SECOND-ORDER UNIFORMLY ACCURATE METHOD FOR THE SEMICLASSICAL NONLINEAR SCHRÖDINGER EQUATION WITH INITIAL DATA IN $\mathbf{H^2}$

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Abstract. This paper studies the numerical solution of the semiclassical nonlinear Schrödinger equation on the d-dimensional torus $\mathbb{T}^d$, with highly oscillatory initial data depending on a small parameter $\varepsilon \in (0,1]$. We first show that a WKB-type approximation attains an $\mathcal{O}(\varepsilon)$ error in the L^2 norm for H^2 initial data theoretically, although its accuracy deteriorates as ε increases. To address this limitation, we propose a numerical scheme that (i) applies a Galilean transform to remove the oscillations in the initial data, (ii) establishes sharp space–time estimates for the transformed equation, and (iii) employs a new low-regularity integrator to achieve second-order accuracy under the minimal H^2 regularity, which is weaker than the regularity assumptions in the literature. Furthermore, our analysis shows that the CFL-type conditions linking h, τ, and ε — typically imposed in the semiclassical regime in the literature — are not required in our scheme to obtain second-order convergence with respect to τ and h, uniformly with respect to ε, under the weaker regularity condition. Numerical experiments support the theoretical results and demonstrate the robustness of the method across a wide range of ε.

Keywords. Nonlinear Schrödinger equation, semiclassical, highly oscillatory, uniformly accurate, Fourier spectral method, optimal-order convergence, weaker regularity condition

1. Introduction

In this paper, we consider the numerical approximation of the weakly nonlinear Schrödinger equation on the torus in the semiclassical regime:

$$\begin{aligned} &i\varepsilon\partial_t u + \frac{\varepsilon^2}{2}\Delta u = \varepsilon\lambda|u|^2u, \quad x \in \mathbb{T}^d,\ t>0,\\ &u(0,x) = e^{i\kappa\cdot x/\varepsilon}a_0(x). \end{aligned} \tag{1}$$

Here, $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ and the sign of $\lambda = \pm 1$ distinguishes the defocusing $(+1)$ and focusing (-1) cases. The function $a_0 \in H^\gamma(\mathbb{T}^d)$, $\gamma > \frac{d}{2}$ denotes the initial amplitude profile, whose derivatives are assumed to be uniformly bounded with respect to ε. The wave vector $\kappa \in \mathbb{R}^d$ is chosen such that $\kappa/\varepsilon \in \mathbb{Z}^d$, ensuring that the initial data is periodic. Otherwise, the equation becomes periodic on a scaled torus, to which the analysis developed in this paper applies equally. Under these conditions, the solution exhibits rapid oscillations in both space and time, with characteristic scales governed by the small parameter ε. The semiclassical weakly nonlinear Schrödinger equation has been the subject of extensive theoretical investigation and arises in various physical contexts, as discussed, for instance, in [8, 9].

The semiclassical nonlinear Schrödinger equation has been extensively studied from a numerical perspective. Traditional finite difference methods, such as the leapfrog and Crank–Nicolson schemes [15], require stringent constraints on the time step and mesh size—specifically, $\tau \ll \varepsilon$ and $h \ll \varepsilon$—to ensure accurate approximations. These constraints lead to high computational costs, and worse, the associated error bounds typically involve negative powers of ε, making such methods inefficient or even impractical in the semiclassical regime $\varepsilon \ll 1$. In contrast,

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time-splitting spectral methods [2, 3, 12, 14] relax these restrictions on step sizes, allowing for $\tau = O(\varepsilon)$ and $h = o(\varepsilon)$. Asymptotic-preserving approaches have also been developed by reformulating the Schrödinger equation using WKB expansions [8, 10], as exemplified by the methods in [1, 4]. Geometric optics techniques can also be applied to obtain leading-order approximations, which allow for discretization parameters independent of ε and provide error estimates with convergence rates in terms of ε. However, these methods are tailored to the semiclassical regime and may fail to be accurate or reliable when $\varepsilon = O(1)$.

To reduce the dependence of the time step and mesh size on the small parameter ε and to achieve uniform convergence over $\varepsilon \in (0, 1]$, [17] proposed weighted finite difference schemes. In particular, assuming that the modulation amplitude $a \in C^4([0, T] \times \mathbb{T})$ has fourth-order partial derivatives bounded independently of ε, they established an ε-uniform error bound of order $O(h^2 + \tau^2)$ in the maximum norm, without any CFL condition in the case of the implicit method. However, the analysis requires relatively high regularity. These results illustrate the inherent difficulty of designing numerical schemes that simultaneously ensure uniform accuracy, high-order convergence, low regularity requirements, and relaxed consistency and stability constraints on the time step τ and mesh size h, uniformly with respect to ε.

In this work, we address several of these challenges by developing a new ε-uniform scheme that achieves second-order accuracy under minimal regularity assumptions H^2 and without imposing any ε-dependent CFL-type restriction. A key ingredient is the exploitation of the highly oscillatory nature of the solution, which is effectively handled through a *Galilean transformation*, defined by

$$\mathcal{G}_\kappa(U) = e^{\frac{i}{\varepsilon}(\kappa\cdot x - \frac{|\kappa|^2}{2}t)} U(t, x - \kappa t), \tag{2}$$

where $\kappa \in \mathbb{R}^d$ is the wave vector determined by the initial phase. Applying the inverse transformation, we consider

$$U(t, x) = \mathcal{G}_\kappa^{-1}(u) = e^{-\frac{i}{\varepsilon}(\kappa\cdot x + \frac{|\kappa|^2}{2}t)} u(t, x + \kappa t).$$

The transformed function $U(t, x)$ satisfies the same form of the semiclassical weakly nonlinear Schrödinger equation:

$$\begin{cases} i\varepsilon\partial_t U + \dfrac{\varepsilon^2}{2}\Delta U = \varepsilon\lambda|U|^2 U, & x \in \mathbb{T}^d,\ t > 0, \\ U(0, x) = a_0(x), \end{cases} \tag{3}$$

but with *non-oscillatory* initial data. Importantly, the solution $U(t, x)$ remains uniformly bounded in the H^γ norm with respect to $\varepsilon \in (0, 1]$, which is a key property for establishing convergence rates that are uniform in ε; see Proposition 3.5 for details. This uniform regularity highlights the essential role of the Galilean transformation in the construction and analysis of uniformly accurate numerical schemes, while preserving the original structure of the equation.

A first algorithm can be derived by retaining only the leading-order term in (3), neglecting the $\varepsilon^2\Delta$ term. This leads to the approximation b of U, defined by the equation

$$\begin{cases} i\partial_t b = \lambda|b|^2 b, & x \in \mathbb{T}^d,\ t > 0, \\ b(0, x) = a_0(x), \end{cases} \tag{4}$$

whose solution is explicitly given by $b(t, x) = e^{-i\lambda|a_0(x)|^2 t} a_0(x)$. The corresponding approximation to u is then obtained by applying the transform $\mathcal{G}_\kappa$ to b. This algorithm is closely related to the classical WKB method. As shown in Theorem 2.1, the error between the exact solution u and the approximation $\mathcal{G}_\kappa(b)$ is of order $\mathcal{O}(\varepsilon)$ for H^2 initial data; however, the approximation may become inaccurate when ε is not small.

To overcome this limitation and ensure accuracy across the full range $\varepsilon \in (0,1]$, we develop a new scheme based on a direct approximation of the full transformed equation (3), retaining the $\varepsilon\Delta$ term in the discretization. Our methodology is briefly outlined below. As a starting point for all low-regularity integrators, we introduce the twisted variable $V(t) = e^{-it\frac{\varepsilon}{2}\Delta}U(t)$ and apply the Duhamel formula with a filtering technique. This yields the following integral formulation for the twisted variable:

$$V(t_{n+1}) = V(t_n) - i\lambda \int_{t_n}^{t_{n+1}} e^{-is\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left| e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(s) \right|^2 e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(s) \right] ds + \mathcal{R}_1^n, \tag{5}$$

where $\mathcal{R}_1^n$ is the high-frequency error, given explicitly in (28), and discarded in the numerical scheme. The approximation of the low-frequency part consists of two components: a second-order approximation of $\Pi_N V(s)$ based on an expansion around $\Pi_N V(t_n)$, and an approximation of the integral of the phase function, which captures the resonant structure of the nonlinear term. For the latter one, our approach is inspired by the phase-based approximation techniques developed in [7,16]. In particular, we adopt the following second-order approximation from [16, (1.5)]:

$$\int_0^\tau e^{is\phi} ds = \int_0^\tau \left[e^{is\alpha} + i\beta e^{is\beta} \cdot \frac{1}{\tau}\int_0^\tau \sigma e^{i\sigma\alpha} d\sigma \right] ds + \mathcal{T}(\alpha,\beta), \tag{6}$$

$$\phi = \frac{\varepsilon}{2}(|k|^2 + |k_1|^2 - |k_2|^2 - |k_3|^2), \quad \alpha = \frac{\varepsilon}{2}|k_1|^2, \quad \beta = \frac{\varepsilon}{2}(k_1\cdot k_2 + k_1 \cdot k_3 + k_3 \cdot k_2). \tag{7}$$

This is a second-order approximation, as reflected in the remainder term $\mathcal{T}(\alpha,\beta)$, which satisfies the bound $|\mathcal{T}(\alpha,\beta)| = \mathcal{O}(\tau^3|\beta|^2)$. The resulting approximation error is denoted by $\mathcal{R}_4^n$ (see the detailed derivation in (36)), with its following Fourier coefficients given by

$$\widehat{\mathcal{R}_4^n}(k) = -i\lambda \sum_{k=k_1+k_2+k_3, |k_i|\le N} \mathcal{T}(\alpha,\beta)\cdot e^{it_n\phi}\,\overline{\widehat{V}}_{k_1}(t_n)\widehat{V}_{k_2}(t_n)\widehat{V}_{k_3}(t_n), \quad |k|\le N.$$

Exploiting the fact that $|\beta|^2$ is a multivariate polynomial in k_1, k_2, and k_3 of degree at most two in each variable and total degree at most four, we derive

$$\begin{aligned} \|\mathcal{R}_4^n\|_{L^2} &\le \tau^3 \Big\| \sum_{k=k_1+k_2+k_3, |k_i|\le N} |\beta|^2 \overline{\hat{V}}_{k_1}(t_n)\hat{V}_{k_2}(t_n)\hat{V}_{k_3}(t_n)\Big\|_{\ell_k^2} \\ &\lesssim \varepsilon^2\tau^3(\|\Delta V(t_n)\|_{L^2}\|\Delta V(t_n)\|_{L^\infty}\|V(t_n)\|_{L^\infty} + \|\Delta V(t_n)\|_{L^2}\|\nabla V(t_n)\|_{L^\infty}\|\nabla V(t_n)\|_{L^\infty}). \end{aligned} \tag{8}$$

This estimate implies that, in order to achieve a second-order temporal convergence in the L^2 norm, i.e., $\|\mathcal{R}_4^n\|_{l_n^1 L^2} = \mathcal{O}(\tau^2)$, it is necessary that $\sup_n |\Delta V(t_n)|_{L^\infty}$ remain bounded. By the Sobolev embedding $H^{2+d/2} \hookrightarrow W^{2,\infty}$, this in turn requires the solution to lie in $H^{2+d/2}$.

In this paper, we address the above difficulty by employing a different analytical technique to estimate the error, which enables us to reduce the regularity requirement on the solution from $H^{2+d/2}$ to H^2. The key idea is to apply the scaled Strichartz estimate (Lemma 3.4) to control the solution in mixed space–time norms, as shown in Proposition 3.5. For example, in the one-dimensional case ($d=1$), Proposition 3.5 yields

$$\|U\|_{L_t^4 H_x^{2,4}([0,T]\times\mathbb{T})} \le C\varepsilon^{-1/4}, \quad \|U\|_{L_t^\infty H_x^2([0,T]\times\mathbb{T})} \le C, \tag{9}$$

which shows that the solution is uniformly bounded in H^2, while the $L_t^4 H_x^{2,4}$-norm carries a negative power of ε. This estimate enables a sharper control of the remainder term $\mathcal{R}_4^n$:

$$\begin{aligned}\sum_{n=0}^{L_0-1} \|\mathcal{R}_4^n\|_{L^2} &\lesssim \varepsilon^{\frac{3}{2}}\tau^3\Big(\|\Delta U\|_{L_t^4L_x^4([0,T]\times\mathbb{T})}\|\Delta U\|_{L_t^4L_x^4([0,T]\times\mathbb{T})}\|U\|_{L_{t,x}^\infty([0,T]\times\mathbb{T})}\\ &\qquad + \|\Delta U\|_{L_t^4L_x^4([0,T]\times\mathbb{T})}\|\nabla U\|_{L_t^4L_x^4([0,T]\times\mathbb{T})}\|\nabla U\|_{L_{t,x}^\infty([0,T]\times\mathbb{T})}\Big) + O(\tau^2)\\ &\lesssim \tau^2,\end{aligned}$$

where the estimates in (9) are applied. Here, the negative powers of ε arising from (9) are compensated by the prefactor $\varepsilon^{3/2}$, which yields a uniform-in-ε bound. Consequently, Theorem 2.2 establishes second-order convergence of the numerical solution in the L^2 norm. We emphasize that this convergence is uniform for all $\varepsilon \in (0,1]$ and does not require any CFL-type condition.

The remainder of this paper is structured as follows. In Section 2, we introduce the notation and state our main results. In Section 3, we establish the existence and boundedness of solutions to (3). Section 4 is devoted to the error analysis and the proof of the main convergence theorem. Finally, Section 5 provides numerical experiments that illustrate and validate the theoretical results.

2. Notation and main results

In this section, we introduce the basic notation, function spaces, and fundamental inequalities that will be used throughout the paper. After that, we state the main results of the paper.

2.1. Notation

In this paper, we adhere to the following convention: the parameter ε denotes the small semiclassical parameter inherent to the dispersive equation, whereas ϵ is reserved for arbitrarily small positive constants appearing in Sobolev embeddings, Strichartz estimates, or error bounds.

Let f be a function defined on the d-dimensional torus $\mathbb{T}^d$. Its Fourier transform and inverse Fourier transform are defined respectively by

$$\hat{f}_k = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x)\, e^{-ik\cdot x}\,dx, \quad k\in\mathbb{Z}^d, \qquad f(x) = \sum_{k\in\mathbb{Z}^d}\hat{f}_k\, e^{ik\cdot x}.$$

The Fourier transform enjoys several useful properties. In particular, Plancherel's identity asserts that

$$\|f\|_{L^2(\mathbb{T}^d)}^2 = (2\pi)^d\sum_{k\in\mathbb{Z}^d}|\hat{f}_k|^2,$$

while the convolution identity for the product of functions reads

$$\widehat{(fg)}_k = \sum_{k_1+k_2=k}\hat{f}_{k_1}\hat{g}_{k_2}.$$

We define the Fourier projection operators as follows: for any integer $N>0$, the low- and high-frequency projection operators Π_N and $\Pi_{>N}$ are defined by

$$\widehat{(\Pi_N f)}_k = \begin{cases}\hat{f}_k, & |k|\le N,\\ 0, & |k|>N,\end{cases} \qquad \widehat{(\Pi_{>N} f)}_k = \begin{cases}0, & |k|\le N,\\ \hat{f}_k, & |k|>N.\end{cases}$$

For $r \in [1,\infty)$ and $s > 0$, the Sobolev space $H^{s,r}(\mathbb{T}^d)$ consists of functions $f \in L^r(\mathbb{T}^d)$ such that the fractional operator $J^s f := (1-\Delta)^{s/2} f$ belongs to $L^r(\mathbb{T}^d)$. Its norm is defined by

$$\|f\|_{H^{s,r}} = \|J^s f\|_{L^r} = \Big\| \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{s/2} \hat{f}_k \, e^{ik\cdot x} \Big\|_{L^r}.$$

When $r = 2$, we write $H^s(\mathbb{T}^d) := H^{s,2}(\mathbb{T}^d)$.

Let $I \subset [0,\infty)$ be a time interval and $1 \le p, q \le \infty$. We define $L_t^q L_x^p(I \times \mathbb{T}^d)$ or $L^q(I; L^p(\mathbb{T}^d))$ as the collection of measurable functions $u : I \to L^p(\mathbb{T}^d)$ satisfying

$$\|u\|_{L_t^q L_x^p(I\times\mathbb{T}^d)} = \Big\| \|u(t,\cdot)\|_{L^p(\mathbb{T}^d)} \Big\|_{L^q(I)} < \infty.$$

We often abbreviate $L_t^q L_x^p(I \times \mathbb{T}^d)$ as $L_{t,x}^p(I \times \mathbb{T}^d)$ for the case $p = q$. Likewise, we write $L_t^\infty H_x^\gamma(I \times \mathbb{T}^d)$ for Sobolev-valued time-dependent spaces. For notational simplicity, when $I = (0,T)$, we simply write $L_t^q L_x^p$ or $L_{t,x}^p$ without the time domain explicitly.

For a sequence $\{a_n\}_{n\in\mathbb{Z}}$, its discrete ℓ^p-norm is defined by

$$\|a_n\|_{l_n^p} := \begin{cases} \left(\sum_{n\in\mathbb{Z}} |a_n|^p\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{n\in\mathbb{Z}} |a_n|, & p = \infty. \end{cases}$$

We write $A \lesssim B$ (or $B \gtrsim A$) if there exists an absolute constant $C > 0$, independent of τ, N, and $n \in \mathbb{Z}$, such that $A \le CB$ holds, where the constant C may change from one occurrence to another. The notation $a+$ denotes $a + \epsilon$ for an arbitrarily small $\epsilon > 0$, and $a-$ denotes $a - \epsilon$ for an arbitrarily small $\epsilon > 0$.

2.2. Main results

In this section, we state two main theorems that summarize the key results of the paper. The first theorem provides an $\mathcal{O}(\varepsilon)$ error estimate for the solution of (4), consistent with classical WKB analysis and highlighting the H^2-regularity requirement on the initial data. The second theorem concerns the convergence of the proposed numerical scheme, which is derived from a Galilean-transformed formulation and tailored to handle low-regularity solutions. For H^2 initial data, it establishes second-order convergence in both space and time, uniformly in $\varepsilon \in [0,1]$, and remains robust throughout the entire semiclassical regime.

Theorem 2.1 (Convergence in the small-ε regime)**.** *Let u and b denote the solutions of (1) and (4), respectively. Suppose $a_0 \in H^2(\mathbb{T}^d)$ with $1 \le d \le 3$. Then, we have*

$$\|u(t) - \mathcal{G}_\kappa(b)(t)\|_{L_t^\infty L_x^2([0,T]\times\mathbb{T}^d)} \le C(T, \|a_0\|_{H^2})\varepsilon,$$

where the interval $[0,T] \subset (-T_*, T^*)$, where $(-T_*, T^*)$ denotes the existence time of the solution, as specified in Proposition 3.5.

Proof. Since the Galilean transformation (15) is an isometry in L^2, it suffices to estimate the difference between $U(t,x)$ and $b(t,x)$. In particular, we have for any $t \in [0,T]$,

$$\|u(t) - \mathcal{G}_\kappa(b)(t)\|_{L^2} = \|U(t,x) - b(t,x)\|_{L^2}.$$

Define the residual $R(t,x) := U(t,x) - b(t,x)$. Then $R(0,x) = 0$, and $R(t,x)$ satisfies the equation

$$i\partial_t R + \frac{\varepsilon}{2}\Delta R = \lambda\left(|U|^2 U - |b|^2 b\right) - \frac{\varepsilon}{2}\Delta b.$$

Testing against $\bar{R}$ and taking the imaginary part yields

$$\partial_t \|R\|_{L^2}^2 = \Im\left(\int_{\mathbb{T}^d} \lambda\left(|U|^2U - |b|^2 b\right)\overline{R}\,dx\right) - \frac{\varepsilon}{2}\Im\left(\int_{\mathbb{T}^d} \Delta b\cdot\overline{R}\,dx\right)$$
$$\leq C\left(\|b\|_{L^\infty}^2 + \|R\|_{L^\infty}^2\right)\|R\|_{L^2}^2 + \frac{\varepsilon}{2}\|\Delta b\|_{L^2}\|R\|_{L^2}.$$

The Sobolev embedding $H^2 \hookrightarrow L^\infty$, together with the Cauchy–Schwarz inequality, implies that

$$\partial_t \|R\|_{L^2}^2 \leq C\left(\|b\|_{L_t^\infty H_x^2}^2 + \|R\|_{L_t^\infty H_x^2}^2 + 1\right)\|R\|_{L^2}^2 + \varepsilon^2\|b\|_{L_t^\infty H_x^2}^2. \tag{10}$$

Now recall that $b(t,x) = e^{-i\lambda|a_0|^2 t}a_0(x)$, then

$$\|b\|_{L_t^\infty H_x^2} \lesssim \|a_0\|_{H^2} + \|a_0\|_{H^2}^5. \tag{11}$$

From Proposition 3.5 below, we also have the bound

$$\|U\|_{L_t^\infty H_x^2} \lesssim \|a_0\|_{H^2} \quad \Rightarrow \quad \|R\|_{L_t^\infty H_x^2} \leq \|U\|_{L_t^\infty H_x^2} + \|b\|_{L_t^\infty H_x^2} \lesssim \|a_0\|_{H^2} + \|a_0\|_{H^2}^5. \tag{12}$$

Since $\|R(0)\|_{L^2} = 0$, applying Gronwall's inequality to (10) and combining it with (11) and (12), we obtain, for any $t \in [0,T]$,

$$\|R(t)\|_{L^2}^2 \leq C\varepsilon^2 t\cdot\exp\left(Ct\right),$$

where the constant C depends only on $\|a_0\|_{H^2}$. This completes the proof. □

The above result ensures that (4) yields a first-order error in ε, making it effective when $\varepsilon \ll 1$. However, a uniform error estimate valid for all $\varepsilon \in (0,1]$ is more desirable, as it guarantees the approximation remains accurate across the entire regime. This motivates the development of the proposed scheme. We first present its derivation and then provide its error estimate. Notably, the convergence of the scheme holds uniformly for all $\varepsilon \in (0,1]$, with time and space discretization parameters chosen independently of ε. This uniformity is essential for accurately and efficiently capturing the rapid oscillations characteristic of the semiclassical regime.

Our scheme is based on the semi-discrete approach developed in [7,16] for the standard nonlinear Schrödinger equation, with several modifications adapted to the transformed equation (3) in the semiclassical regime.

Let $t_n = n\tau$ for $n = 0,1,\ldots,L_0$ be the temporal grid points over the interval $[0,T] \subset (-T_*,T^*)$. Here, $\tau = T/L_0$ is the time step size, and U^n denotes the numerical approximation of the transformed solution U at time t_n. After initializing with the projected exact data $U^0 = \Pi_N a_0$, the fully discrete scheme is given by:

$$U^{n+1} = e^{i\tau\frac{\varepsilon}{2}\Delta}U^n - i\tau\lambda\, e^{i\tau\frac{\varepsilon}{2}\Delta}\Pi_N\left\{\left[\varphi\left(-2i\tau\tfrac{\varepsilon}{2}\Delta\right) + \psi\left(-2i\tau\tfrac{\varepsilon}{2}\Delta\right)\right]\Pi_N\overline{U^n}\cdot\left(\Pi_N U^n\right)^2\right\}$$
$$+ i\tau\lambda\Pi_N\left\{\left[e^{i\tau\frac{\varepsilon}{2}\Delta}\psi\left(-2i\tau\tfrac{\varepsilon}{2}\Delta\right)\Pi_N\overline{U^n}\right]\left(e^{i\tau\frac{\varepsilon}{2}\Delta}\Pi_N U^n\right)^2\right\} - \frac{\tau^2}{2}e^{i\tau\frac{\varepsilon}{2}\Delta}\Pi_N\left[\left|\Pi_N U^n\right|^4\Pi_N U^n\right], \tag{13}$$

where the functions φ and ψ are defined by

$$\varphi(z) = \begin{cases}\frac{e^z-1}{z}, & z\neq 0,\\ 1, & z = 0,\end{cases} \qquad \psi(z) = \begin{cases}\frac{e^z-1-ze^z}{z^2}, & z\neq 0,\\ -\frac{1}{2}, & z = 0.\end{cases}$$

The complete scheme proceeds by first recursively updating the transformed approximation U^n, and then recovering

$$u^n = \mathcal{G}_\kappa^n(U^n), \tag{14}$$

where Galilean-type transformation $\mathcal{G}_\kappa^n(U^n)$ is defined by

$$\mathcal{G}_\kappa^n(U) = e^{\frac{i}{\varepsilon}\left(\kappa\cdot x - \frac{|\kappa|^2}{2}t_n\right)}U(x-\kappa t_n), \quad \text{for spatial function } U. \tag{15}$$

We now present the convergence results of the above numerical scheme.

Theorem 2.2 (Uniform-in-ε convergence). *Let $T > 0$ and $a_0 \in H^2(\mathbb{T}^d)$ with $1 \le d \le 3$. Let $u(t)$ be the exact solution to the weakly nonlinear Schrödinger equation (1), and u^n the numerical approximation given by the fully discrete scheme (13)–(14) with time step τ and truncation parameter N, both chosen independently of the semiclassical parameter $\varepsilon \in (0, 1]$.*

Then, there exist constants $\tau_0 > 0$ and $N_0 > 0$, independent of ε, such that for all $0 < \tau \le \tau_0$ and $N > N_0$, for any $\varepsilon \in (0, 1]$, the error satisfies

$$\|u(t_n) - u^n\|_{L^2(\mathbb{T}^d)} \le C \begin{cases} N^{-2} + \tau^2, & d = 1, \\ N^{-2} + \tau^{2-}, & d = 2, 3 \end{cases}$$

where the constant C is independent of τ, N and $\varepsilon \in (0, 1]$.

Remark 2.3. The convergence estimates above are uniform with respect to the semiclassical parameter $\varepsilon \in (0, 1]$, ensuring that the scheme remains accurate not only in the highly oscillatory regime $\varepsilon \ll 1$, but also when ε is close to 1. This highlights its robustness and stability across the entire semiclassical range. Furthermore, the discretization parameters τ and N can be chosen independently of ε, free from any CFL-type condition, making the scheme particularly efficient for semiclassical computations. Finally, we remark that the δ-order loss in temporal convergence for dimensions $d = 2, 3$ arises from the use of Strichartz estimates on the torus.

At the same time, our regularity assumption on the initial data is minimal compared to existing results. Remarkably, even for the classical nonlinear Schrödinger equation with $\varepsilon = 1$, no existing methods are known to achieve second-order convergence in both space and time under merely H^2 initial data for high-dimensional problems. This underscores the novelty and effectiveness of our approach.

Remark 2.4. In this paper, we restrict ourselves to a single-phase highly oscillatory initial datum of the form

$$u(0, x) = a_0(x)\, e^{-i\kappa\cdot x/\varepsilon}, \qquad \kappa \in \mathbb{R}^d \backslash \{0\}.$$

The extension to the multiphase case

$$u(0, x) = \sum_{m=1}^{M} a_{m,0}(x)\, e^{-i\kappa_m\cdot x/\varepsilon}, \qquad \kappa_m \in \mathbb{R}^d \backslash \{0\}.$$

remains unclear and is of particular interest. We refer to [17] for related work in this direction.

3. Uniform boundedness of the exact solution to equation (3)

Since the proposed algorithm consists of approximation of the transformed solution U, the existence and boundedness of solutions of (3) plays an important role and that is the main focus of this section. The transformed solution U can be proved in the space $C([0, T]; H^\gamma(\mathbb{T}^d))$, and we obtain a priori estimates that are uniform in the semiclassical parameter ε with respect to the H^γ norm. In contrast, its bounds in time-space norms may grow as $\varepsilon \to 0$, and we provide a specific rate of growth for this dependence on ε.

3.1. Basic lemmas

Lemma 3.1 (Bernstein's inequality [11]). *Let $s \ge 0$ and $1 < p < \infty$. For any function $f \in H^{s,p}(\mathbb{T}^d)$, the following estimates hold:*

$$\|\Pi_N J^s f\|_{L^p} \lesssim N^s \|f\|_{L^p}, \qquad \|\Pi_{>N} f\|_{L^p} \lesssim N^{-s} \|J^s f\|_{L^p}.$$

In particular, when $s = 0$, the projection operators $\Pi_{\le N}$ and $\Pi_{>N}$ are bounded on $L^p(\mathbb{T}^d)$:

$$\|\Pi_N f\|_{L^p} \lesssim \|f\|_{L^p}, \qquad \|\Pi_{>N} f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Lemma 3.2 (The Kato–Ponce inequality [13])**.** *Let $s > \frac{d}{2}$. For $f, g \in H^s(\mathbb{T}^d)$, it holds that*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}.$$

We now present the following Strichartz-type estimate for the Schrödinger equation on the torus $\mathbb{T}^d$.

Lemma 3.3 (Strichartz-type estimate [5,6])**.** *In the one-dimensional case $d = 1$, there exists a constant $C = C(T)$ such that*

$$\|e^{it\Delta} f\|_{L^4_{t,x}([-T,T]\times\mathbb{T})} \le C\|f\|_{L^2(\mathbb{T})}. \tag{16}$$

Moreover, for $d \ge 1$, $p \ge \frac{2(d+2)}{d}$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\theta = \frac{d}{2} - \frac{d+2}{p} + \epsilon$ with any arbitrarily small $\epsilon > 0$, we have

$$\|e^{it\Delta} f\|_{L^p_{t,x}([-T,T]\times\mathbb{T}^d)} \le C\|f\|_{H^\theta(\mathbb{T}^d)}, \tag{17}$$

$$\Big\| \int_0^T e^{i(t-s)\Delta} F(s)\, ds \Big\|_{L^2(\mathbb{T}^d)} \le C\|J^\theta F\|_{L^{p'}_{t,x}([-T,T]\times\mathbb{T}^d)}. \tag{18}$$

Lemma 3.4 (Scaled Strichartz estimate)**.** *In the one-dimensional case $d = 1$, there exists a constant $C = C(T)$ such that*

$$\big\|e^{it\frac{\varepsilon}{2}\Delta} f\big\|_{L^4_{t,x}([-T,T]\times\mathbb{T})} \le C\varepsilon^{-\frac{1}{4}}\|f\|_{L^2(\mathbb{T})}.$$

Moreover, for $d \ge 1$, $p \ge \frac{2(d+2)}{d}$ and $\theta = \frac{d}{2} - \frac{d+2}{p} + \epsilon$ with any arbitrarily small $\epsilon > 0$, we have

$$\big\|e^{it\frac{\varepsilon}{2}\Delta} f\big\|_{L^p_{t,x}([-T,T]\times\mathbb{T}^d)} \le C\varepsilon^{-\frac{1}{p}}\|f\|_{H^\theta(\mathbb{T}^d)},$$

$$\Big\| \int_0^T e^{i(t-s)\frac{\varepsilon}{2}\Delta} F(s)\, ds \Big\|_{L^2(\mathbb{T}^d)} \le C\varepsilon^{-\frac{1}{p}}\|J^\theta F\|_{L^{p'}_{t,x}([-T,T]\times\mathbb{T}^d)}.$$

Proof. The proof of Lemma 3.4 follows directly from Lemma 3.3 by a time rescaling argument. Indeed, the propagator $e^{it\frac{\varepsilon}{2}\Delta}$ can be viewed as a rescaled version of $e^{it\Delta}$ with $t \mapsto \frac{\varepsilon}{2}t$, and the estimates adjust accordingly due to the scaling properties of the norms involved. We therefore omit the proof. □

3.2. Uniform boundedness of solution to equation (3)

With the above preparations in place, we are now ready to prove the existence and uniform boundedness of solutions to equation (3).

Proposition 3.5. *Let $\gamma > \frac{d}{2}$ and $a_0 \in H^\gamma(\mathbb{T}^d)$. Define $\theta = \frac{d}{2} - \frac{d+2}{p} + \epsilon$, for any $p \ge \frac{2(d+2)}{d}$ and arbitrarily small $\epsilon > 0$. Then there exists a life span $(-T_*, T^*)$, with $T_*, T^* > 0$ independent of $\varepsilon \in (0, 1]$, such that the transformed equation (3) admits a unique solution*

$$U \in C((-T_*, T^*); H^\gamma(\mathbb{T}^d)) \cap L^p((-T_*, T^*); H^{\gamma-\theta,p}(\mathbb{T}^d)).$$

Moreover, for any time interval I with $I \subsetneq (-T_*, T^*)$, there exists a constant $C > 0$, depending only on $|I|$ and $\|a_0\|_{H^\gamma}$, but independent of ε, such that the solution satisfies the bounds

$$\|U\|_{L^\infty_t H^\gamma_x(I\times\mathbb{T}^d)} \le C, \tag{19}$$

$$\|U\|_{L^p_t H^{\gamma-\theta,p}_x(I\times\mathbb{T}^d)} \le C\varepsilon^{-1/p}. \tag{20}$$

In the special case $d = 1$, the above statement also holds with $p = 4$ and $\theta = 0$, that is,

$$\|U\|_{L_t^4 H_x^{\gamma,4}(I\times\mathbb{T})} \le C\varepsilon^{-1/4}.$$

Proof. Our proof strategy begins with a time rescaling. Specifically, we define the rescaled function,

$$w(t,x) = U(2\varepsilon^{-1}t, x),$$

which satisfies a nonlinear Schrödinger equation with a nonlinear term of order ε^{-1}, but posed on a shorter time interval $2^{-1}\varepsilon I$, instead of the original interval I:

$$\begin{cases} i\partial_t w + \Delta w = 2\varepsilon^{-1}\lambda |w|^2 w, \\ w(0,x) = a_0(x). \end{cases} \tag{21}$$

Next, we apply the fixed-point theorem to establish the existence and uniqueness of w.

We first consider the case $d = 1$. Fix $T_1, M > 0$, to be determined later, and define the function space

$$X = \{w \mid \|w\|_X \le M\}, \tag{22}$$

with the norm

$$\|w\|_X = \|w\|_{L_t^\infty H_x^\gamma \cap L_t^4 H_x^{\gamma,4}([-\varepsilon T_1, \varepsilon T_1]\times\mathbb{T})}. \tag{23}$$

Under this norm, $(X, \|\cdot\|_X)$ is a Banach space. We define the mapping P by

$$Pw(t) = e^{it\Delta}a_0 - i\frac{2\lambda}{\varepsilon}\int_0^t e^{i(t-s)\Delta}\left[|w(s)|^2 w(s)\right] ds.$$

We first show that P maps X into itself. By the Strichartz estimate (16) in Lemma 3.3 and the definition of P, we have for $t \in [-\varepsilon T_1, \varepsilon T_1]$:

$$\begin{aligned} \|Pw\|_X &\le \left\|e^{it\Delta}a_0\right\|_X + \frac{2|\lambda|}{\varepsilon}\int_0^{\varepsilon T_1}\left\|e^{i(t-s)\Delta}\left[|w|^2 w\right]\right\|_X ds \\ &\le C\|a_0\|_{H^\gamma} + \frac{2C}{\varepsilon}\cdot \varepsilon T_1 \left\||w|^2 w\right\|_{L_t^\infty H_x^\gamma([-\varepsilon T_1,\varepsilon T_1]\times\mathbb{T})}. \end{aligned}$$

Applying the Kato–Ponce inequality and using $\gamma > d/2$, we obtain that for any $w \in X$,

$$\|Pw\|_X \le C\|a_0\|_{H^\gamma} + 2CT_1\|w\|^3_{L_t^\infty H_x^\gamma([-\varepsilon T_1,\varepsilon T_1]\times\mathbb{T})} \le C\|a_0\|_{H^\gamma} + 2CT_1\cdot M^3.$$

By choosing $M = 2C\|a_0\|_{H^\gamma}$ in (22) and selecting T_1 sufficiently small such that $2CT_1M^2 \le 2^{-1}$, we ensure that $Pw \in X$.

Next, we show that P is a contraction. For any $w_1, w_2 \in X$,

$$\begin{aligned} \|Pw_1 - Pw_2\|_X &= \frac{2|\lambda|}{\varepsilon}\left\|\int_0^t e^{i(t-s)\Delta}\left[|w_1|^2 w_1 - |w_2|^2 w_2\right] ds\right\|_X \\ &\le 2CT_1\|w_1 - w_2\|_{H^\gamma}\left(\|w_1\|^2_{H^\gamma} + \|w_2\|^2_{H^\gamma}\right) \\ &\le 2CT_1M^2\|w_1 - w_2\|_{H^\gamma}. \end{aligned}$$

Choosing T_1 as above implies that P is a contraction on X. Hence, by the Banach fixed point theorem, there exists a unique fixed point $w^* \in X$ such that $w^* = Pw^*$, which establishes the local existence of a solution of (21) on $[-\varepsilon T_1, \varepsilon T_1]$. By iteratively applying this local existence result (with the number of iterations independent of ε), we obtain a solution defined on a maximal interval $(-2^{-1}\varepsilon T_*, 2^{-1}\varepsilon T^*)$, i.e.,

$$w \in C((-2^{-1}\varepsilon T_*, 2^{-1}\varepsilon T^*); H^\gamma(\mathbb{T})) \cap L^4((-2^{-1}\varepsilon T_*, 2^{-1}\varepsilon T^*); H^{\gamma,4}(\mathbb{T})). \tag{24}$$

Finally, according to the inverse change of variables:

$$U(t,x) = w(2^{-1}\varepsilon t, x), \tag{25}$$

we establish the existence and uniqueness of a solution U of (3) on the interval $(-T_*, T^*)$ for the special case $d=1$. Moreover, again by the transformation (25), we obtain that for any time interval I with $I \subsetneq (-T_*, T^*)$, there exists a constant $C = C(|I|, \|a_0\|_{H^\gamma})$ such that

$$\|U\|_{L_t^4 H_x^{\gamma,4}(I\times\mathbb{T})} \le \varepsilon^{-1/4}\|w\|_{L_t^4 H_x^{\gamma,4}(2^{-1}\varepsilon I\times\mathbb{T})} \le C\varepsilon^{-1/4}.$$

For the high-dimensional case $d>1$, we modify the norm in (23) as follows:

$$\|w\|_X = \|w\|_{L_t^\infty H_x^\gamma \cap L_t^p H_x^{\gamma-\theta,p}([-\varepsilon T_1, \varepsilon T_1]\times\mathbb{T}^d)}.$$

The same fixed-point argument applies, using the admissible Strichartz pair $(p, \gamma-\theta)$ (see the Strichartz estimate (17)). This leads to

$$w \in C((-2^{-1}\varepsilon T_*, 2^{-1}\varepsilon T^*); H^\gamma(\mathbb{T}^d)) \cap L^p((-2^{-1}\varepsilon T_*, 2^{-1}\varepsilon T^*); H^{\gamma-\theta,p}(\mathbb{T}^d)),$$

for some T_*, T^* independent of $\varepsilon \in (0,1]$.

Finally, according to the inverse change of variables (25): the equation (3) admits a unique solution U that satisfies

$$U \in C((-T_*, T^*); H^\gamma(\mathbb{T}^d)) \cap L^p((-T_*, T^*); H^{\gamma-\theta,p}(\mathbb{T}^d)),$$

and the following estimates hold:

$$\|U\|_{L_t^\infty H_x^\gamma(I\times\mathbb{T}^d)} = \|w\|_{L_t^\infty H_x^\gamma(2^{-1}\varepsilon I\times\mathbb{T}^d)} \le C,$$
$$\|U\|_{L_t^p H_x^{\gamma-\theta,p}(I\times\mathbb{T}^d)} = \varepsilon^{-1/p}\|w\|_{L_t^p H_x^{\gamma-\theta,p}(2^{-1}\varepsilon I\times\mathbb{T}^d)} \le C\varepsilon^{-1/p},$$

where the conditions on p and θ are the same as those in Proposition 3.5. □

4. Construction and local error analysis of the numerical scheme

In this section, we construct a fully discrete numerical scheme, building upon the semi-discrete scheme developed in [7,16]. We then proceed to derive the corresponding local error estimates. A key distinction from the previous semi-discrete scheme is that our primary focus lies in deriving error bounds in the L^2-norm, rather than the H^γ-norm. Consequently, the existing error analysis is not directly applicable, as it can not lead to sharp regularity conditions in the L^2-norm. To overcome this challenge, our analysis utilizes the boundedness of the solution in the space-time norm.

4.1. Construction of the scheme

We begin by defining the twisted variable $V(t) = e^{-it\frac{\varepsilon}{2}\Delta}U(t)$. By applying Duhamel's formula and performing a high- and low-frequency decomposition, we obtain the following expression for $V(t_{n+1})$:

$$V(t_{n+1}) = V(t_n) - i\lambda\int_{t_n}^{t_{n+1}} e^{-is\frac{\varepsilon}{2}\Delta}\Pi_N\left[\left|e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(s)\right|^2 e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(s)\right]ds + \mathcal{R}_1^n, \tag{26}$$

where $\mathcal{R}_1^n$ represents the error introduced by truncating high-frequency components in the approximation and is defined as

$$\mathcal{R}_1^n = -i\lambda\int_{t_n}^{t_{n+1}} e^{-is\frac{\varepsilon}{2}\Delta}\Big[|U|^2U - \Pi_N\big(|\Pi_N U|^2\Pi_N U\big)\Big](s)ds. \tag{27}$$

To construct a higher-order approximation of the integral in (28), we begin by approximating $\Pi_N V(s)$ for $s \in I_n$, using Duhamel's formula once more:

$$\Pi_N V(s) = \Pi_N V(t_n) - i\lambda \int_{t_n}^{s} e^{-i\sigma\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{i\sigma\frac{\varepsilon}{2}\Delta}V(\sigma)\right|^2 e^{i\sigma\frac{\varepsilon}{2}\Delta}V(\sigma)\right] d\sigma. \tag{28}$$

Denote the first term by $\mathcal{W}_1$ and approximate the integral by $\mathcal{W}_2$ by freezing the phase factor $e^{\pm i\sigma\frac{\varepsilon}{2}\Delta}$ at $\sigma = s$ and replacing $V(\sigma)$ with $\Pi_N V(t_n)$. The remainder is collected into $\mathcal{W}_3$. This gives the decomposition $\Pi_N V(s) = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3$ with

$$\mathcal{W}_1 = \Pi_N V(t_n), \tag{29}$$

$$\mathcal{W}_2 = -i\lambda(s-t_n)\, e^{-is\frac{\varepsilon}{2}\Delta}\,\Pi_N \left[e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right]^2 e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n), \tag{30}$$

$$\mathcal{W}_3 = -i\lambda \int_{t_n}^{s} e^{-i\sigma\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{i\sigma\frac{\varepsilon}{2}\Delta}V(\sigma)\right|^2 e^{i\sigma\frac{\varepsilon}{2}\Delta}V(\sigma)\right] d\sigma - \mathcal{W}_2. \tag{31}$$

By substituting the decomposition of $\Pi_N V(s)$ into the cubic nonlinear term in (26), we obtain the following expression:

$$V(t_{n+1}) = V(t_n) + I_1(t_n) + I_2(t_n) + \mathcal{R}_1^n + \mathcal{R}_2^n,$$

where $I_1(t_n)$ and $I_2(t_n)$ are given by

$$I_1(t_n) = -i\lambda \int_{t_n}^{t_{n+1}} e^{-is\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right|^2 e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right] ds; \tag{32}$$

$$I_2(t_n) = -\int_{t_n}^{t_{n+1}} (s-t_n) e^{-is\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right|^4 e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right] ds, \tag{33}$$

and the remainder term $\mathcal{R}_2^n$ is defined as

$$\mathcal{R}_2^n = -i\lambda \sum_{\substack{j,k,l\\ j+k+l\geq 5}} \int_{t_n}^{t_{n+1}} e^{-is\frac{\varepsilon}{2}\Delta}\Pi_N \left[e^{-is\frac{\varepsilon}{2}\Delta}\overline{\mathcal{W}_j}\cdot e^{is\frac{\varepsilon}{2}\Delta}\mathcal{W}_k \cdot e^{is\frac{\varepsilon}{2}\Delta}\mathcal{W}_l\right] ds. \tag{34}$$

By freezing the phase factor in I_2 at $s = t_n$, we obtain

$$I_2(t_n) = -\frac{\tau^2}{2} e^{-it_n\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{it_n\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right|^4 e^{it_n\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right] + \mathcal{R}_3^n, \tag{35}$$

where the first term will be retained in the numerical scheme. Next, we focus on the approximation of $I_1(t_n)$ by exploiting the resonance structure of the nonlinear term. To this end, we work in Fourier space. Let $\widehat{V}_k$ denote the k-th Fourier coefficient of V, and denote

$$\Lambda_k = \{(k_1,k_2,k_3) \mid k_1+k_2+k_3 = k, |k| \leq N, |k_i| \leq N,\ i = 1,2,3\},$$

then the Fourier transform of $I_1(t_n)$ can be written as

$$\widehat{I_1}(t_n,k) = -i\lambda \sum_{\Lambda_k} \left(\int_0^{\tau} e^{is\phi} ds\right) \cdot e^{it_n\phi}\widehat{\overline{V}}_{k_1}(t_n)\widehat{V}_{k_2}(t_n)\widehat{V}_{k_3}(t_n).$$

Substituting the approximation of the phase function in (6)–(7), we obtain

$$\widehat{I_1}(t_n,k) = -i\lambda \sum_{\Lambda_k} \int_0^{\tau} \left[e^{is\alpha} + i\beta e^{is\beta}\cdot \frac{1}{\tau}\int_0^{\tau} \sigma e^{i\sigma\alpha} d\sigma\right] ds \cdot e^{it_n\phi}\widehat{\overline{V}}_{k_1}(t_n)\widehat{V}_{k_2}(t_n)\widehat{V}_{k_3}(t_n) + \widehat{\mathcal{R}_4^n}(k),$$

where the remainder term is defined as

$$\widehat{\mathcal{R}_4^n}(k) = -i\lambda \sum_{\Lambda_k} \mathcal{T}(\alpha,\beta)\cdot e^{it_n\phi}\,\widehat{\overline{V}}_{k_1}(t_n)\widehat{V}_{k_2}(t_n)\widehat{V}_{k_3}(t_n), \tag{36}$$

$$\mathcal{T}(\alpha,\beta) = \int_0^\tau \Big[e^{is\phi} - e^{is\alpha} - i\beta e^{is\beta}\cdot\frac{1}{\tau}\int_0^\tau \sigma e^{i\sigma\alpha}d\sigma\Big]ds. \tag{37}$$

By transforming $\widehat{I_1}(t_n,k)$ back to physical space by using the following relation:

$$\int_0^\tau e^{is\alpha}ds = \tau\varphi(i\tau\alpha), \quad \frac{1}{\tau}\int_0^\tau se^{is\alpha}ds = -\tau\psi(i\tau\alpha),$$

and neglecting the residual terms $\mathcal{R}_i^n$, $i=1,2,3,4$, we arrive at the fully discretized numerical scheme

$$V^{n+1} = V^n + \Phi^n(V^n), \qquad V^0 = \Pi_N a_0, \tag{38}$$

where Φ^n is the numerical propagator defined by

$$\begin{aligned}\Phi^n(f) = &-i\lambda\tau\, e^{-it_n\frac{\varepsilon}{2}\Delta}\Pi_N\left\{\left[\varphi\left(-2i\tau\tfrac{\varepsilon}{2}\Delta\right)+\psi\left(-2i\tau\tfrac{\varepsilon}{2}\Delta\right)\right]e^{-it_n\frac{\varepsilon}{2}\Delta}\Pi_N\overline{f}\cdot\left(e^{it_n\frac{\varepsilon}{2}\Delta}\Pi_N f\right)^2\right\}\\ &+ i\lambda\tau e^{-it_{n+1}\frac{\varepsilon}{2}\Delta}\Pi_N\left\{\left[e^{-i(t_n-\tau)\frac{\varepsilon}{2}\Delta}\psi\left(-2i\tau\tfrac{\varepsilon}{2}\Delta\right)\Pi_N\overline{f}\right]\cdot\left(e^{it_{n+1}\frac{\varepsilon}{2}\Delta}\Pi_N f\right)^2\right\}\\ &-\frac{\tau^2}{2}e^{-it_n\frac{\varepsilon}{2}\Delta}\Pi_N\left[\left|e^{it_n\frac{\varepsilon}{2}\Delta}\Pi_N f\right|^4 e^{it_n\frac{\varepsilon}{2}\Delta}\Pi_N f\right].\end{aligned} \tag{39}$$

By twisting back, one obtains the fully discrete scheme of $U^n := e^{it_n\frac{\varepsilon}{2}\Delta}V^n$ in (13). Furthermore, we derive that

$$V(t_{n+1}) = V(t_n) + \Phi^n(V(t_n)) + \sum_{i=1}^4 \mathcal{R}_i^n, \quad 0\le n\le L_0. \tag{40}$$

4.2. Analysis of local errors

This section is devoted to deriving bounds for local error terms $\mathcal{R}_i^n$, $i=1,2,3,4$. We begin with an estimate for the first term $\mathcal{R}_1^n$. For simplicity, we denote the interval $[t_n,t_{n+1}]$ by I_n and the interval $[t_{n-1},t_{n+2}]$ by J_n in the following.

Lemma 4.1. *For $a_0\in H^2(\mathbb{T}^d)$ with $1\le d\le 3$, there exists a constant $C>0$ depending only on $\|a_0\|_{H^2}$ and T such that for all $N\ge 1$:*

$$\|\mathcal{R}_1^n\|_{L^2(\mathbb{T}^d)} \le C\tau N^{-2}.$$

Moreover, for any $\frac{d}{2}<r<2$, the following estimate holds:

$$\|\mathcal{R}_1^n\|_{H^r(\mathbb{T}^d)} \le C\tau N^{-2+r}.$$

Proof. By the Sobolev embedding $H^2(\mathbb{T}^d)\hookrightarrow L^\infty(\mathbb{T}^d)$ and Bernstein's inequality, we have

$$\begin{aligned}\|\mathcal{R}_1^n\|_{L^2(\mathbb{T}^d)} &\le \left\|\int_{t_n}^{t_{n+1}} e^{-is\frac{\varepsilon}{2}\Delta}\Big[|U|^2U - \Pi_N(|\Pi_N U|^2\Pi_N U)\Big](s)ds\right\|_{L^2(\mathbb{T}^d)}\\ &\le \tau\sup_{t\in I_n}\||U|^2U - \Pi_N(|\Pi_N U|^2\Pi_N U)\|_{L^2(\mathbb{T}^d)}\\ &\le C\tau\big(\|\Pi_{>N}(|U|^2U)\|_{L_t^\infty L_x^2} + \||U|^2U - |\Pi_N U|^2\Pi_N U\|_{L_t^\infty L_x^2}\big)\\ &\le C\tau N^{-2}\|U\|^3_{L_t^\infty H_x^2}.\end{aligned}$$

For the H^r estimate with $\frac{d}{2} < r < 2$, we proceed in a similar manner. By applying the Kato–Ponce inequality together with Bernstein's inequality, we obtain:

$$\begin{aligned}\|\mathcal{R}_1^n\|_{H^r(\mathbb{T}^d)} &\le \tau \sup_{s\in I_n} \||U|^2U - \Pi_N(|\Pi_N U|^2\Pi_N U)\|_{H^r(\mathbb{T}^d)}\\ &\le C\tau\big(\|\Pi_{>N}(|U|^2U)\|_{L_t^\infty H_x^r} + \||U|^2U - |\Pi_N U|^2\Pi_N U\|_{L_t^\infty H_x^r}\big)\\ &\le C\tau N^{-2+r}\|U\|^3_{L_t^\infty H_x^2}.\end{aligned}$$

Finally, the desired result follows from the uniform bound $\|U\|_{L_t^\infty H_x^2} \le C\|a_0\|_{H^2}$. □

Lemma 4.2. *Let $a_0 \in H^2(\mathbb{T}^d)$ with $1 \le d \le 3$. Then there exists a constant $C > 0$, depending only on $\|a_0\|_{H^2}$ and T, such that the following estimates:*

$$\begin{aligned}\|\mathcal{R}_2^n\|_{H^2(\mathbb{T}^d)} &\le C\tau^2,\\ \|\mathcal{R}_2^n\|_{L^2(\mathbb{T}^d)} &\le C\tau^3 + C\tau^2 N^{-2}.\end{aligned}$$

Proof. According to the definition of $\mathcal{R}_2^n$ in (34), we apply the Kato-Ponce inequality to obtain

$$\|\mathcal{R}_2^n\|_{H^2(\mathbb{T}^d)} \lesssim \tau \sum_{i+j+k\ge 5} \sup_{s\in I_n} \|\mathcal{W}_i\|_{H^2}\|\mathcal{W}_j\|_{H^2}\|\mathcal{W}_k\|_{H^2}. \tag{41}$$

The estimates for W_i for $i = 1, 2, 3$ follow directly from their expressions in (29)–(31). Specifically, there exists a constant C, depending only on T and $\|V\|_{L^\infty(0,T;H^2)}$, such that

$$\sup_{s\in I_n} \|\mathcal{W}_1\|_{H^2} \le C, \qquad \sup_{s\in I_n} \|\mathcal{W}_i\|_{H^2} \le C\tau, \quad i = 2, 3. \tag{42}$$

Since for positive integers i, j, k satisfying $i + j + k \ge 5$, at least one of the indices must be 2 or 3; for instance, the case $(i, j, k) = (1, 1, 3)$ or its permutations. Combined with (42), this implies that the worst contribution in (41) is of order τ^2. Substituting these bounds into (41) yields the first inequality.

For the second inequality, we apply Hölder's inequality to (34), assigning the L^2-norm to the factor with the largest index, while the remaining terms are controlled via the Sobolev embedding $H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ with $1 \le d \le 3$. This yields

$$\begin{aligned}\|\mathcal{R}_2^n\|_{L^2} &\lesssim \tau \sum_{\substack{i+j+k\ge 5\\ i\le j\le k}} \sup_{s\in I_n} \|\mathcal{W}_i\|_{H^2}\|\mathcal{W}_j\|_{H^2}\|\mathcal{W}_k\|_{L^2}\\ &\le C\tau^3 + C\tau \sup_{s\in I_n} \|\mathcal{W}_3\|_{L^2},\end{aligned} \tag{43}$$

where the last inequality is deduced from (42). Thus, it remains to estimate $\sup_{s\in I_n} \|\mathcal{W}_3\|_{L^2}$. By decomposing the integral expression of $\mathcal{W}_3$ in (31) into low- and high-frequency contributions and invoking the estimates from Lemma 4.1 for the high-frequency part, we obtain

$$\begin{aligned}\mathcal{W}_3 = &-i\lambda \int_{t_n}^s e^{-i\sigma\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{i\sigma\frac{\varepsilon}{2}\Delta}\Pi_N V(\sigma)\right|^2 e^{i\sigma\frac{\varepsilon}{2}\Delta}\Pi_N V(\sigma)\right] d\sigma\\ &+ i\lambda \int_{t_n}^s e^{-is\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right|^2 e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right] d\sigma + \mathcal{O}(\tau N^{-2}).\end{aligned} \tag{44}$$

Note that the propagator $e^{i\sigma\frac{\varepsilon}{2}\Delta}$ is an isometry in L^2, and the difference between the propagators satisfies the following estimate:

$$\left\|\big(e^{i\sigma\frac{\varepsilon}{2}\Delta} - e^{is\frac{\varepsilon}{2}\Delta}\big)f\right\|_{L^2} \le C\varepsilon\tau\|f\|_{H^2}. \tag{45}$$

Furthermore, applying Duhamel's formula (28) gives:

$$\|V(\sigma) - V(t_n)\|_{L^2} \le C\tau \|V\|^3_{L^\infty_t H^2_x}.$$

Then we decompose the difference in the integrand of (44) by inserting suitable intermediate terms. Applying the above properties to these decomposed terms, this leads to the following estimate:

$$\sup_{s\in I_n} \|\mathcal{W}_3\|_{L^2} \le C\tau^2 + C\tau N^{-2}.$$

Substituting this estimate into (43) completes the proof of the second inequality. □

Lemma 4.3. *Let $a_0 \in H^2(\mathbb{T}^d)$ with $1 \le d \le 3$. Then there exists a constant $C > 0$, depending only on $\|a_0\|_{H^2}$ and T, such that the following estimates hold:*

$$\|\mathcal{R}_3^n\|_{H^2(\mathbb{T}^d)} \le C\tau^2,$$
$$\|\mathcal{R}_3^n\|_{L^2(\mathbb{T}^d)} \le C\varepsilon\tau^3.$$

Proof. From (33) and (35), we have

$$\begin{aligned}\mathcal{R}_3^n = &-\int_{t_n}^{t_{n+1}} (s-t_n) e^{-is\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right|^4 e^{is\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right] ds \\ &+ \int_{t_n}^{t_{n+1}} (s-t_n) e^{-it_n\frac{\varepsilon}{2}\Delta}\Pi_N \left[\left|e^{it_n\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right|^4 e^{it_n\frac{\varepsilon}{2}\Delta}\Pi_N V(t_n)\right] ds.\end{aligned}$$

Estimating the H^2 norm of each term in $\mathcal{R}_3^n$ by applying the Kato-Ponce inequality (3.2), we obtain

$$\|\mathcal{R}_3^n\|_{H^2(\mathbb{T}^d)} \le C\tau^2 \|V\|^5_{L^\infty_t H^2_x}.$$

Next, using arguments similar to those in Lemma 4.2, we insert appropriate intermediate terms and apply inequality (45). This results in

$$\|\mathcal{R}_3^n\|_{L^2(\mathbb{T}^d)} \le C\varepsilon\tau^3.$$

This completes the proof of the lemma. □

Lemma 4.4. *Let $a_0 \in H^2(\mathbb{T}^d)$ with $1 \le d \le 3$. Then for any $\epsilon > 0$, the following estimates hold:*

$$\left\| e^{it\frac{\varepsilon}{2}\Delta} J^2 V(t_n)\right\|_{l^4_n L^4_{t,x}(J_n\times\mathbb{T})} \le C\varepsilon^{-\frac{1}{4}}, \quad d = 1, \tag{46}$$

$$\left\| e^{it\frac{\varepsilon}{2}\Delta} J^{2-\epsilon} V(t_n)\right\|_{l^q_n L^q_{t,x}(J_n\times\mathbb{T}^d)} \le C\varepsilon^{-\frac{1}{q}}, \quad d = 2,3, \tag{47}$$

where $q = \frac{2(d+2)}{d}$, and the constant C is independent of ε.

Proof. From the Duhamel's formula (28), we obtain

$$V(t_n) = V(t) + i\lambda \int_{t_n}^{t} e^{-is\frac{\varepsilon}{2}\Delta}\left[\left|e^{is\frac{\varepsilon}{2}\Delta}V(s)\right|^2 e^{is\frac{\varepsilon}{2}\Delta}V(s)\right] ds. \tag{48}$$

By applying $e^{it\frac{\varepsilon}{2}\Delta}\Delta$ to the above equality and using $V(t) = e^{-it\frac{\varepsilon}{2}\Delta}U(t)$, we have

$$e^{it\frac{\varepsilon}{2}\Delta}J^2 V(t_n) = J^2 U(t) + i\lambda \int_{t_n}^{t} e^{i(t-s)\frac{\varepsilon}{2}\Delta} J^2 \left[|U(s)|^2 U(s)\right] ds. \tag{49}$$

Then we use this equality to derive the estimates in (46)–(47). We first consider the case $d = 1$. By Proposition 3.5, and note that $\cup_{n=0}^{L_0-1} J_n = [-\tau, T + 2\tau]$, we have the bound

$$\|J^2 U\|_{l_n^4 L_{t,x}^4(J_n\times\mathbb{T})} \le C\|J^2U\|_{L_{t,x}^4([-\tau,T+2\tau]\times\mathbb{T})} \le C\varepsilon^{-\frac{1}{4}}.$$

For the integral term in (49), we apply Minkowski's inequality and Lemma 3.4 with $p = 4$ to obtain

$$\begin{aligned}
&\left\|\int_{t_n}^t e^{i(t-s)\frac{\varepsilon}{2}\Delta} J^2\left[|U(s)|^2U(s)\right] ds\right\|_{l_n^4 L_{t,x}^4(J_n\times\mathbb{T})}\\
&\le \left\|\int_{t_n}^{t_{n+1}} \left\|e^{i(t-s)\frac{\varepsilon}{2}\Delta} J^2\left[|U(s)|^2U(s)\right]\right\|_{L_{t,x}^4(J_n\times\mathbb{T})} ds\right\|_{l_n^4}\\
&\lesssim \varepsilon^{-\frac{1}{4}}\left\|J^2\left[|U|^2U\right]\right\|_{L_t^\infty L_x^2} \lesssim \varepsilon^{-\frac{1}{4}}\|U\|^3_{L_t^\infty H_x^2}.
\end{aligned}$$

Hence, for $d = 1$, we conclude

$$\left\|e^{it\frac{\varepsilon}{2}\Delta} J^2 \Pi_N V(t_n)\right\|_{l_n^4 L_{t,x}^4(J_n\times\mathbb{T})} \le C\varepsilon^{-\frac{1}{4}}.$$

For $d = 2, 3$, we apply the same strategy, using $p = \frac{2(d+2)}{d}$ in Lemma 3.4, which yields the desired result with $q = \frac{2(d+2)}{d}$; that is, $q = 4$ for $d = 2$ and $q = \frac{10}{3}$ for $d = 3$. This completes the proof of the lemma. □

Lemma 4.5. *Let $a_0 \in H^2(\mathbb{T}^d)$ with $1 \le d \le 3$. Then there exists a constant $C > 0$, depending only on $\|a_0\|_{H^2}$ and T, such that the following estimates hold.*

(1) *For $\max\{1, \frac{d}{2}\} < r < 2$, we have*

$$\|\mathcal{R}_4^n\|_{H^r(\mathbb{T}^d)} \le C\varepsilon^{2-r}\tau^{3-r}.$$

(2) *For the L^2 estimate, we have for arbitrary sufficiently small δ:*

$$\Big\|\sum_{n=0}^{L_0-1}\mathcal{R}_4^n\Big\|_{L^2(\mathbb{T}^d)} \le C\varepsilon\begin{cases}\tau^2, & d=1,\\ \tau^{2-\delta}, & d=2,3.\end{cases}$$

Proof. (1) Without loss of generality, we assume $|k_1| \ge |k_2| \ge |k_3|$. Then for $k = k_1 + k_2 + k_2$, this implies $|k| \lesssim |k_1|$. From the definition of β in (7) and the above assumption leads to $|\beta| \lesssim \varepsilon^2|k_1||k_2|$. Recalling the definition of $\mathcal{T}(\alpha,\beta)$ in (37), we have the following reformulation:

$$\mathcal{T}(\alpha,\beta) = \int_0^\tau e^{is\alpha}(e^{is\beta}-1)\,ds - (e^{i\tau\beta}-1)\cdot\frac{1}{\tau}\int_0^\tau s e^{is\alpha}\,ds. \tag{50}$$

Notice that $2 - r \in (0, 1)$, we have

$$|\mathcal{T}(\alpha,\beta)| \le \tau\sup_{s\in[0,\tau]}|e^{is\beta}-1| \le \tau^{3-r}|\beta|^{2-r} \lesssim \varepsilon^{2-r}\tau^{3-r}|k_1|^{2-r}|k_2|^{2-r}. \tag{51}$$

Substituting (51) into (36) and applying Plancherel's theorem yields, for $\max\{1, \frac{d}{2}\} < r < 2$,

$$\begin{aligned}
\|\mathcal{R}_4^n\|_{H^r(\mathbb{T}^d)} &= \|(1+|k|^2)^{r/2}\widehat{\mathcal{R}_4^n}(k)\|_{l_k^2}\\
&\le C\varepsilon^{2-r}\tau^{3-r}\left\|\sum_{\Lambda_k}(1+|k_1|^2)|k_2|^{2-r}|\widehat{\overline{V}}_{k_1}(t_n)||\widehat{V}_{k_2}(t_n)||\widehat{V}_{k_3}(t_n)|\right\|_{l_k^2}\\
&\le C\varepsilon^{2-r}\tau^{3-r}\left\|J^2V\right\|_{L_t^\infty L_x^2}\left\|J^{2-r}\mathcal{F}^{-1}(|\widehat{V}_k|)\right\|_{L_t^\infty L_x^\infty}\left\|\mathcal{F}^{-1}(|\widehat{V}_k|)\right\|_{L_t^\infty L_x^\infty}.
\end{aligned}$$

Since $r > d/2$, the result in (i) can be obtained by embedding $H^r \hookrightarrow L^\infty$.

(2) We introduce $T_1 = \Pi_{\tau^{-1}}$ and $T_2 = \Pi_{>\tau^{-1}}$ and decompose $\mathcal{R}_4^n$ in (36) into

$$\widehat{\mathcal{R}_4^n}(k) = -i\lambda \sum_{\Lambda_k} \sum_{i+j+l\geq 4} \mathcal{T}(\alpha,\beta)\cdot e^{it_n\phi}\,\widehat{\overline{T_i V}}_{k_1}(t_n)\widehat{T_j V}_{k_2}(t_n)\widehat{T_l V}_{k_3}(t_n)$$
$$- i\lambda \sum_{\Lambda_k} \mathcal{T}(\alpha,\beta)\cdot e^{it_n\phi}\,\widehat{\overline{T_1 V}}_{k_1}(t_n)\widehat{T_1 V}_{k_2}(t_n)\widehat{T_1 V}_{k_3}(t_n) = \widehat{\mathcal{R}_{41}^n}(k) + \widehat{\mathcal{R}_{42}^n}(k).$$

For estimate of $\mathcal{R}_{41}^n$, we note that for $i+j+l \geq 4$, at least one of T_i, T_j or T_k is T_2. Moreover, according to (50), we obtain $|\mathcal{T}(\alpha,\beta)| \lesssim \tau$. Consequently, in this case, by embedding $H^2 \hookrightarrow L^\infty$, we obtain

$$\|\mathcal{R}_{41}^n\|_{L^2} \lesssim \tau \|T_2 V(t_n)\|_{L^2} \|V(t_n)\|_{H^2}^2 \lesssim \tau(\tau^{-1})^{-2} = \tau^3.$$

For estimate of $\mathcal{R}_{42}^n$, we recast $\mathcal{T}(\alpha,\beta)$ into a sum of two triple integrals, namely

$$\mathcal{T}(\alpha,\beta) = \int_0^\tau e^{is\alpha}(e^{is\beta} - 1 - is\beta)\,ds + i\beta \int_0^\tau (1 - e^{is\beta})\,ds \cdot \frac{1}{\tau}\int_0^\tau s e^{is\alpha}\,ds$$
$$= \int_0^\tau \int_0^s \int_0^\sigma e^{is\alpha} e^{i\theta\beta}(-\beta^2)\,d\theta\,d\sigma\,ds + \frac{1}{\tau}\int_0^\tau \int_0^\tau \int_0^\sigma s e^{is\alpha} e^{i\theta\beta}\beta^2\,d\theta\,d\sigma\,ds. \tag{52}$$

Since β^2 is a multivariate polynomial in k_1, k_2, and k_3, with degree at most two in each variable and total degree at most four, substituting (52) into $\mathcal{R}_{42}^n$ yields many terms. However, their treatment is similar, so we focus on a representative term for estimation; the estimates for the remaining terms are analogous and are omitted for brevity. Specifically, we consider the contribution associated with the $(k_1 \cdot k_2)^2$ term in β^2, namely,

$$\mathcal{T}_2(\alpha,\beta) = -\varepsilon^2 \int_0^\tau \int_0^s \int_0^\sigma e^{is\alpha} e^{i\theta\beta}(k_1\cdot k_2)^2\,d\theta\,d\sigma\,ds.$$

By abbreviating $\mathcal{V}_N^n = T_1 \Pi_N V(t_n)$, the term corresponding to $\mathcal{T}_2(\alpha,\beta)$ is denoted as $\mathcal{A}^n$, obtained by replacing $\mathcal{T}$ in $\mathcal{R}_{42}^n$ with $\mathcal{T}_2$. By changing of variable, we express $\mathcal{A}^n$ in the physical space as:

$$\mathcal{A}^n = \varepsilon^2 \sum_{i,j=1}^d \int_0^\tau \int_0^s \int_0^\sigma e^{-i(t_n+\theta)\frac{\varepsilon}{2}\Delta}\Big[\overline{e^{i(t_n+2s-\theta)\frac{\varepsilon}{2}\Delta}\partial_{ij}\mathcal{V}_N^n} \cdot e^{i(t_n+\theta)\frac{\varepsilon}{2}\Delta}\partial_{ij}\mathcal{V}_N^n \cdot e^{i(t_n+\theta)\frac{\varepsilon}{2}\Delta}\mathcal{V}_N^n\Big]\,d\theta\,d\sigma\,ds$$
$$= \varepsilon^2 \sum_{i,j=1}^d \int_{I_n} \int_{t_n}^s \int_{t_n}^\sigma e^{-i\theta\frac{\varepsilon}{2}\Delta}\Big[\overline{e^{i(2s-\theta)\frac{\varepsilon}{2}\Delta}\partial_{ij}\mathcal{V}_N^n} \cdot e^{i\theta\frac{\varepsilon}{2}\Delta}\partial_{ij}\mathcal{V}_N^n \cdot e^{i\theta\frac{\varepsilon}{2}\Delta}\mathcal{V}_N^n\Big]\,d\theta\,d\sigma\,ds, \tag{53}$$

where ∂_{ij} denotes the second-order mixed derivative $\partial_{x_i}\partial_{x_j}$.

For $d = 1, 2$, to estimate its L_x^2 norm, we first apply the Minkowski inequality to exchange the order of the spatial norm and the integral, and then bound the outer s-integral by taking the L_s^∞ norm. Applying Hölder's inequality then yields

$$\|\mathcal{A}^n\|_{L_x^2} \lesssim \varepsilon^2\tau^2 \sup_{s\in I_n} \int_{t_n}^{t_{n+1}} \big\|J^2 e^{i(2s-\theta)\Delta}\mathcal{V}_N^n\big\|_{L_x^4} \big\|J^2 e^{i\theta\Delta}\mathcal{V}_N^n\big\|_{L_x^4} \big\|e^{i\theta\frac{\varepsilon}{2}\Delta}\mathcal{V}_N^n\big\|_{L_x^\infty}\,d\theta$$
$$\lesssim \varepsilon^2\tau^{\frac{5}{2}} \sup_{s\in I_n} \Big\|J^2 e^{i(2s-\theta)\frac{\varepsilon}{2}\Delta}\mathcal{V}_N^n\Big\|_{L^4_{\theta,x}(I_n)} \Big\|J^2 e^{i\theta\frac{\varepsilon}{2}\Delta}\mathcal{V}_N^n\Big\|_{L^4_{\theta,x}(I_n)} \Big\|e^{i\theta\frac{\varepsilon}{2}\Delta}\mathcal{V}_N^n\Big\|_{L^\infty_{\theta,x}(I_n)}$$
$$\lesssim \varepsilon^2\tau^{\frac{5}{2}} \Big\|e^{it\frac{\varepsilon}{2}\Delta}J^2\mathcal{V}_N^n\Big\|^2_{L_t^4 L_x^4(J_n\times\mathbb{T}^d)} \|V(t)\|_{L_t^\infty H_x^2},$$

where in the last step we have used that the time variables $2s-\theta$ and θ both vary over J_n and I_n respectively, $I_n \subseteq J_n$.

Summing over n, for $d=1$, by applying a discrete Hölder inequality together with Lemma 4.4, we obtain

$$\sum_{n=0}^{L_0-1} \|\mathcal{A}^n\|_{L^2(\mathbb{T}^d)} \le C\varepsilon^2\tau^2 \left\| e^{it\frac{\varepsilon}{2}\Delta} J^2 \mathcal{V}_N^n \right\|^2_{l_n^4 L_{t,x}^4(J_n\times\mathbb{T}^d)} \le C\varepsilon^{\frac{3}{2}}\tau^2. \tag{54}$$

For $d=2$, we apply Lemma 3.1 and Lemma 4.4, with the choice $\delta = 2\epsilon$, to obtain

$$\sum_{n=0}^{L_0-1} \|\mathcal{A}^n\|_{L^2(\mathbb{T}^d)} \le C\varepsilon^2\tau^{2-2\epsilon} \left\| e^{it\frac{\varepsilon}{2}\Delta} J^{2-\epsilon} \mathcal{V}_N^n \right\|^2_{l_n^4 L_{t,x}^4(J_n\times\mathbb{T}^d)} \le C\varepsilon^{\frac{3}{2}}\tau^{2-\delta}. \tag{55}$$

For $d=3$, we need to reformulate the summation of $\mathcal{A}^n$ before taking the L^2 norm. To be specific, we apply Fubini's theorem to (53) to obtain

$$\sum_{n=0}^{L_0-1} \mathcal{A}^n = \varepsilon^2 \sum_{i,j=1}^{d} \sum_{n=0}^{L_0-1} \int_{I_n} e^{-i\theta\frac{\varepsilon}{2}\Delta} \int_\theta^{t_{n+1}}\int_\theta^s \left[\overline{e^{i(2s-\theta)\frac{\varepsilon}{2}\Delta}\partial_{ij}\mathcal{V}_N^n} \cdot e^{i\theta\frac{\varepsilon}{2}\Delta}\partial_{ij}\mathcal{V}_N^n \cdot e^{i\theta\frac{\varepsilon}{2}\Delta}\mathcal{V}_N^n\right] d\sigma\, ds\, d\theta.$$

Thus, applying the scaled Strichartz estimate in Lemma 3.4 with $p' = \frac{10}{7}$, and using the embedding $L_{t,x}^{\frac{5}{3}} \hookrightarrow L_{t,x}^{\frac{10}{7}}$, we then proceed with the same argument as above to obtain

$$\begin{aligned}
\Big\| \sum_{n=0}^{L_0-1} \mathcal{A}^n \Big\|_{L^2} &\lesssim \varepsilon^{2-\frac{3}{10}} \left\| \sum_{i,j=1}^{d} \int_\theta^{t_{n+1}}\int_\theta^s \left[\overline{e^{i(2s-\theta)\frac{\varepsilon}{2}\Delta}\partial_{ij}\mathcal{V}_N^n} \cdot e^{i\theta\frac{\varepsilon}{2}\Delta}\partial_{ij}\mathcal{V}_N^n \cdot e^{i\theta\frac{\varepsilon}{2}\Delta}\mathcal{V}_N^n\right] d\sigma ds \right\|_{l_n^{\frac{5}{3}} L_{\theta,x}^{\frac{5}{3}}(I_n\times\mathbb{T}^d)} \\
&\lesssim \varepsilon^{\frac{17}{10}}\tau^2 \sup_{s\in I_n} \left\| e^{i(2s-\theta)\frac{\varepsilon}{2}\Delta} J^2\mathcal{V}_N^n \right\|_{L_n^{\frac{10}{3}} L_{\theta,x}^{\frac{10}{3}}(I_n)} \left\| e^{i\theta\frac{\varepsilon}{2}\Delta} J^2\mathcal{V}_N^n \right\|_{L_n^{\frac{10}{3}} L_{\theta,x}^{\frac{10}{3}}(I_n)} \left\| e^{i\theta\frac{\varepsilon}{2}\Delta} \mathcal{V}_N^n \right\|_{L_n^\infty L_{\theta,x}^\infty(I_n)} \\
&\lesssim \varepsilon^{\frac{17}{10}}\tau^{2-2\epsilon} \left\| e^{it\frac{\varepsilon}{2}\Delta} J^{2-\epsilon}\mathcal{V}_N^n \right\|^2_{l_n^{\frac{10}{3}} L_{t,x}^{\frac{10}{3}}(J_n\times\mathbb{T}^d)} \|V(t)\|_{L_t^\infty H_x^2} \lesssim \varepsilon\tau^{2-\delta},
\end{aligned}$$

where we used Lemma 4.4 in the last inequality. Combining this estimate with the bound in (54), we obtain the proof of (2).

□

5. Proof of the convergence results

We begin this section by establishing a uniform bound for the numerical solution in the Sobolev space H^r, $r > \frac{d}{2}$. Then, we present the proof of Theorem 2.2.

Proposition 5.1. *Let V^n be defined in (38). Suppose that the initial data $a_0 \in H^2(\mathbb{T}^d)$, $\max\{1, \frac{d}{2}\} < r < 2$. Then there exists a constant $C > 0$, depending only on the final time T and the initial norm $\|a_0\|_{H^2}$, such that for sufficiently large N and sufficiently small τ, the following bound holds:*

$$\|V^n\|_{H^r} \le C, \quad \text{for all } 0 \le n \le L_0.$$

Proof. We introduce the error function $e_n := V(t_n) - V^n$, which measures the deviation between the exact and numerical solutions at time t_n. By construction (40), the error at step $n+1$ can be decomposed as

$$\begin{aligned}
e_{n+1} &= e_n + V(t_{n+1}) - \Phi^n(V(t_n)) + \Phi^n(V(t_n)) - \Phi^n(V^n) \\
&= e_n + \sum_{i=1}^{4} \mathcal{R}_i^n + \Phi^n(V(t_n)) - \Phi^n(V^n).
\end{aligned} \tag{56}$$

Applying Lemmas 4.1–4.3 and 4.5, and using the condition $\frac{d}{2} < r < 2$, we obtain the following uniform bound on the sum of the remainder terms:

$$\sum_{i=1}^{4} \|\mathcal{R}_i^n\|_{H^r(\mathbb{T}^d)} \le C\left(\tau N^{-2+r} + \tau^2 + \varepsilon^{2-r}\tau^{3-r}\right),$$

where the constant $C > 0$ depends only on the final time T and the norm $\|a_0\|_{H^2}$.

Next, we analyze the stability properties of the numerical flow Φ^n in (39). By the boundedness of the operators φ and ψ on $H^r(\mathbb{T}^d)$, we have

$$\|\varphi\left(-2i\tau\tfrac{\varepsilon}{2}\Delta\right) f\|_{H^r} \le C\|f\|_{H^r}, \quad \|\psi\left(-2i\tau\tfrac{\varepsilon}{2}\Delta\right) f\|_{H^r} \le C\|f\|_{H^r},$$

uniformly in τ. Consequently, the numerical flow Φ^n satisfies a Lipschitz-type stability estimate:

$$\|\Phi^n(V(t_n)) - \Phi^n(V^n)\|_{H^r} \le C\tau(\|e_n\|_{H^r} + \|e_n\|_{H^r}^5).$$

Substituting these estimates into (56) yields the following recursive inequality:

$$\|e_{n+1}\|_{H^r} \le (1 + C\tau)\|e_n\|_{H^r} + C\tau\|e_n\|_{H^r}^5 + C\left(\tau N^{-2+r} + \tau^2 + \varepsilon^{2-r}\tau^{3-r}\right).$$

We now proceed by mathematical induction to establish a uniform bound on the error. At the initial step $n = 0$, V^0 is obtained via a spectral projection of a_0, hence

$$\|e_0\|_{H^r} \le CN^{-2+r}, \quad \text{and} \quad \|e_0\|_{H^r} \le 1$$

by taking N sufficiently large. Assuming that at step n, the inductive hypothesis $\|e_n\|_{H^r} \le 1$ holds, we aim to show that the same bound holds at step $n + 1$. Under the assumption $\|e_n\|_{H^r} \le 1$, it follows immediately that $\|e_n\|_{H^r}^5 \le \|e_n\|_{H^r}$. To bound the accumulation of errors over successive steps, we apply a discrete Gronwall inequality to the recursive estimate. Noting that $\tau L_0 \le T$ remains uniformly bounded, this yields

$$\begin{aligned}\|e_{n+1}\|_{H^r} &\le (1 + 2C\tau)\|e_n\|_{H^r} + C\left(\tau N^{-2+r} + \tau^2 + \varepsilon^{2-r}\tau^{3-r}\right) \\ &\le Ce^{2CT}\left(N^{-2+r} + \tau + \tau^{2-r}\right).\end{aligned}$$

Therefore, by choosing N sufficiently large and τ sufficiently small, the inductive step is closed, and the bound $\|e_n\|_{H^r} \le 1$ holds uniformly.

Finally, recalling that the exact solution $V(t)$ remains uniformly bounded in $H^r(\mathbb{T}^d)$ over the interval $[0, T]$ $(r < 2)$, we conclude that

$$\|V^n\|_{H^r} \le \|V(t_n)\|_{H^r} + \|e_n\|_{H^r} \le C,$$

for all $0 \le n \le L_0$, where C depends only on the given data. This completes the proof. □

Proof of Theorem 2.2. We restrict our attention to the case $d = 1$, as the cases $d = 2$ and $d = 3$ can be handled similarly. We begin by expressing the global error using the telescoping identity (56):

$$e_{n+1} = e_0 + \sum_{i=1}^{4}\sum_{j=0}^{n} \mathcal{R}_i^j + \sum_{j=0}^{n}\Big(\Phi^j\big(V(t_j)\big) - \Phi^j\big(V^j\big)\Big).$$

By the projection estimate $\|e_0\|_{L^2} = \|\Pi_{>N} V(0)\|_{L^2} \le N^{-2}$ and the bounds in Lemmas 4.1–4.3 and 4.5 on the remainder terms, we obtain

$$\begin{aligned}\|e_{n+1}\|_{L^2} &\le \|e_0\|_{L^2} + \sum_{i=1}^{4} \Big\| \sum_{j=0}^{n} \mathcal{R}_i^j \Big\|_{L^2} + \sum_{j=0}^{n} \big\| \Phi^j\big(V(t_j)\big) - \Phi^j\big(V^j\big)\big\|_{L^2} \\ &\le C(N^{-2} + \tau^2) + \sum_{j=0}^{n} \big\| \Phi^j\big(V(t_j)\big) - \Phi^j\big(V^j\big)\big\|_{L^2}.\end{aligned}$$

It remains to estimate the stability term. Using the boundedness of the nonlinear maps φ and ψ in L^2, Proposition 5.1, and the Sobolev embedding $H^r(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ for $r > \frac{d}{2}$, we deduce

$$\begin{aligned}\big\| \Phi^j\big(V(t_j)\big) - \Phi^j\big(V^j\big)\big\|_{L^2} &\le C\tau \|e_j\|_{L^2} (\|V(t_j)\|_{H^r}^2 + \|V^j\|_{H^r}^2) \\ &\quad + C\tau \|e_j\|_{L^2} (\|V(t_j)\|_{H^r}^4 + \|V^j\|_{H^r}^4) \\ &\le C\tau \|e_j\|_{L^2}.\end{aligned}$$

Substituting this estimate into the global error bound, we obtain

$$\|e_{n+1}\|_{L^2} \le C(N^{-2} + \tau^2) + C\tau \sum_{j=0}^{n} \|e_j\|_{L^2}.$$

By the discrete Gronwall inequality, this yields

$$\|e_{n+1}\|_{L^2} \le C(N^{-2} + \tau^2),$$

which completes the proof. □

6. Numerical examples

In this section, we present numerical experiments to demonstrate the performance of the proposed numerical scheme for the weakly nonlinear Schrödinger equation in the semiclassical regime. The numerical experiments are carried out in both one and two spatial dimensions. The results validate the accuracy and stability of the method under various parameter settings.

We fix λ in (1) as 1 throughout and consider initial data of the form

$$u(0, x) = a_0(x) e^{i\kappa \cdot x/\varepsilon},$$

where $\kappa = 1$ in one dimension and $\kappa = (1, 2) \in \mathbb{R}^2$ in two dimension. The function a_0 is chosen from the Sobolev space H^γ and is defined by the following expression:

$$a_0(x) = \frac{1}{4} \sum_{k \in \mathbb{Z}^d} q_k \langle k \rangle^{-\gamma - d/2 - 0.001} e^{ik \cdot x}, \quad \langle k \rangle = (1 + |k|^2)^{1/2}, \tag{57}$$

where the coefficients q_k are randomly generated. This construction ensures that the initial function $a_0(x)$ lies in H^γ but not in $H^{\gamma+0.001}$, and serves as a suitable test case for verifying the sharpness of our theoretical error estimates under low regularity assumptions.

6.1. Convergence in the small-ε regime

Theorem 2.1 establishes a convergence result in the small-ε regime for approximating (1) by solutions to (4), under the regularity assumption $a_0 \in H^2$. In this setting, the method achieves optimal first-order convergence with respect to ε. To numerically verify this result, we take the initial function defined in (57) with $\gamma = 2$. Since exact solutions of (1) are not available, we compute reference solutions by applying the scheme defined in (13)–(14), using $N = 2^{12}$, $\tau = 2^{-12}$ in one dimension and $N = 2^9$, $\tau = 2^{-9}$ in two dimensions.

We measure the L^2 error described in Theorem 2.1 at final time $T = 1$, for $\varepsilon = 2^{-m}$, with $m = 6, 7, \ldots, 14$. The left panel of Figure 1 presents the results in one spatial dimension for two types of initial data: one in $H^2(\mathbb{T})$, which satisfies the assumptions of Theorem 2.3, and one in $H^1(\mathbb{T})$, which falls outside the theoretical framework. As predicted, the error decays linearly with ε for $a_0 \in H^2$, confirming the expected optimal first-order convergence. In contrast, for $a_0 \in H^1$, we observe a convergence rate of approximately $\mathcal{O}(\varepsilon^{1/2})$, indicating a reduction in accuracy due to the lower regularity. Although this case lies beyond the reach of the theory, it offers heuristic insight into the method's sensitivity to initial data regularity.

The right panel shows the results in two spatial dimensions with $a_0 \in H^2(\mathbb{T}^2)$. The method again exhibits first-order convergence with respect to ε, consistent with the one-dimensional H^2 case. These results provide numerical confirmation of the convergence behavior predicted by Theorem 2.3 under optimal regularity assumptions.

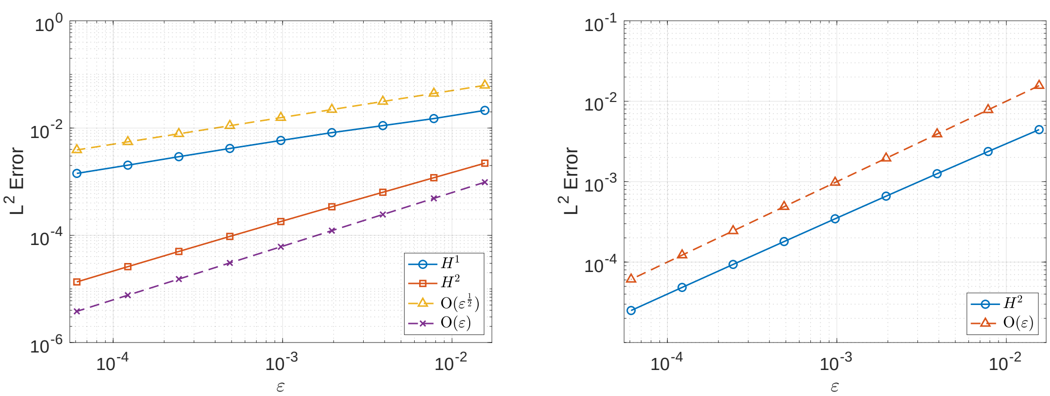

FIGURE 1. L^2 errors versus ε for initial data for $d = 1$ (left) and $d = 2$ (right).

6.2. Uniform-in-ε convergence in 1D

To illustrate how the parameter ε affects the solution, we visualize numerical solutions of equation (1) for different values of ε. Figure 2 shows the results in one spatial dimension. The initial data is chosen as in (57) with $\gamma = 2$, and the computation is carried out using the numerical scheme (13)–(14) with $N = 2^{12}$, $\tau = 2^{-12}$.

The experiment highlights how varying ε affects the modulus of solutions: the left panel corresponds to $\varepsilon = 0.1$, and the right panel to $\varepsilon = 0.0001$. In the left panel, the relatively large ε leads to stronger dispersion effects, causing the modulus of the solution to exhibit both translational motion and visible spreading over time. In contrast, the smaller ε in the right panel enhances nonlinear effects, resulting in a nearly pure translation of the initial profile with invisible spreading—indicating that dispersion is largely suppressed. This is evident in the space-time plot: the tilt of the color bands (i.e., the slope of the level curves of the solution modulus in the x-t plane) reflects a propagation speed close to 1. These results demonstrate how the balance between dispersion and nonlinearity in the NLS equation changes with ε, significantly impacting the wave dynamics.

We next examine the uniform-in-ε space-time accuracy of the proposed numerical scheme (13)–(14) for equation (1). This property is essential for accurately capturing oscillations in the semiclassical regime without requiring the temporal and spatial discretizations to resolve the finest $\mathcal{O}(\varepsilon)$-scale structures. To this end, we examine both the temporal and spatial convergence

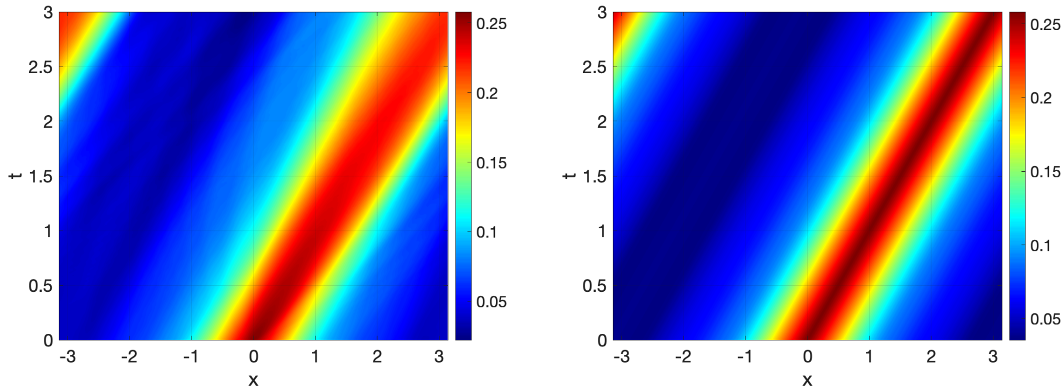

FIGURE 2. Numerical solution with $\varepsilon = 0.1$ (left) and $\varepsilon = 0.0001$ (right)

at the final time $T = 1$ for $\gamma = 1, 2$ and various semiclassical parameters $\varepsilon = 10^{-m}$ with $m = 0, 1, 2, 3, 4$, thereby assessing the scheme's performance across a wide range of ε values.

For the temporal convergence test, we choose the time step sizes independent of ε, specifically $\tau = 2^{-k}$ for $k = 6, \cdots, 11$. Since the exact solution is not available, the temporal error is computed by comparing the numerical solution against a reference solution obtained using $N = 2^{10}$ Fourier modes and $\tau_{\text{ref}} = 2^{-13}$ time steps. The right panel of Figure 3 presents the results for $\gamma = 2$, where uniform second-order convergence in time with respect to ε is observed, consistent with the theoretical prediction in Theorem 2.2.

For the more delicate case $\gamma = 1$, the left panel of Figure 3 shows that the scheme initially exhibits first-order convergence when ε is relatively large. However, as $\varepsilon \to 0$, the convergence rate improves and approaches second order. This phenomenon can be heuristically understood by noting that, in the limiting case $\varepsilon = 0$, the PDE (1) reduces to an ODE system. For such systems, the temporal discretization achieves its optimal second-order accuracy. A rigorous justification of this limiting behavior remains an interesting direction for future analysis.

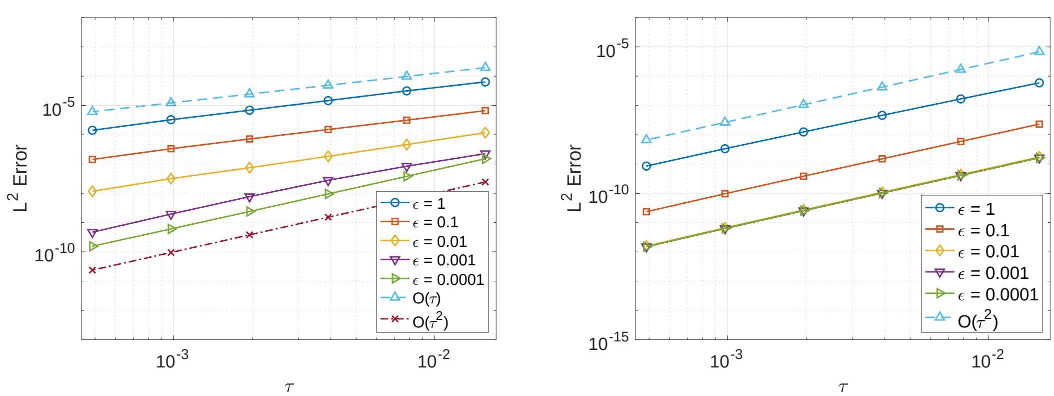

FIGURE 3. Temporal L^2 errors for initial data in H^1 (left) and H^2 (right).

Figure 4 presents the spatial convergence results. Similar to the temporal case, errors are measured by comparing numerical solutions to reference solutions computed with $N_{\text{ref}} = 2^{12}$ Fourier modes and $\tau = 2^{-10}$ time steps. The spatial errors are evaluated for $N = 2^m$ with $m = 6, \ldots, 11$. As shown in Figure 4, for both H^γ initial data with $\gamma = 1, 2$, the spatial convergence rate clearly follows $N^{-\gamma}$. This matches the theoretical prediction given in Theorem 2.2 for the case $\gamma = 2$.

Combined with the temporal convergence results, these findings confirm that the proposed numerical scheme achieves uniform accuracy in both space and time across a wide range of ε values.

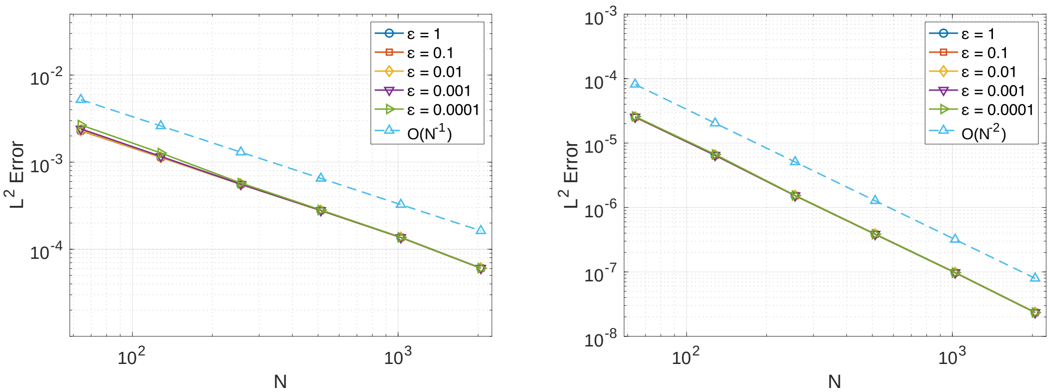

FIGURE 4. Spatial L^2 errors for initial data in H^1 (left) and H^2 (right).

6.3. Uniform-in-ε convergence in 2D

We now turn to two-dimensional examples to further validate the proposed numerical scheme. Figure 5 visualizes the modulus of numerical solutions at times $T = 0, 0.5, 1$ for equation (1) in two spatial dimensions with the H^2 initial data in form of (57) and $\varepsilon = 0.2$. The numerical solution is computed by choosing $N = 2^9 = L_0$. As shown in the Figure 5, the modulus of the solution exhibits a combination of translational motion and dispersive spreading. Specifically, the wave profile propagates along the vector $\kappa = (1, 2)$, while the dispersion corresponding to $\varepsilon = 0.2$ causes visible spreading of the wave packet—similar to the behavior observed in the one-dimensional case.

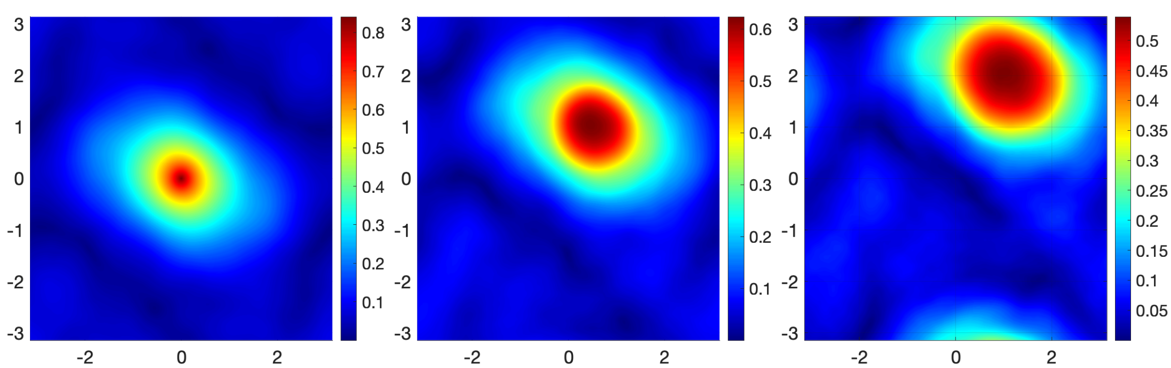

FIGURE 5. Modulus of the numerical solution at $T = 0$ (left), $T = 0.5$ (center), and $T = 1$ (right).

Next, we consider the same H^2 initial data to assess the convergence of the proposed scheme. As in the 1D setting, we examine both temporal and spatial convergence using discretizations

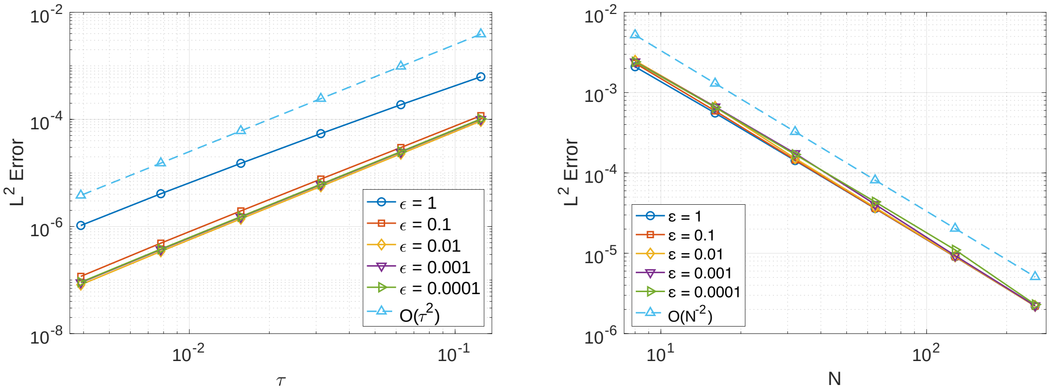

FIGURE 6. Temporal (left) and spatial (right) L^2 errors for initial data in H^2 .

that are independent of ε. For temporal convergence, we fix the spatial resolution to $N = 2^6$ and compute errors for time step sizes $\tau = 2^{-k}$, $k = 3, \ldots, 8$. The reference solution is obtained with $\tau_{\text{ref}} = 2^{-10}$ time steps. The results, shown in the left panel of Figure 6, confirm second-order convergence in time uniformly with respect to ε, consistent with the theoretical results in Theorem 2.2.

For spatial convergence, we fix the temporal resolution to $\tau = 2^{-7}$ and vary the number of Fourier modes $N = 2^m$, $m = 3, \ldots, 8$. The reference solution is computed with $N_{\text{ref}} = 2^9$ spatial modes. The right panel of Figure 6 reports the results, showing second-order convergence in space, again in agreement with theoretical expectations. These two-dimensional experiments further verify the uniform accuracy of the proposed scheme across a range of ε values, both in time and space.

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