{with} \quad \frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{p_*}, \qquad (2.61)$$

where $p_* = \infty$ for $N = 2$ and $p_* = 6$ for $N = 3$. For both $N = 2$ and $N = 3$, the parameter θ determined by (2.61) satisfies the following relation:

$$\frac{N}{p} - \frac{N}{2} = \frac{\theta}{1-\theta}. \qquad (2.62)$$

We can choose p sufficiently close to 2 as mentioned below (2.58). Since

$$C \|\chi(v - R_h v)\|_{H^1(\Omega)} \leqslant C \|\nabla(v - R_h v)\|_{L^2(D_j^2)} + C d_j^{-1} \|v - R_h v\|_{L^2(D_j^2)} \qquad (2.63)$$

it follows that

$$\|v - R_h v\|_{L^2(D_j^1)}$$
$$\leqslant \|v - R_h v\|_{L^p(D_j^2)}^{1-\theta} \big(C \|\nabla(v - R_h v)\|_{L^2(D_j^2)} + C d_j^{-1} \|v - R_h v\|_{L^2(D_j^2)} \big)^{\theta}$$
$$= \big(\epsilon^{-\frac{\theta}{1-\theta}} \|v - R_h v\|_{L^p(D_j^2)} \big)^{1-\theta} \big(C \epsilon \|\nabla(v - R_h v)\|_{L^2(D_j^2)} + C \epsilon d_j^{-1} \|v - R_h v\|_{L^2(D_j^2)} \big)^{\theta}$$
$$\leqslant C \epsilon^{-\frac{\theta}{1-\theta}} \|v - R_h v\|_{L^p(D_j^2)} + C \epsilon \|\nabla(v - R_h v)\|_{L^2(D_j^2)} + C \epsilon d_j^{-1} \|v - R_h v\|_{L^2(D_j^2)},$$

where ϵ can be an arbitrary positive number.

By choosing $\epsilon = d_j (\rho/d_j)^\sigma$ with a fixed $\sigma \in (0, 1)$, we obtain

$$\|v - R_h v\|_{L^2(D_j^1)} \leqslant C \left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - R_h v\|_{L^p(D_j^1)} \qquad (2.64)$$

$$+ \left(\frac{\rho}{d_j}\right)^{\sigma} \big(C d_j \|\nabla(v - R_h v)\|_{L^2(D_j^2)} + C \|v - R_h v\|_{L^2(D_j^2)} \big).$$

Hence,

$$\rho^{-\frac{N}{2}} h^{-\frac{1}{2}} \sum_{j=0}^{J} d_j^{\frac{N-3}{2}} \|v - R_h v\|_{L^2(D_j^1)}$$

$$\leqslant C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}\left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}}d_j^{-\frac{\theta}{1-\theta}+\frac{N-3}{2}}\|v-R_hv\|_{L^p(D_j^2)}$$

$$+C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}\left(\frac{\rho}{d_j}\right)^{\sigma}\left(d_j^{\frac{N-1}{2}}\|\nabla(v-R_hv)\|_{L^2(D_j^2)}+Cd_j^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(D_j^2)}\right)$$

$$\leqslant C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}\left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}}d_j^{-\frac{\theta}{1-\theta}+\frac{N-3}{2}}\|v-R_hv\|_{L^p(D_j^2)}$$

$$+C_\kappa+C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\left(\frac{\rho}{d_J}\right)^{\sigma}\sum_{j=0}^{J}d_j^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(D_j^3)}, \tag{2.65}$$

where we have used (2.59) and the fact $\frac{\rho}{d_j}\leqslant\frac{\rho}{d_J}$ in deriving the last inequality. Note that

$$\sum_{j=0}^{J}d_j^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(D_j^3)}\leqslant Cd_J^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(S_{\kappa\rho}(x_0))}+3\sum_{j=0}^{J}d_j^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(D_j^1)}.$$

Combining the last two estimates, we obtain

$$\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}d_j^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(D_j^1)}\leqslant C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}\left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}}d_j^{-\frac{\theta}{1-\theta}+\frac{N-3}{2}}\|v-R_hv\|_{L^p(D_j^2)}$$

$$+C_\kappa+C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\left(\frac{\rho}{d_J}\right)^{\sigma}d_J^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(S_{\kappa\rho}(x_0))}$$

$$+C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\left(\frac{\rho}{d_J}\right)^{\sigma}\sum_{j=0}^{J}d_j^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(D_j^1)}.$$

For the fixed $\sigma\in(0,1)$, by choosing a sufficiently large parameter κ we have $\left(\frac{\rho}{d_J}\right)^{\sigma}\leqslant\frac{C}{\kappa^\sigma}$, and therefore the last term of the inequality above can be absorbed by the left-hand side. From now on we fix the parameter κ. Then we have

$$\sum_{j=0}^{J}\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}d_j^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(D_j^1)}\leqslant\sum_{j=0}^{J}C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}}d_j^{-\frac{\theta}{1-\theta}+\frac{N-3}{2}}\|v-R_hv\|_{L^p(D_j^2)}$$

$$+C_\kappa+C\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}\left(\frac{\rho}{d_J}\right)^{\sigma}d_J^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(S_{\kappa\rho}(x_0))}.$$
$$\tag{2.66}$$

It remains to estimate $\|v-R_hv\|_{L^p(D_j^1)}$ and $\|v-R_hv\|_{L^2(S_{\kappa\rho}(x_0))}$. This is done by applying Lemma 2.19 (with $q'=p$ therein), Lemma 2.18 (with $q=p$ therein) and Hölder's inequality, i.e.,

$$\|v-R_hv\|_{L^p(\Omega)}\leqslant Ch^2\|\varphi\|_{L^p(\Omega)}\leqslant Ch^2\rho^{\frac{N}{p}-\frac{N}{2}}, \tag{2.67}$$

$$\|v-R_hv\|_{L^2(\Omega)}\leqslant Ch^2\quad\text{(setting }q'=q=2\text{ in Lemma 2.19 and Lemma 2.18)}. \tag{2.68}$$

Then, substituting these estimates into (2.66), we obtain

$$\sum_{j=0}^{J}\rho^{-\frac{N}{2}}h^{-\frac{1}{2}}d_j^{\frac{N-3}{2}}\|v-R_hv\|_{L^2(D_j^1)}\leqslant\sum_{j=0}^{J}C\left(\frac{h}{\rho}\right)^{\frac{N}{2}}\left(\frac{h}{d_j}\right)^{\frac{3-N}{2}}\left(\frac{\rho}{d_j}\right)^{\frac{N}{p}-\frac{N}{2}-\frac{\theta\sigma}{1-\theta}}d_j^{\frac{N}{p}-\frac{N}{2}-\frac{\theta}{1-\theta}}$$

$$+ C_\kappa + C\left(\frac{h}{\rho}\right)^{\frac{N}{2}}\left(\frac{h}{d_J}\right)^{\frac{3-N}{2}}\left(\frac{\rho}{d_J}\right)^\sigma \tag{2.69}$$

By choosing $p < 2$ to be sufficiently close to 2 (so that $q' = p$ satisfies the condition of Lemma 2.19) and using the relation $\frac{N}{p} - \frac{N}{2} = \frac{\theta}{1-\theta}$ as shown in (2.62), we obtain

$$\sum_{j=0}^{J} \rho^{-\frac{N}{2}} h^{-\frac{1}{2}} d_j^{\frac{N-3}{2}} \|v - R_h v\|_{L^2(D_j^1)} \leqslant C. \tag{2.70}$$

Then, substituting the last inequality into the right-hand side of (2.60), we obtain

$$\rho^{-\frac{N}{2}} h^{-1} \|\nabla(v - R_h v)\|_{L^1(\Lambda_h)} \leqslant C.$$

This proves (2.26) for sufficiently small mesh size, say $h \leqslant h_0$. This condition is required when we use Corollary 2.14, Lemma 2.18 and Lemma 2.19 in this subsection.

In the case $h \geqslant h_0$, we denote by $\tilde{g}_h \in S_h(\Omega_h)$ the isoparametric finite element function satisfying $\tilde{g}_h = u_h$ on $\partial\Omega_h$ and $\tilde{g}_h = 0$ at the interior nodes of the domain Ω_h. Then the following estimate holds:

$$\|\tilde{g}_h\|_{L^\infty(\Omega_h)} \leqslant C\|u_h\|_{L^\infty(\partial\Omega_h)}.$$

Since $\chi_h = u_h - \tilde{g}_h \in \mathring{S}_h(\Omega_h)$, it follows from (1.1) that

$$0 = \int_{\Omega_h} \nabla u_h \cdot \nabla(u_h - \tilde{g}_h) = \|\nabla(u_h - \tilde{g}_h)\|_{L^2(\Omega_h)}^2 + \int_{\Omega_h} \nabla\tilde{g}_h \cdot \nabla(u_h - \tilde{g}_h),$$

and therefore

$$\|\nabla(u_h - \tilde{g}_h)\|_{L^2(\Omega_h)}^2 = -\int_{\Omega_h} \nabla\tilde{g}_h \cdot \nabla(u_h - \tilde{g}_h) \leqslant C\|\nabla\tilde{g}_h\|_{L^2(\Omega_h)}\|\nabla(u_h - \tilde{g}_h)\|_{L^2(\Omega_h)}.$$

Thus, by using the inverse inequality and the condition $h \geqslant h_0$, we have

$$\|\nabla(u_h - \tilde{g}_h)\|_{L^2(\Omega_h)} \leqslant C\|\nabla\tilde{g}_h\|_{L^2(\Omega_h)} \leqslant Ch^{-1}\|\tilde{g}_h\|_{L^2(\Omega_h)} \leqslant Ch_0^{-1}\|\tilde{g}_h\|_{L^\infty(\Omega_h)}$$
$$\leqslant Ch_0^{-1}\|u_h\|_{L^\infty(\partial\Omega_h)}.$$

By using the inverse inequality again, we obtain

$$\|u_h - \tilde{g}_h\|_{L^\infty(\Omega_h)} \leqslant Ch^{-\frac{N}{2}}\|u_h - \tilde{g}_h\|_{L^2(\Omega_h)}$$
$$\leqslant Ch^{-\frac{N}{2}}\|\nabla(u_h - \tilde{g}_h)\|_{L^2(\Omega_h)}$$
$$\leqslant Ch_0^{-\frac{N}{2}-1}\|u_h\|_{L^\infty(\partial\Omega_h)}.$$

By the triangle inequality, this proves

$$\|u_h\|_{L^\infty(\Omega_h)} \leqslant \|\tilde{g}_h\|_{L^\infty(\Omega_h)} + \|u_h - \tilde{g}_h\|_{L^\infty(\Omega_h)} \leqslant C\|u_h\|_{L^\infty(\partial\Omega_h)}$$

for $h \geqslant h_0$.

Combining the two cases $h \leqslant h_0$ and $h \geqslant h_0$, we obtain the result of Theorem 1.1. $\qquad\square$

3. Proof of Theorem 1.2

In this section, we adapt Schatz's argument in [39] to the proof of maximum-norm stability of isoparametric finite element solutions of the Poisson equation in the curvilinear polyhedron considered here. The argument is based on the weak maximum principle established in Theorem 1.1 and the following technical result, which asserts that the $W^{1,\infty}$ regularity estimate of the Poisson equation can hold in a family of larger perturbed domains Ω^t, $t \in [0, \delta]$, such that $\text{dist}(\partial\Omega^t, \partial\Omega) \sim t$ and the $W^{1,\infty}$ estimate is uniformly with respect to $t \in [0, \delta]$.

Remark 3.1. Here we make a remark on the idea of our proof. To prove Theorem 1.2, we observe that the numerical solution u_h is in fact the Ritz projection of $u^{(h)} \in H_0^1(\Omega_h)$ which is the exact solution of the Poisson equation on Ω_h:

$$-\Delta u^{(h)} = f \quad \text{in } \Omega_h \quad (f \text{ is extended by zero outside } \Omega),$$

in the sense that

$$R_h\big(u^{(h)} \circ \Phi_h^{-1}\big) = u_h \circ \Phi_h^{-1}.$$

Using the weak maximum principle established in Theorem 1.1, one can imitate the proof of [32, Theorem 5.1] to show that there holds L^∞ stability for our Ritz projection R_h. It follows that

$$\|u^{(h)} - u_h\|_{L^\infty(\Omega_h)} \leqslant C\|u^{(h)} - I_h u^{(h)}\|_{L^\infty(\Omega_h)}.$$

Now we can obtain the result of Theorem 1.2 as long as we establish the estimate

$$\|u - u^{(h)}\|_{L^\infty(\Omega_h)} \leqslant Ch^{r+1}\|f\|_{L^p(\Omega)} \quad (p > N),$$

where we have extended u by zero outside Ω. To this end, we consider employing the maximum principle of harmonic functions since $\Delta(u^{(h)} - u) = 0$ in $\Omega \cap \Omega_h$. Here technically we introduce larger perturbed domain Ω^t and solution u^t

$$-\Delta u^t = f \quad \text{in } \Omega^t,$$

in the larger perturbed domain Ω^t. Then using maximum principle, we compare u and $u^{(h)}$ with u^t respectively, for example we have

$$\|u - u^t\|_{L^\infty(\Omega)} \leqslant \|u^t\|_{L^\infty(\partial\Omega)} \leqslant Ch^{r+1}\|u^t\|_{W^{1,\infty}(\Omega^t)}.$$

This explains the motivation of establishing Proposition 3.1.

Proposition 3.1. *Let Ω be a curvilinear polyhedron with edge openings smaller than π, and define*

$$\Omega(\varepsilon) := \{x \in \mathbb{R}^N \ : \ \mathrm{dist}(x, \Omega) < \varepsilon\},$$

which is an ε neighborhood of Ω. Then there exist constants $\delta > 0$ and $\lambda > 0$ and a family of larger bounded domains Ω^t satisfying

$$\Omega(\lambda t) \subseteq \Omega^t \subseteq \Omega(\lambda^{-1}t) \quad \forall t \in [0, \delta],$$

such that the weak solution $u^t \in H_0^1(\Omega^t)$ of the Poisson equation

$$-\Delta u^t = f \quad \text{in } \Omega^t, \quad \text{with } f \in L^p(\Omega^t) \text{ for some } p > N, \tag{3.1}$$

satisfies the following estimate:

$$\|u^t\|_{W^{1,\infty}(\Omega^t)} \leqslant C_p\|f\|_{L^p(\Omega^t)} \quad \text{for } t \in [0, \delta], \tag{3.2}$$

where C_p is some constant which is independent of $t \in [0, \delta]$.

Proof. In a standard convex polyhedron $\hat{\Omega}$, the following estimate holds for $p > N$ (cf. [35, Lemma 2.1]):

$$\|\nabla w\|_{L^\infty(\hat{\Omega})} \leqslant C_p\|\nabla \cdot (a\nabla w)\|_{L^p(\hat{\Omega})} \quad \forall w \in H_0^1(\hat{\Omega}) \text{ such that } \nabla \cdot (a\nabla w) \in L^2(\hat{\Omega}). \tag{3.3}$$

where $a = (a_{ij})$ is any symmetric positive definite matrix in $W^{1,q}(\hat{\Omega})$ with $q > N$, satisfying the following estimate:

$$C^{-1}|\xi|^2 \leqslant a\xi \cdot \xi \leqslant C|\xi|^2. \tag{3.4}$$

On the curvilinear polyhedron Ω considered in this article, by using a partition of unity we can reduce the problem to an open subset of Ω which is diffeomorphic to a convex polyhedral cone. Therefore, the following result still holds for $p > N$:

$$\|\nabla w\|_{L^\infty(\Omega)} \leqslant C_p \|\nabla \cdot (a\nabla w)\|_{L^p(\Omega)} \quad \forall w \in H_0^1(\Omega) \text{ such that } \nabla \cdot (a\nabla w) \in L^2(\Omega). \qquad (3.5)$$

If there exists a smooth diffeomorphism $\Psi_t : \Omega \to \Omega^t$ (smooth uniformly with respect to $t \in [0, \delta]$), then we can pull the Poisson equation on $\Omega^t = \Psi_t(\Omega)$ back to the curvilinear polyhedron Ω as an elliptic equation with some coefficient matrix a satisfying (3.4), and then use the result in (3.5). This would prove (3.2). If the partial derivatives of the diffeomorphism from Ω to Ω^t can be uniformly bounded with respect to $t \in [0, \delta]$, then the constant in (3.2) is independent of $t \in [0, \delta]$.

It remains to prove the existence of a smooth diffeomorphism $\Psi_t : \Omega \to \Omega^t = \Psi_t(\Omega)$. This is presented in the following lemma. $\qquad \square$

Lemma 3.2. *Let Ω be a curvilinear polyherdon. Then there exist constants $\delta > 0$ and $\lambda > 0$ (which only depend on Ω), and a family of diffeomorphisms $\Psi_t : \mathbb{R}^N \to \mathbb{R}^N$ for $t \in [0, \delta]$, such that*

(1) *$\Omega(\lambda t) \subseteq \Psi_t(\Omega) \subseteq \Omega(\lambda^{-1} t)$ for $t \in [0, \delta]$ and some constant $\lambda > 0$.*
(2) *The partial derivatives of Ψ_t are bounded uniformly with respect to $t \in [0, \delta]$, i.e.,*

$$|\nabla^k \Psi_t(x)| \leqslant C_k \quad \forall x \in \mathbb{R}^N, \ \forall k \geqslant 1, \ \text{where } C_k \text{ is independent of } t \in [0, \delta].$$

Proof. It is known that any given smooth and compactly supported vector field X on $\mathbb{R}$ induces a flow map

$$\Psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \quad (t, x) \mapsto \Phi(t, x),$$

such that each $\Psi_t = \Psi(t, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is a diffeomorphism of $\mathbb{R}^N$ for sufficiently small t, say $|t| \leqslant \delta$. Moreover, $\Psi_0 = \mathrm{Id}$, $\Psi_{t+s} = \Psi_t \circ \Psi_s$ for $t, s \in \mathbb{R}$, and the partial derivatives of Ψ_t are uniformly bounded by constants which only depend on X and δ (independent of t).

Therefore, in order to prove Lemma 3.2, it suffices to construct a compactly supported smooth vector field X, such that the flow map induced by X satisfies $\Omega(\lambda t) \subseteq \Psi_t(\Omega) \subseteq \Omega(\lambda^{-1} t)$ for $t \in [0, \delta]$ (with some constants $\lambda > 0$ and $\delta > 0$). This can be proved by utilizing the following result, which provides a criteria for the construction of such a vector field.

Lemma 3.3. *Let Ω be a curvilinear polyhedron, and let X be a smooth and compactly supported vector field on $\mathbb{R}^N$ satisfying the following conditions:*

(1) *$X|_{\Omega'} \equiv 0$ for some nonempty open subset $\Omega' \subset\subset \Omega$.*
(2) *$\langle X(x), N_x \rangle \geqslant c$ at all smooth points $x \in \partial\Omega$, where N_x denotes the unit outward normal vector at $x \in \partial\Omega$ and $c > 0$ is some constant.*
(3) *$|X(x)| \leqslant 1 \quad \forall x \in \mathbb{R}^N$*

Then there are constants $\lambda > 0$ and $\delta > 0$, which only depend on X and Ω, such that the flow map Ψ_t induced by the vector field X has the following property:

$$\Omega(\lambda t) \subseteq \Psi_t(\Omega) \subseteq \Omega(\lambda^{-1} t) \quad \text{for } t \in [0, \delta].$$

Let us temporarily assume that Lemma 3.3 holds, and use it to prove Lemma 3.2. To this end, it suffices to construct a vector field which satisfies the conditions in Lemma 3.3.

From the definition of the curvilinear polyhedron we know that for every $x \in \partial\Omega$ there exists a map $\varphi_x : U_x \to B_{\varepsilon_x}(0)$ which is a diffeomorphism from a neighborhood U_x of x in $\mathbb{R}^N$ to a ball centered at 0 with radius ε_x, such that $\varphi_x(x) = 0$ and $\varphi_x(U_x \cap \Omega) = K_x \cap B_0(\varepsilon_x)$, where $K_x = \{y \in \mathbb{R}^3 : y/|y| \in \Theta\}$ is a cone corresponding to a spherical region $\Theta \subset \mathbb{S}^2$ which is contained in an open half sphere, say $\mathbb{S}_+^2 = \{x \in \mathbb{R}^3 : |x| = 1, \ x_3 > 0\}$. We shall use the following terminology:

(1) By composing φ_x with an additional linear transformation if necessary, we can assume that $\nabla\varphi_x(x) = I$ (which holds only at the point x in U_x).

(2) If p is a smooth point on ∂K_x (not on the edges or vertex of ∂K_x), then we denote by $N_{x,p}$ the unit outward normal vector of ∂K_x at p, and define

$$\hat{N}_x = \{N_{x,p} : p \text{ in some smooth piece of } \partial K_x\}$$

to be the set of all outward unit normal vectors on the smooth faces of ∂K_x. When x is a smooth point of $\partial\Omega$, $\hat{N}_x$ consists of only one vector, i.e., the usual unit normal vector N_x. Therefore, the set $\hat{N}_x$ can be viewed as generalization of normal vector at x when x is not a smooth point.

(3) Let y be an interior point in the polyhedral cone K_x. Then the unit vector $V_x = -y/|y|$ satisfies that $\langle V_x, N_{x,p}\rangle > 0$ for all $N_{x,p} \in \hat{N}_x$.

We will construct a smooth vector field X on $\mathbb{R}^N$ as follows, by using a partition of unity. By the three properties above and the compactness of $\partial\Omega$, there is constant $c > 0$ only dependent on Ω such that for each $x \in \partial\Omega$, there is a unit vector $V_x \in \mathbb{R}^N$ such that

$$\langle V_x, N_{x,p}\rangle \geqslant 2c \quad \forall N_{x,p} \in \hat{N}_x.$$

Since the normal vector at a smooth point of $\partial\Omega$ changes continuously in a smooth piece of $\partial\Omega$, one can shrink the neighborhood U_x of $x \in \partial\Omega$ so that

$$\langle V_x, N_y\rangle \geqslant c \text{ for all smooth points } y \in \partial\Omega \cap U_x,$$

where N_y denotes the unit outward normal vector at $y \in \partial\Omega \cap U_x$. We define a smooth vector field X_x on U_x by

$$X_x(y) = V_x \quad \forall y \in U_x,$$

and choose a finite covering $\{U_{x_\ell}\}_{1\leqslant\ell\leqslant L}$ of $\partial\Omega$ from these U_x, $x \in \partial\Omega$, and a family of smooth cut-off functions $\{\chi_\ell\}_{1\leqslant\ell\leqslant L}$ such that $0 \leqslant \chi_\ell \leqslant 1$ and

$$\text{supp}(\chi_\ell) \subseteq U_{x_\ell} \text{ and } \sum_{1\leqslant\ell\leqslant L} \chi_\ell(x) = 1, \quad \forall x \in \partial\Omega.$$

Then we denote by X_{x_ℓ} the above-mentioned vector field defined on U_{x_ℓ}, and define

$$X = \sum_{\ell=1}^{L} \chi_\ell X_{x_\ell},$$

so that X is a compactly supported smooth vector field such that

$$\langle X(y), N_y\rangle = \sum_{\chi_\ell(y)\neq 0} \chi_\ell(y)\langle X_{x_\ell}, N_y\rangle \geqslant c, \quad \text{for all smooth point } y \in \partial\Omega.$$

and clearly $|X(x)| \leqslant 1$, $\forall x \in \mathbb{R}^N$. This proves the existence of a desired vector field X, and therefore completes the proof of Proposition 3.1. $\qquad\square$

Proof of Lemma 3.3. For each $x \in \partial\Omega$, let $\varphi_x : U_x \to B_{\varepsilon_x}(0)$ be the map as in the definition of the curvilinear polyhedron. Here we do not require $\varphi_x(U_x)$ to be a ball so that we can assume U_x to be convex.

By composing φ_x with an additional linear transformation if necessary, we can assume that $\nabla\varphi_x(x) = I$ (as in the proof of Lemma 3.2). Since $c \leqslant \langle X(x), N_x\rangle \leqslant 1$ (as a the condition in Lemma 3.3), we can shrink the neighborhood U_x small enough so that

$$\frac{c}{2} \leqslant \langle(\nabla\varphi_x(y))^\top X(y), N_{x,p}\rangle \leqslant 2 \quad \forall y \in U_x, \ p \in \varphi_x(U_x \cap \partial\Omega) = \varphi_x(U_x) \cap \partial K_x,$$

$$p \text{ is a smooth point.} \qquad (3.6)$$

Moreover, since $(\nabla\varphi_x)^\top = I$ at x, we can shrink U_x so that the following equivalence relation holds:

$$d(y_1, y_2) \sim d(\varphi_x(y_1), \varphi_x(y_2)) \quad \forall y_1, y_2 \in U_x,$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{R}^N$. As a result,

$$d(y, U_x \cap \Omega) \sim d(\varphi_x(y), \varphi_x(U_x \cap \Omega)) \quad \forall y \in U_x.$$

We can choose a finite covering $\{U_{x_\ell}\}_{1 \leqslant \ell \leqslant L}$ of $\partial\Omega$ from these U_x. Then there exists a sufficiently small $\delta > 0$ such that for any $x \in \partial\Omega$ there exists $1 \leqslant \ell \leqslant L$ such that for all $t \in [0, \delta]$,

$$\Psi_t(x) \in U_{x_\ell} \quad \text{for some } 1 \leqslant \ell \leqslant L.$$

Moreover,

$$d(\Psi_t(x), \Omega) = d(\Psi_t(x), U_\ell \cap \Omega) \tag{3.7}$$

and

$$d(\Psi_t(x), \Omega) \sim d(\varphi_{x_\ell}(\Psi_t(x)), \varphi_{x_\ell}(U_\ell \cap \Omega)). \tag{3.8}$$

Let $Y_\ell = (\nabla\varphi_{x_\ell})^\top X|_{U_\ell}$ be the pushforward vector field under φ_{x_ℓ}, then $\varphi_{x_\ell}(\Psi_t(x))$ is the integral curve of vector field Y_ℓ, with initial value point $\varphi_{x_\ell}(x)$. From (3.6) we know that

$$\frac{c}{2} \leqslant \langle Y_\ell(z), N_{x_\ell,p} \rangle \leqslant 2 \quad \forall z \in \varphi_{x_\ell}(U_{x_\ell}), \ \forall p \in \varphi_{x_\ell}(U_{x_\ell} \cap \partial\Omega) = \varphi_{x_\ell}(U_{x_\ell}) \cap \partial K_{x_\ell},$$

which implies that the integral curve $\varphi_{x_\ell}(\Psi_t(x))$ is flowing outside $\varphi_{x_\ell}(U_{x_\ell} \cap \Omega)$, i.e.,

$$\frac{ct}{2} \leqslant d(\varphi_{x_\ell}(\Psi_t(x)), \varphi_{x_\ell}(U_{x_\ell} \cap \Omega)) \leqslant 2t.$$

Then, from the equivalence of distance as shown in (3.7)–(3.8), we conclude that there exists a constant $\lambda > 0$ such that

$$2\lambda t \leqslant d(\Psi_t(x), \Omega) \leqslant \frac{1}{2}\lambda^{-1}t \quad \forall t \in [0, \delta], \ \forall x \in \partial\Omega.$$

We consider the domain $\Omega(\lambda t) := \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \lambda t\} \supset \Omega$. On the one hand, since $X|_{\Omega'} = 0$ for some subdomain $\Omega' \subset\subset \Omega$ it follows that $\Psi_t(\Omega) \cap \Omega(\lambda t) \neq \varnothing$. On the other hand, since $d(\Psi_t(x), \Omega) > \lambda t$ for all $x \in \partial\Omega$, the boundaries of $\Psi_t(\Omega)$ and $\Omega(\lambda t)$ are disjoint. It follows that $\Omega(\lambda t) \subseteq \Psi_t(\Omega)$ for $t \in [0, \delta]$. Similarly, one can prove that $\Omega(\lambda^{-1}t) \supset \Psi_t(\Omega)$. This completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *Let Ω^t be the domain in Proposition 3.1, satisfying $\Omega(\lambda t) \subseteq \Omega^t \subseteq \Omega(\lambda^{-1}t)$ for $t \in [0, \delta]$, with $\Omega(\lambda t) = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \lambda t\}$. Suppose that $f \in L^p(\Omega^t)$ for some $p > N$, and $\Omega_h \subset \Omega^t$ for some $t = O(h^{r+1})$ and $h \leqslant h_1$, where $h_1 > 0$ is some constant. Let $u \in H_0^1(\Omega)$ and $u^{(h)} \in H_0^1(\Omega_h)$ be the weak solutions of the following PDE problems:*

$$-\Delta u = f \quad \text{in } \Omega,$$

$$-\Delta u^{(h)} = f \quad \text{in } \Omega_h,$$

and extend u and $u^{(h)}$ by zero to the larger domain Ω^t. Then there exists $h_2 > 0$ such that for $h \leqslant h_2$ the following estimate holds:

$$\|u - u^{(h)}\|_{L^\infty(\Omega^t)} \leqslant Ch^{r+1}\|f\|_{L^p(\Omega^t)} \tag{3.9}$$

Proof. Since $\max_{x \in \Omega_h} |\Phi_h(x) - x| \leqslant C_0 h^{r+1}$ for some constant C_0, it follows that $\Omega_h \subset \Omega(C_0 h^{r+1}) \subset \Omega^t$ for $t = C_0\lambda^{-1}h^{r+1}$. When h is sufficiently small we have $t = C_0\lambda^{-1}h^{r+1} \leqslant \delta$ and therefore Ω^t is well defined. Let $u^t \in H_0^1(\Omega^t)$ be a weak solution of the Poisson equation

$$-\Delta u^t = f \quad \text{in } \Omega^t.$$

Proposition 3.1 implies that

$$\|u^t\|_{W^{1,\infty}(\Omega^t)} \leqslant C\|f\|_{L^p(\Omega^t)}. \tag{3.10}$$

Since $u^t - u$ is harmonic in $\Omega \subset \Omega^t$ and $u^t - u^{(h)}$ is harmonic in $\Omega_h \subset \Omega^t$, the maximum principle of the continuous problem implies that

$$
\begin{aligned}
\|u^t - u^{(h)}\|_{L^\infty(\Omega_h)} &\leqslant \|u^t - u^{(h)}\|_{L^\infty(\partial\Omega_h)} \\
&= \|u^t\|_{L^\infty(\partial\Omega_h)} \qquad\qquad \text{(since } u^{(h)} = 0 \text{ on } \partial\Omega_h) \\
&\leqslant Ch^{r+1}\|u^t\|_{W^{1,\infty}(\Omega^t)} \\
&\leqslant Ch^{r+1}\|f\|_{L^p(\Omega^t)}, \tag{3.11}
\end{aligned}
$$

where we have used the fact that $\operatorname{dist}(x, \partial\Omega^t) \leqslant 2C_0 h^{r+1}$ for $x \in \partial\Omega_h$. Therefore,

$$
\begin{aligned}
\|u^t - u^{(h)}\|_{L^\infty(\Omega^t)} &\leqslant \|u^t - u^{(h)}\|_{L^\infty(\Omega_h)} + \|u^t\|_{L^\infty(\Omega^t\backslash\Omega_h)} \\
&\leqslant Ch^{r+1}\|f\|_{L^p(\Omega^t)} + Ch^{r+1}\|u^t\|_{W^{1,\infty}(\Omega^t)} \\
&\leqslant Ch^{r+1}\|f\|_{L^p(\Omega^t)}. \tag{3.12}
\end{aligned}
$$

The following result can be proved in the same way:

$$\|u^t - u\|_{L^\infty(\Omega^t)} \leqslant Ch^{r+1}\|f\|_{L^p(\Omega^t)}. \tag{3.13}$$

The result of Lemma 3.4 follows from (3.12)–(3.13) and the triangle inequality. $\qquad\square$

In the following, we prove Theorem 1.2 by using the technical result in Proposition 3.1.

Let Ω^t be the domain in Proposition 3.1, satisfying $\Omega(\lambda t) \subseteq \Omega^t \subseteq \Omega(\lambda^{-1} t)$ for $t \in [0, \delta]$, with $\Omega(\lambda t) = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < \lambda t\}$. For the simplicity of notation, we still denote by $f \in L^p(\Omega^t)$ an extension of $\tilde{f} \in L^p(\Omega \cup \Omega_h)$ satisfying $\|f\|_{L^p(\Omega^t)} \leqslant C\|\tilde{f}\|_{L^p(\Omega \cup \Omega_h)} \leqslant C\|f\|_{L^p(\Omega)}$.

Under assumption 1.1, the curvilinear polyhedral domain Ω can be extended to a larger convex polyhedron Ω_* with a piecewise flat boundary such that $\overline{\Omega} \subset \Omega_*$ and the triangulation $\mathscr{K}$ can be extended to a quasi-uniform triangulation $\mathscr{K}_*$ on Ω_* (thus the triangulation in $\Omega_*\backslash\overline{\Omega}$ is also isoparametric on its boundary $\partial\Omega$).

Let $\tilde{u}$ be an extension of $u^{(h)}$ such that $\tilde{u} = u^{(h)}$ on Ω_h and $\tilde{u} = 0$ in $\Omega_*\backslash\Omega_h$. Let $\mathring{S}_h(\Omega_*) \subset H^1_0(\Omega_*)$ be the H^1-conforming isoparametric finite element space on Ω_* with triangulation $\mathscr{K}_*$. Let $\tilde{u}_h \in \mathring{S}_h(\Omega_*)$ be the Ritz projection of $\tilde{u}$ defined by

$$\int_{\Omega_*} \nabla(\tilde{u} - \tilde{u}_h) \cdot \nabla\chi_h = 0 \quad \forall\, \chi_h \in \mathring{S}_h(\Omega_*).$$

Then

$$
\begin{aligned}
\|u^{(h)} - u_h\|_{L^\infty(\Omega_h)} &= \|\tilde{u} - u_h\|_{L^\infty(\Omega_h)} \\
&\leqslant \|\tilde{u} - \tilde{u}_h\|_{L^\infty(\Omega_h)} + \|\tilde{u}_h - u_h\|_{L^\infty(\Omega_h)} \\
&\leqslant \|\tilde{u} - \tilde{u}_h\|_{L^\infty(\Omega_*)} + \|\tilde{u}_h - u_h\|_{L^\infty(\Omega_h)}, \tag{3.14}
\end{aligned}
$$

where $\|\tilde{u} - \tilde{u}_h\|_{L^\infty(\Omega_*)}$ is the error of the Ritz projection of an H^1-conforming FEM in a standard convex polyhedron and therefore can be estimated by using the result on a standard convex polyhedron (or using the interior maximum-norm estimate as in [40, Theorem 5.1] and [32, Proof of Theorem 5.1]), i.e.,

$$
\begin{aligned}
\|\tilde{u} - \tilde{u}_h\|_{L^\infty(\Omega_*)} &\leqslant C\ell_h \|\tilde{u} - I_h\tilde{u}\|_{L^\infty(\Omega_*)} \\
&\leqslant C\ell_h \|u^{(h)} - I_h u^{(h)}\|_{L^\infty(\Omega_h)} \\
&\leqslant C\ell_h \|u - I_h u\|_{L^\infty(\Omega_h)} + C\ell_h h^{r+1}\|f\|_{L^p(\Omega^t)}, \tag{3.15}
\end{aligned}
$$

where the last inequality uses the triangle inequality and (3.9), and $I_h \widetilde{u}$ is the interpolation operator associated with the larger triangulation $\mathscr{K}_*$ which extends the interpolation operator $I_h : C(\overline{\Omega_h}) \to S_h(\Omega_h)$ associated with $\mathscr{K}$. Since $\widetilde{u}_h - u_h$ is discrete harmonic in Ω_h, i.e.,

$$\int_{\Omega_h} \nabla(\widetilde{u}_h - u_h) \cdot \nabla \chi_h \, \mathrm{d}x = \int_{\Omega_h} \nabla(\widetilde{u} - u^{(h)}) \cdot \nabla \chi_h \, \mathrm{d}x = 0 \quad \forall \, \chi_h \in \mathring{S}_h(\Omega_h),$$

it follows from Theorem 1.1 that $\widetilde{u}_h - u_h$ satisfies the discrete maximum principle, i.e.,

$$
\begin{aligned}
\|\widetilde{u}_h - u_h\|_{L^\infty(\Omega_h)} &\leqslant C\|\widetilde{u}_h - u_h\|_{L^\infty(\partial \Omega_h)} \\
&= C\|\widetilde{u}_h\|_{L^\infty(\partial \Omega_h)} \\
&= C\|\widetilde{u}_h - \widetilde{u}\|_{L^\infty(\partial \Omega_h)} \quad (\text{since } \widetilde{u}|_{\partial \Omega_h} = 0) \\
&\leqslant C\|\widetilde{u}_h - \widetilde{u}\|_{L^\infty(\Omega_*)}.
\end{aligned}
\tag{3.16}
$$

Substituting (3.15) and (3.16) into (3.14) yields

$$\|u^{(h)} - u_h\|_{L^\infty(\Omega_h)} \leqslant C\ell_h \|u - I_h u\|_{L^\infty(\Omega_h)} + C\ell_h h^{r+1} \|f\|_{L^p(\Omega^t)}.$$

Since $u^{(h)} = u_h = 0$ in $\Omega \backslash \Omega_h$, it follows that

$$\|u^{(h)} - u_h\|_{L^\infty(\Omega)} = \|u^{(h)} - u_h\|_{L^\infty(\Omega \cap \Omega_h)} \leqslant C\ell_h \|u - I_h u\|_{L^\infty(\Omega_h)} + C\ell_h h^{r+1} \|f\|_{L^p(\Omega^t)}.$$

Then, combining this with (3.9), we obtain the following error bound:

$$\|u - u_h\|_{L^\infty(\Omega)} \leqslant C\ell_h \|u - I_h u\|_{L^\infty(\Omega_h)} + C\ell_h h^{r+1} \|f\|_{L^p(\Omega^t)}.$$

Finally, we note that

$$
\begin{aligned}
\|u - \check{I}_h u\|_{L^\infty(\Omega)} &= \|u \circ \Phi_h - I_h(u \circ \Phi_h)\|_{L^\infty(\Omega_h)} \\
&\geqslant \|u - I_h u\|_{L^\infty(\Omega_h)} - C\|u - u \circ \Phi_h\|_{L^\infty(\Omega_h)} \\
&\geqslant \|u - I_h u\|_{L^\infty(\Omega_h)} - C\|u\|_{W^{1,\infty}(\mathbb{R}^d)} \|\Phi_h - \mathrm{Id}\|_{L^\infty(\Omega_h)} \\
&\geqslant \|u - I_h u\|_{L^\infty(\Omega_h)} - Ch^{r+1} \|u\|_{W^{1,\infty}(\mathbb{R}^d)} \\
&\geqslant \|u - I_h u\|_{L^\infty(\Omega_h)} - Ch^{r+1} \|f\|_{L^p(\Omega^t)}.
\end{aligned}
$$

This proves the result of Theorem 1.2. $\qquad \square$

4. Conclusion

We have proved the weak maximum principle of the isoparametric FEM for the Poisson equation in curvilinear polyhedral domains with edge openings smaller than π, which include smooth domains and smooth deformations of convex polyhedra. The proof requires using a duality argument for an elliptic equation with some discontinuous coefficients arising from the use of isoparametric finite elements. Hence, the standard H^2 elliptic regularity does not hold for the solution of the corresponding dual problem. We have overcome the difficulty by decomposing the solution into a smooth H^2 part and a nonsmooth $W^{1,p}$ part, separately, and replaced the H^2 regularity required in a standard duality argument by some $W^{1,p}$ estimates for the nonsmooth part of the solution.

As an application of the weak maximum principle, we have proved an L^∞-norm best approximation property of the isoparametric FEM for the Poisson equation. All the analysis for the Poisson equation in this article can be extended to elliptic equations with $W^{1,\infty}$ coefficients. However, the current analysis does not allow us to extend the results to curvilinear polyhedral domains with edge openings bigger than π (smooth deformations of nonconvex polyhedra) or graded mesh in three dimensions. These would be the subject of future research.

There are other approaches to the maximum principle of finite element methods for elliptic equations using non-obtuse meshes, which is restricted to piecewise linear finite elements and Poisson equation with constant coefficients; see [19]. The approach in the current manuscript is applicable to elliptic equations with $W^{1,\infty}$ coefficients, general quasi-uniform meshes, and high-order finite elements, and therefore requires completely different analysis from the approaches using non-obtuse meshes.

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Buyang Li. Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong.
E-mail address: buyang.li@polyu.edu.hk

Weifeng Qiu. Department of Mathematics, City University of Hong Kong, Hung Hom, Hong Kong.
E-mail address: weifeqiu@cityu.edu.hk

Yupei Xie. Corresponding author. Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong.
E-mail address: yupei.xie@polyu.edu.hk

Wenshan Yu. Division of Science and Technology, United International College (BNU-HKBU), Zhuhai, 519087, P.R. China.
E-mail address: yuwenshan@uic.edu.cn