Improved error estimates for a modified exponential Euler method for the semilinear stochastic heat equation with rough initial data*

Xinping Gui†, Buyang Li‡, Jilu Wang§

Abstract

A class of stochastic Besov spaces $B^pL^2(\Omega;\dot H^\alpha(\mathcal{O}))$, $1\leqslant p\leqslant\infty$ and $\alpha\in[-2,2]$, is introduced to characterize the regularity of the noise in the semilinear stochastic heat equation

$$\mathrm{d}u-\Delta u\mathrm{d}t=f(u)\mathrm{d}t+\mathrm{d}W(t),$$

under the following conditions for some $\alpha\in(0,1]$:

$$\Big\|\int_0^t e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}\leqslant Ct^{\frac{\alpha}{2}}\quad\text{and}\quad\Big\|\int_0^t e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{B^\infty L^2(\Omega;\dot H^\alpha(\mathcal{O}))}\leqslant C.$$

The conditions above are shown to be satisfied by both trace-class noises (with $\alpha=1$) and one-dimensional space-time white noises (with $\alpha=\frac12$). The latter would fail to satisfy the conditions with $\alpha=\frac12$ if the stochastic Besov norm $\|\cdot\|_{B^\infty L^2(\Omega;\dot H^\alpha(\mathcal{O}))}$ is replaced by the classical Sobolev norm $\|\cdot\|_{L^2(\Omega;\dot H^\alpha(\mathcal{O}))}$, and this often causes reduction of the convergence order in the numerical analysis of the semilinear stochastic heat equation. In this article, the convergence of a modified exponential Euler method, with a spectral method for spatial discretization, is proved to have order α in both time and space for possibly nonsmooth initial data in $L^4(\Omega;\dot H^\beta(\mathcal{O}))$ with $\beta>-1$, by utilizing the real interpolation properties of the stochastic Besov spaces and a class of locally refined stepsizes to resolve the singularity of the solution at $t=0$.

Key words: semilinear stochastic heat equation, additive noise, space-time white noise, exponential Euler method, spectral method, strong convergence, stochastic Besov space, real interpolation

MSC (2020): 65M15

1 Introduction

We consider the numerical approximation to the semilinear stochastic heat equation

$$\begin{cases}\mathrm{d}u-\Delta u\mathrm{d}t=f(u)\mathrm{d}t+\mathrm{d}W(t) & \text{for } t\in(0,T],\\ \qquad u(0)=u^0,\end{cases}\tag{1.1}$$

in a convex polygonal domain $\mathcal{O}\subset\mathbb{R}^d$, $d\geqslant1$, up to a given time $T>0$, with a given nonlinear drift function $f:\mathbb{R}\to\mathbb{R}$ and a given initial value u^0, where $\Delta: H^1_0(\mathcal{O})\cap H^2(\mathcal{O})\to L^2(\mathcal{O})$ is the Dirichlet Laplacian operator, and $W(t)$ is an $L^2(\mathcal{O})$-valued Q-Wiener process on a

*This work is supported in part by the National Natural Science Foundation of China (NSFC grants U1930402 and 12071020) and a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (GRF Project No. PolyU15300519).

†Beijing Computational Science Research Center, Beijing, China. E-mail address: gui@csrc.ac.cn

‡Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong. Email address: buyang.li@polyu.edu.hk

§School of Science, Harbin Institute of Technology, Shenzhen 518055, China. E-mail address: wangjilu@hit.edu.cn

probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. In particular, $W(t)$ has the following expression:

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\mu_k}\phi_k W_k(t), \tag{1.2}$$

where $W_k(t)$, $k = 1, 2, \dots$, are real-valued independent Brownian motions, and ϕ_k, $k = 1, 2, \dots$, are common eigenfunctions of the operators Q and $A = -\Delta$, i.e.,

$$Q\phi_k = \mu_k \phi_k \quad \text{and} \quad A\phi_k = \lambda_k \phi_k \quad \text{for } k = 1, 2, \dots$$

It is well known that $\mathrm{tr}(Q) = \sum_{k=1}^{\infty} \mu_k < \infty$ decides a genuine Wiener process which determines a trace-class noise, and $Q = I$ gives a cylindrical Wiener process which determines a space-time white noise; see Prato & Zabczyk [10] for more details. The trace-class noise is much smoother than the space-time white noise, and therefore the analysis for the latter is often more challenging.

When the initial value is sufficiently smooth, it is well known that problem (1.1) has a unique mild solution satisfying the integral equation

$$u(t) = e^{-tA}u^0 + \int_0^t e^{-(t-s)A} f(u(s))\mathrm{d}s + \int_0^t e^{-(t-s)A}\mathrm{d}W(s), \tag{1.3}$$

in $L^2(\Omega; L^2(\mathcal{O}))$ for all $t \in (0, T]$, where $\{e^{-tA} : t \geqslant 0\}$ is the analytic semigroup generated by the operator A; see [23]. Based on the expression in (1.3), Jentzen & Kloeden [20] proposed the exponential Euler method for semilinear stochastic problems, with the spectral Galerkin method in space. For an abstract semilinear stochastic equation in a Hilbert space H, under the assumptions

$$\begin{aligned} &|f'(x) - f'(y)| \leqslant C|x - y|, \quad |A^{-r} f'(x) A^r v| \leqslant C|v|, \\ &|A^{-1} f''(x)(v, w)| \leqslant C|A^{-\frac{1}{2}} v||A^{-\frac{1}{2}} w|, \end{aligned}$$

where $x, y, v, w \in H$ and $r \in \{0, \frac{1}{2}, 1\}$, Jentzen & Kloeden [20] proved the strong convergence with an error bound of $O(\tau \ln(\frac{1}{\tau}) + M^{-\frac{1}{2}+\varepsilon})$ for one-dimensional space-time white noises, where ε can be an arbitrary small number. These assumptions exclude nonlinear Nemytskii operators and therefore cannot be applied to the semilinear stochastic heat equations. For the stochastic heat equations, Wang & Qi [38] proved that the exponential Euler method in [20] has an L^2-norm error bound of $O(M^{-1} + \tau)$ and $O(M^{-\frac{1}{2}+\varepsilon} + \tau^{\frac{1}{2}-\varepsilon})$ for trace-class noises and one-dimensional space-time white noises, respectively. Recently, Wang [36] proposed a nonlinearity-tamed exponential integrator for the stochastic Allen–Cahn equation with a locally Lipschitz nonlinear drift function and proved an L^2-norm error bound of $O(M^{-\frac{1}{2}+\varepsilon} + \tau^{\frac{1}{2}-\varepsilon})$.

In addition to the exponential Euler method, many other numerical methods for problem (1.1) have also been studied in the literature, including the semi-implicit Euler method in time and the finite difference/element method in space; see [6, 12, 13, 29, 30, 35, 39]. The temporal convergence orders proved in these articles are not greater than $\frac{1}{2} - \varepsilon$ for additive one-dimensional space-time white noises and not greater than $\frac{1}{4}$ for multiplicative one-dimensional space-time white noises. The sharp order $\frac{1}{4}$ for multiplicative space-time white noises was proved for sufficiently smooth initial value $u^0 \in C^3(\overline{\mathcal{O}})$ in [12, 13]. Recently, Anton, Cohen & Quer-Sardanyons proved the sharp convergence order $\frac{1}{4}$ for multiplicative space-time white noises noises for initial data only in $H^1(\mathcal{O})$; see [2].

For all the methods mentioned above, the suboptimal-order error bound $O(M^{-\frac{1}{2}+\varepsilon} + \tau^{\frac{1}{2}-\varepsilon})$ for additive one-dimensional space-time white noises was proved for initial data at least in $H^1(\mathcal{O})$, and the sharper error bound $O(M^{-\frac{1}{2}} + \tau^{\frac{1}{2}})$ has not been proved yet. The main reason is that the following conditions were often used to characterize the regularity of the noises:

$$\Big\|\int_0^t e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}\leqslant Ct^{\frac{\alpha}{2}}\quad\text{and}\quad\Big\|\int_0^t e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{L^2(\Omega;\dot H^\alpha(\mathcal{O}))}\leqslant C,\tag{1.4}$$

which can be satisfied by the space-time white noise with $\alpha=\frac{1}{2}-\varepsilon$, but cannot be satisfied with the sharp order $\alpha=\frac{1}{2}$.

Numerical approximations of related equations to (1.1) were also extensively studied. For example, different kinds of stochastic differential equations were considered in [7, 16–19, 24], and the semilinear stochastic wave equations were discussed in [1, 3, 8, 9, 21, 32, 37], where convergence rates are higher than that for the semilinear stochastic heat equation due to the better regularity of the solution to the stochastic wave equations.

In this article, we show that by modifying the exponential Euler method at the starting time level and use proper variable stepsizes locally refined towards $t=0$, the temporal and spatial convergence orders of the numerical solution can be improved to $\frac{1}{2}$ for additive one-dimensional space-time white noises, while the regularity of the initial data can be relaxed to $H^\beta(\mathcal{O})$ with $\beta>-1$. This wider class of initial data includes discontinuous functions and measures in one dimension, such as the Dirac delta measure. In particular, the following error bound is proved:

$$\|U_M^n-u(t_n)\|_{L^2(\Omega;L^2(\mathcal{O}))}\leqslant C(\tau^\alpha+M^{-\alpha}t_n^{-\frac{\alpha-\beta}{2}})\quad\text{for } t_n\in(0,T],\tag{1.5}$$

for $u^0\in L^4(\Omega;H^\beta(\mathcal{O}))$ with some constant $\beta>-1$, where α characterizes the regularity of the noise, as described in Section 2.3. The result in (1.5), which includes the sharp order $\alpha=\frac{1}{2}$ for one-dimensional space-time white noises, is obtained by using a class of locally refined variable stepsizes to resolve the singularity of the solution at $t=0$, and by utilizing the real interpolation properties of a class of stochastic Besov spaces $B^qL^p(\Omega;\dot H^s(\mathcal{O}))$, $s\in\mathbb{R}$, $1\leqslant p,q\leqslant\infty$, defined in this article, replacing the condition (1.4) by

$$\Big\|\int_0^t e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}\leqslant Ct^{\frac{\alpha}{2}}\quad\text{and}\quad\Big\|\int_0^t e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{B^\infty L^2(\Omega;\dot H^\alpha(\mathcal{O}))}\leqslant C,\tag{1.6}$$

which incorporates the space-time white noise with $\alpha=\frac{1}{2}$.

The rest of this article is organized as follows. In Section 2, we first introduce the basic notations to be used in this article, as well as the definition and properties of the stochastic Besov spaces. Then we present the numerical methods and the main theoretical results. The proof of the main theorem is presented in Section 3. Numerical results for several different initial data and noises are presented in Section 4 to support the theoretical analysis. Conclusions are presented in Section 5.

2 Main results

2.1 Basic notations

For $s\in[0,2]$, we denote by $A^{\frac{s}{2}}:D(A^{\frac{s}{2}})\to L^2(\mathcal{O})$ the linear operator with domain

$$D(A^{\frac{s}{2}})=\Big\{v=\sum_{k=1}^\infty v_k\phi_k:\|A^{\frac{s}{2}}v\|_{L^2(\mathcal{O})}^2=\sum_{k=1}^\infty\lambda_k^s v_k^2<\infty\Big\},$$

where

$$A^{\frac{s}{2}}v:=\sum_{k=1}^\infty\lambda_k^{\frac{s}{2}}v_k\phi_k.$$

It is known that $D(A^{\frac{s}{2}})$ coincides with the following real interpolation space with equivalent norms:

$$D(A^{\frac{s}{2}})\cong\dot H^s(\mathcal{O}):=(L^2(\mathcal{O}),H_0^1(\mathcal{O})\cap H^2(\mathcal{O}))_{\frac{s}{2},2}\quad\text{for } s\in(0,2).$$

Therefore, we simply define the norm of $\dot H^s(\mathcal{O})$ to be the same as $D(A^{\frac{s}{2}})$, i.e.,

$$\|v\|_{\dot H^s(\mathcal{O})} = \Big(\sum_{k=1}^{\infty} \lambda_k^s v_k^2\Big)^{\frac12}.$$

The operator $A^{\frac{s}{2}} : \dot H^s(\mathcal{O}) \to L^2(\mathcal{O})$ is obviously invertible, and its inverse is given by

$$A^{-\frac{s}{2}} v = \sum_{k=1}^{\infty} \lambda_k^{-\frac{s}{2}} v_k \phi_k. \tag{2.7}$$

The dual space of $\dot H^s(\mathcal{O})$ is denoted by $\dot H^{-s}(\mathcal{O})$. In particular, $v \in \dot H^{-s}(\mathcal{O})$ if and only if $v = \sum_{k=1}^{\infty} v_k \phi_k$ with

$$\|v\|_{\dot H^{-s}(\mathcal{O})} = \Big(\sum_{k=1}^{\infty} \lambda_k^{-s} v_k^2\Big)^{\frac12} < \infty.$$

We denote by $\langle \cdot,\cdot\rangle_{\dot H^s, \dot H^{-s}}$ the pairing between $\dot H^s(\mathcal{O})$ and $\dot H^{-s}(\mathcal{O})$. Namely, for $g = \sum_{k=1}^{\infty} g_k \phi_k \in \dot H^s(\mathcal{O})$ and $h = \sum_{k=1}^{\infty} h_k \phi_k \in \dot H^{-s}(\mathcal{O})$, we have

$$\langle g, h\rangle_{\dot H^s(\mathcal{O}), \dot H^{-s}(\mathcal{O})} := \sum_{k=1}^{\infty} g_k h_k,$$

which is well-defined as the series $\sum_{k=1}^{\infty} g_k h_k$ is absolutely convergent, i.e.,

$$\sum_{k=1}^{\infty} |g_k h_k| = \sum_{k=1}^{\infty} \lambda_k^{\frac{s}{2}} |g_k| \lambda_k^{-\frac{s}{2}} |h_k| \leqslant \Big(\sum_{k=1}^{\infty} \lambda_k^s g_k^2\Big)^{\frac12} \Big(\sum_{k=1}^{\infty} \lambda_k^{-s} h_k^2\Big)^{\frac12} = \|g\|_{\dot H^s(\mathcal{O})} \|h\|_{\dot H^{-s}(\mathcal{O})}.$$

In the case $g \in \dot H^s(\mathcal{O}) \subset L^2(\mathcal{O})$ and $h \in L^2(\mathcal{O}) \subset \dot H^{-s}(\mathcal{O})$, we have

$$\langle g, h\rangle_{\dot H^s(\mathcal{O}), \dot H^{-s}(\mathcal{O})} = (g, h),$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$. Therefore, $\langle \cdot,\cdot\rangle_{\dot H^s(\mathcal{O}), \dot H^{-s}(\mathcal{O})}$ is an extension of the $L^2(\mathcal{O})$ inner product $(\cdot,\cdot)$. For this reason and the simplicity of notation, we simply use $(\cdot,\cdot)$ to denote the pairing between $\dot H^s(\mathcal{O})$ and $\dot H^{-s}(\mathcal{O})$ in this article.

The operator $A^{-\frac{s}{2}} : \dot H^{-s}(\mathcal{O}) \to L^2(\mathcal{O})$ is well defined by (2.7) and coincident with the adjoint operator of $A^{-\frac{s}{2}} : L^2(\mathcal{O}) \to \dot H^s(\mathcal{O})$, i.e.,

$$\begin{aligned}
(u, A^{-\frac{s}{2}} w) &= \Big(\sum_{i=1}^{\infty} u_i \phi_i, \sum_{j=1}^{\infty} \lambda_i^{-\frac{s}{2}} w_j \phi_j\Big) \\
&= \sum_{i=1}^{\infty} \lambda_i^{-\frac{s}{2}} u_i w_i = (A^{-\frac{s}{2}} u, w) \qquad \forall\, u \in L^2(\mathcal{O}) \text{ and } w \in \dot H^{-s}(\mathcal{O}),
\end{aligned}$$

where the right-hand side denotes the pairing between $\dot H^s(\mathcal{O})$ and $\dot H^{-s}(\mathcal{O})$.

For a random variable v that takes values in a Banach space X and is measurable from $(\Omega, \mathcal{F})$ to $(X, \mathcal{B}(X))$, we define the following norm:

$$\|v\|_{L^p(\Omega;X)} := \Big(\int_\Omega \|v\|_X^p \, P(\mathrm{d}\omega)\Big)^{\frac1p} \qquad \forall\, 1 \leqslant p \leqslant \infty.$$

If we denote by $\|\cdot\|_{L^2(\mathcal{O}) \to L^2(\mathcal{O})}$ the operator norm on $L^2(\mathcal{O})$, then the following estimate holds (see [23, Appendix B.2]):

$$\begin{aligned}
&\|A^s e^{-tA}\|_{L^2(\mathcal{O}) \to L^2(\mathcal{O})} \leqslant C t^{-s} && \forall\, t > 0,\ \forall\, s \geqslant 0, \\
&\|A^{-s}(e^{-tA} - I)\|_{L^2(\mathcal{O}) \to L^2(\mathcal{O})} \leqslant C t^s && \forall\, t \geqslant 0,\ \forall\, s \in [0, 1]
\end{aligned}$$

Let $\mathcal{L}_2$ be the space of Hilbert Schmidt operators Φ from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$, with the following norm:

$$\|\Phi\|_{\mathcal{L}_2} := \Big(\sum_{k=1}^{\infty} \|\Phi \phi_k\|_{L^2(\mathcal{O})}^2\Big)^{\frac12},$$

Let $\mathcal{L}_2^0$ be the space of Hilbert Schmidt operators Ψ from $Q^{\frac{1}{2}}L^2(\mathcal{O})$ to $L^2(\mathcal{O})$, with the following norm:

$$\|\Psi\|_{\mathcal{L}_2^0} := \Big(\sum_k \mu_k \|\Psi\phi_k\|^2_{L^2(\mathcal{O})}\Big)^{\frac{1}{2}},$$

where we have adopted the notation $\mathcal{L}_2^0$ and $Q^{\frac{1}{2}}L^2(\mathcal{O})$ in [10, pages 54 and 96]. Let $\mathcal{N}_W^2(0,T;L^2(\mathcal{O}))$ be the space of predictable processes $\Phi : [0,T]\times\Omega \to \mathcal{L}_2^0$ satisfying

$$\int_0^T \|\Phi(s)\|^2_{L^2(\Omega;\mathcal{L}_2^0)}\mathrm{d}s < \infty.$$

For $0 \leqslant t \leqslant T$, it is known that the following Itô's isometry holds (cf. [26, 31]):

$$\Big\|\int_0^t \Phi(s)\mathrm{d}W(s)\Big\|^2_{L^2(\Omega;L^2(\mathcal{O}))} = \int_0^t \|\Phi(s)\|^2_{L^2(\Omega;\mathcal{L}_2^0)}\mathrm{d}s \quad \forall\,\Phi \in \mathcal{N}_W^2(0,T;L^2(\mathcal{O})), \tag{2.8}$$

and the following Burkholder–Davis–Gundy-type inequality holds (cf. [10, 23])

$$\Big\|\int_0^t \Phi(s)\mathrm{d}W(s)\Big\|_{L^p(\Omega;L^2(\mathcal{O}))} \leqslant C_p\Big(\mathbb{E}\int_0^t \|\Phi(s)\|^2_{\mathcal{L}_2^0}\mathrm{d}s\Big)^{\frac{1}{2}} \quad \forall\,\Phi \in \mathcal{N}_W^2(0,T;L^2(\mathcal{O})), \tag{2.9}$$

where C_p is a constant dependent of p.

Throughout this article, we denote by C a generic positive constant which may be different at different occurrences but always independent of τ (time stepsize), n (time level) and M (degrees of freedom in each spatial direction). We denote by "$A \sim B$' the statement "$C^{-1}B \leqslant A \leqslant CB$ for some constant C".

2.2 Stochastic Besov Spaces

Let $1 \leqslant p, q \leqslant \infty$ and $s \in [-2,2]$. Since any function $v \in L^p(\Omega;\dot{H}^{-2}(\mathcal{O}))$ can be decomposed into

$$v = \sum_{k=1}^{\infty} v_k\phi_k \quad \text{with} \quad v_k = (v,\phi_k) \in L^p(\Omega),$$

we can define a projection operator $\Pi_j : L^p(\Omega;\dot{H}^{-2}(\mathcal{O})) \to L^p(\Omega;\dot{H}^2(\mathcal{O}))$ by

$$\Pi_j v = \sum_{k=2^{j-1}}^{2^j-1} v_k\phi_k.$$

Then the stochastic Besov space $B^qL^p(\Omega;\dot{H}^s(\mathcal{O}))$ is defined as the space of functions $v \in L^p(\Omega;\dot{H}^{-2}(\mathcal{O}))$ such that

$$\|v\|_{B^qL^p(\Omega;\dot{H}^s(\mathcal{O}))} < \infty,$$

with

$$\|v\|_{B^qL^p(\Omega;\dot{H}^s(\mathcal{O}))} := \begin{cases} \Big(\sum_{j=1}^{\infty} \|\Pi_j v\|^q_{L^p(\Omega;\dot{H}^s(\mathcal{O}))}\Big)^{\frac{1}{q}} & \text{if } q \in [1,\infty), \\ \max_{j\in\mathbb{N}^+} \|\Pi_j v\|_{L^p(\Omega;\dot{H}^s(\mathcal{O}))} & \text{if } q = \infty. \end{cases}$$

2.3 Assumptions on the nonlinearity and noise

We consider the semilinear stochastic heat equation (1.1) with additive noises under the following assumptions.

Assumption 2.1 (1) The function $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, i.e.,

$$|f(x) - f(y)| \leqslant C|x-y|, \quad \forall\, x, y \in \mathbb{R}.$$

(2) There exists $\eta \in (0,2)$ such that
$$\|(-A)^{-\frac{\eta}{2}}[f''(u)vw]\|_{L^2(\mathcal{O})} \leqslant C\|v\|_{L^2(\mathcal{O})}\|w\|_{L^2(\mathcal{O})} \quad \forall\, u,v,w \in L^2(\mathcal{O}).$$

(3) There exists an $\alpha \in (0,1]$ such that for $t \in [0,T]$
$$\Big\| \int_0^t e^{-(t-s)A} \mathrm{d}W(s) \Big\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant Ct^{\frac{\alpha}{2}}, \tag{2.10}$$
$$\Big\| \int_0^t e^{-(t-s)A} \mathrm{d}W(s) \Big\|_{B^\infty L^2(\Omega;\dot{H}^\alpha(\mathcal{O}))} \leqslant C. \tag{2.11}$$
In the case $\alpha = 1$, we additionally assume that the noise is trace class.

(4) The initial value is $\mathcal{F}_0$ measurable and satisfies that $u^0 \in L^4(\Omega;\dot{H}^\beta(\mathcal{O}))$ for some constant $\beta \in (-1,\alpha]$.

Remark 2.1 Assumption 2.1 (2) holds for $d \in \{1,2,3\}$ if $f : \mathbb{R} \to \mathbb{R}$ is a function with bounded derivatives up to second order; see [34, Example 3.2]. In one dimension, a large number of measure-valued functions satisfy Assumption 2.1 (4). Actually, each measure μ (including the Dirac delta function, i.e., the Dirac measure) corresponds to a linearly bounded functional $\Lambda(f) := \int_{\mathcal{O}} f \mathrm{d}\mu$ on the continuous function space $C(\mathcal{O})$; see [27, page 61] and [33, page 32]. And it is well-known that $\dot{H}^{\frac{1}{2}+\varepsilon}(\mathcal{O}) \hookrightarrow C(\mathcal{O})$; see [28, page 86, Theorem 3.26]. Therefore μ can regarded as a linearly bounded functional on $\dot{H}^{\frac{1}{2}+\varepsilon}(\mathcal{O})$, i.e., $\mu \in \dot{H}^{-\frac{1}{2}-\varepsilon}(\mathcal{O})$. Assumption 2.1 (3) naturally holds for trace-class noises with $\alpha = 1$ and one-dimensional space-time white noises with $\alpha = 1/2$, as shown below.

1. If the operator Q is of trace class, i.e., $\mathrm{tr}(Q) = \sum_{k=1}^{\infty} \mu_k < \infty$, then Assumption 2.1 (3) holds with $\alpha = 1$. In fact, according to (2.8) the following relation holds
$$\begin{aligned}\Big\| \int_0^t e^{-(t-s)A} \mathrm{d}W(s) \Big\|^2_{L^2(\Omega;L^2(\mathcal{O}))} &= \int_0^t \|e^{-(t-s)A}\|^2_{\mathcal{L}_2^0} \mathrm{d}s \\ &= \sum_{k=1}^{\infty} \int_0^t e^{-2(t-s)\lambda_k} \mu_k \mathrm{d}s \\ &= \sum_{k=1}^{\infty} \mu_k \frac{1-e^{-2t\lambda_k}}{2\lambda_k} \leqslant Ct \sum_{k=1}^{\infty} \mu_k.\end{aligned} \tag{2.12}$$
The equivalence relation in (2.11) can be shown similarly as (2.12), i.e.,
$$\begin{aligned}\Big\| \int_0^t e^{-(t-s)A} \mathrm{d}W(s) \Big\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))} &= \Big\| \int_0^t A^{\frac{1}{2}} e^{-(t-s)A} \mathrm{d}W(s) \Big\|^2_{L^2(\Omega;L^2(\mathcal{O}))} \\ &= \int_0^t \|A^{\frac{1}{2}} e^{-(t-s)A}\|^2_{\mathcal{L}_2^0} \mathrm{d}s \\ &= \sum_{k=1}^{\infty} \frac{\mu_k}{2}(1-e^{-2t\lambda_k}) \\ &\leqslant \sum_{k=1}^{\infty} \mu_k,\end{aligned} \tag{2.13}$$
which implies that
$$\Big\| \Pi_j \int_0^t e^{-(t-s)A} \mathrm{d}W(s) \Big\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))} \leqslant \Big\| \int_0^t e^{-(t-s)A} \mathrm{d}W(s) \Big\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))} \leqslant \sum_{k=1}^{\infty} \mu_k.$$
This proves (2.10)–(2.11) in the case $\alpha = 1$.

2. If $d = 1$ and $\mu_k = 1$ (for a space-time white noise), then Assumption 2.1 (3) holds with $\alpha = \frac{1}{2}$. In fact, Weyl's law (see Evans [11, Page 358]) says that the eigenvalues of the Laplacian operator have the following asymptotic behaviour:

$$\lim_{j\to\infty} \frac{\lambda_j^{\frac{d}{2}}}{j} = \frac{(2\pi)^d}{|\mathcal{O}|\alpha(d)}, \tag{2.14}$$

where $|\mathcal{O}|$ denotes the volume of $\mathcal{O}$ and $\alpha(d)$ denotes the volume of the unit ball in $\mathbb{R}^d$. Therefore, $\lambda_k = O(k^{-2})$ in one dimension, and

$$\begin{aligned}\Big\| \int_0^t e^{-(t-s)A} \mathrm{d}W(s) \Big\|_{L^2(\Omega;L^2(\mathcal{O}))}^2 &\sim C(1-e^{-2t}) + C\sum_{k=2}^{\infty} \frac{1-e^{-2tk^2}}{k^2} \\ &\leqslant Ct^{\frac{1}{2}} + C\int_1^{\infty} \frac{1-e^{-2ts^2}}{s^2} ds \\ &= Ct^{\frac{1}{2}} + C\int_1^{\infty} \frac{1-e^{-2tz}}{z^{\frac{3}{2}}} dz \\ &\leqslant Ct^{\frac{1}{2}} + Ct^{\frac{1}{2}}\Gamma\Big(\frac{1}{2}\Big) \leqslant Ct^{\frac{1}{2}}. \end{aligned} \tag{2.15}$$

This proves (2.10) with $\alpha = \frac{1}{2}$ for a space-time white noise.

For $\alpha = \frac{1}{2}$ we also see that

$$\max_{j\in\mathbb{N}^+} \Big(\sum_{k=2^{j-1}}^{2^j-1} \mu_k \lambda_k^{\alpha-1}(1-e^{-2t\lambda_k}) \Big)^{\frac{1}{2}} \leqslant \max_{j\in\mathbb{N}^+} \Big(C \sum_{k=2^{j-1}}^{2^j-1} 2^{2(j-1)(\frac{1}{2}-1)} \Big)^{\frac{1}{2}} \leqslant C,$$

where the last inequality holds because there are only 2^{j-1} terms in the summation. This proves (2.11).

Under Assumption 2.1 the existence, uniqueness and regularity of mild solutions to problem (1.1) are summarized below. The proof of these results is presented in Appendix.

Proposition 2.1 Under Assumption 2.1, problem (1.1) has a unique mild solution u in the space

$$X = \Big\{ v \in L^1\big(0,T; L^2(\Omega; L^2(\mathcal{O}))\big) : \sup_{t\in(0,T]} (1+t^{\frac{\beta}{2}})^{-1} \|v(t)\|_{L^2(\Omega;L^2(\mathcal{O}))} < \infty \Big\}.$$

For $2 \leqslant p \leqslant 4$ the mild solution has the following qualitative regularity:

$$u \in C\big([0,T]; L^p(\Omega; \dot{H}^{\min\{\beta,0\}}(\mathcal{O}))\big) \cap C\big([\varepsilon,T]; L^p(\Omega; L^2(\mathcal{O}))\big)$$

which holds for arbitrary $\varepsilon \in (0,T)$. Moreover, the following quantitative estimates hold:

$$\|u(t)\|_{L^p(\Omega;L^2(\mathcal{O}))} \leqslant C(1+t^{\frac{\beta}{2}}) \quad \text{for } t \in (0,T], \tag{2.16}$$

$$\|u(t)\|_{B^\infty L^p(\Omega;\dot{H}^\alpha(\mathcal{O}))} \leqslant Ct^{-\frac{\alpha-\beta}{2}} \quad \text{for } t \in (0,T]. \tag{2.17}$$

Remark 2.2 For the trace-class noise, since (2.13) holds with the classical Sobolev space, it follows that estimate (2.17) could be replaced by the following stronger result (with the classical Sobolev space for $\alpha = 1$):

$$\|u(t)\|_{L^p(\Omega;\dot{H}^1(\mathcal{O}))} \leqslant Ct^{-\frac{1-\beta}{2}} \quad \text{for } t \in (0,T].$$

2.4 The numerical method

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the time interval $[0,T]$ with stepsizes $\tau_n = t_n - t_{n-1} \sim t_n^\gamma \tau$ for $n = 1,2,\cdots,N$. The variable stepsizes defined in this way have the following properties:

1. $\tau_n \sim \tau_{n-1}$ for two consecutive stepsizes.

2. $\tau_1 = \tau^{\frac{1}{1-\gamma}}$. Hence, the starting stepsize is much smaller than the maximal stepsize. This can resolve the solution's singularity at $t = 0$.

3. The total number of time levels is $O(T/\tau)$. Hence, the total computational cost is equivalent to using a uniform stepsize τ.

This type of stepsizes was shown to be able to resolve the singularity at $t = 0$ for semilinear parabolic equations with nonsmooth initial data; see [25].

By using the variable stepsizes shown above, we consider the following modified exponential Euler method in time:

$$\begin{aligned} u^1 &= e^{-\tau_1 A} u^0, \\ u^n &= e^{-\tau_n A} u^{n-1} + \int_{t_{n-1}}^{t_n} e^{-(t_n - s)A} f(u^{n-1}) \mathrm{d}s + \int_{t_{n-1}}^{t_n} e^{-(t_n-s)A} \mathrm{d}W(s) \quad \text{for } n \geqslant 2, \end{aligned} \tag{2.18}$$

where we assume that $\int_{t_{n-1}}^{t_n} e^{-(t_n-s)A}\mathrm{d}W(s)$ can be computed sufficiently accurately. This can be done by a spectral method in space with sufficiently many terms, as shown below. The semidiscrete scheme in (2.18) differs from the *exponential Euler method* only at the first time level, where we drop the nonlinear term and the noise term. This is because that the nonlinear term $f(u^0)$ is generally not well defined for a nonsmooth initial value $u^0 \in \dot H^\beta(\mathcal{O})$.

Let P_M be the L^2-orthogonal projection onto $S_M = \mathrm{span}\{\phi_1, \dots, \phi_{M^d}\}$, defined by

$$P_M f = \sum_{k=1}^{M^d} f_k \phi_k \quad \text{for} \quad f = \sum_{k=1}^{\infty} f_k \phi_k \in L^2(\mathcal{O}).$$

On a general bounded domain $\mathcal{O}$, we consider the following fully discrete spectral Galerkin method:

$$U_M^1 = e^{-\tau_1 A} U_M^0 \qquad \text{with } U_M^0 = P_M u^0, \tag{2.19a}$$

$$U_M^n = e^{-\tau_n A} U_M^{n-1} + \int_{t_{n-1}}^{t_n} e^{-(t_n-s)A} P_M f(U_M^{n-1}) ds + \int_{t_{n-1}}^{t_n} e^{-(t_n-s)A} P_M dW(s) \quad \text{for } n \geqslant 2. \tag{2.19b}$$

In the one-dimensional case, e.g., $\mathcal{O} = (0,1)$, we can change to consider the following fast method which utilizes trigonometric interpolation and fast Fourier transform (FFT). Let P_M and I_M be the L^2-orthogonal projection and trigonometric interpolation operator (defined below) onto the finite dimensional space

$$S_M = \Big\{ \sum_{j=1}^{M} f_j \sin(j\pi x) : f_j \in \mathbb{R} \Big\}.$$

Then we can consider the Fourier sine collocation method:

$$U_M^1 = e^{-\tau_1 A} U_M^0 \qquad \text{with } U_M^0 = P_M u^0, \tag{2.20a}$$

$$U_M^n = e^{-\tau_n A} U_M^{n-1} + \int_{t_{n-1}}^{t_n} e^{-(t_n-s)A} I_M f(U_M^{n-1}) ds + \int_{t_{n-1}}^{t_n} e^{-(t_n-s)A} P_M dW(s) \quad \text{for } n \geqslant 2, \tag{2.20b}$$

which only requires computing the trigonometric interpolation of the nonlinear function $I_M f(U_M^{n-1}) := I_M^*[f(U_M^{n-1}) - f(0)] + f(0) P_M 1$ instead of the L^2 projection $P_M f(U_M^{n-1})$, where I_M^* denotes the standard trigonometric interpolation operator onto S_M. The definition of I_M guarantees that the standard trigonometric sine interpolation operator I_M^* only acts on a function in $\dot H^1$ (satisfying the zero boundary condition) and therefore has optimal-order convergence; see the discussion below (3.36). The evaluation of the trigonometric interpolation $I_M f(U_M^{n-1})$ could be done with $O(M \log M)$ operations at every time level by using FFT. This fast algorithm using trigonometric interpolation and FFT can also be extended to d-dimensional rectangular domains under the homogeneous Dirichlet boundary condition.

Theorem 2.1 Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with $d \geqslant 1$. Under Assumption 2.1, by choosing γ satisfying the following condition (γ is the constant from the relation $\tau_n \sim t_n^\gamma \tau$):

$$\max\left\{\frac{1}{2}, 1 - \frac{1+\beta}{\alpha}\right\} < \gamma < 1, \tag{2.21}$$

the numerical solution given by the fully discrete spectral Galerkin method (2.19) has the following error bound:

$$\|U_M^n - u(t_n)\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant C\tau^\alpha + CM^{-\alpha} t_n^{-\frac{\alpha-\beta}{2}}.$$

Theorem 2.2 Let $d = 1$ and $\mathcal{O} = (0,1)$. Under Assumption 2.1, by choosing γ satisfying the following condition (γ is the constant from the relation $\tau_n \sim t_n^\gamma \tau$):

$$\max\left\{\frac{1}{2}, 1 - \frac{1+\beta}{\alpha}\right\} < \gamma < 1, \tag{2.22}$$

the numerical solution given by the fully discrete Fourier sine collocation method (2.20) has the following error bound:

$$\|U_M^n - u(t_n)\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant C\tau^\alpha + CM^{-\alpha} t_n^{-\frac{\alpha-\beta}{2}}.$$

Since the spatial discretizations in (2.19) and (2.20) are both by spectral methods, the proofs for Theorems 2.1 and 2.2 are similar. Therefore, we present the proof for Theorem 2.2 in the next section and omit the detailed proof for Theorem 2.1.

3 Proof of Theorem 2.2

The proof of Theorem 2.2 is divided into five subsections. In Section 3.1, we present the real interpolation properties of the stochastic Besov spaces $B^q L^p(\Omega; \dot H^s(\mathcal{O}))$, $s \in [0,2]$. These properties are used to prove the sharp convergence order in the case of one-dimensional space-time white noise, which satisfies the second condition in Assumption 2.1 (3) with $\alpha = \frac{1}{2}$ for the stochastic Besov norm $\|\cdot\|_{B^\infty L^2(\Omega;\dot H^\alpha(\mathcal{O}))}$, but not for the Sobolev norm $\|\cdot\|_{L^2(\Omega;\dot H^\alpha(\mathcal{O}))}$. The error estimates are presented in Sections 3.2–3.4.

3.1 Real interpolation results

The K-functional and J-functional are defined as

$$K(t,f;X_0,X_1) = \inf_{f=f_0+f_1} \|f_0\|_{X_0} + t\|f_1\|_{X_1} \qquad \forall\, f \in X_0 + X_1,\ \ \forall\, t > 0,$$
$$J(t,f;X_0,X_1) = \max\{\|f\|_{X_0}, t\|f\|_{X_1}\} \qquad \forall f \in X_0 \cap X_1,\ \ \forall\, t > 0.$$

Definition 3.1 (Discrete $K_{\theta,q}$-functor [5, Page 41, Lemma 3.1.3]) Let $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$ and let (X_0, X_1) be a compatible couple. The interpolation space $(X_0, X_1)_{\theta,q;K}$ consists of functions $f \in X_0 + X_1$ such that $\|f\|_{(X_0,X_1)_{\theta,q;K}} < \infty$, where

$$\|f\|_{(X_0,X_1)_{\theta,q;K}} := \begin{cases} \Big[\sum_{j\in\mathbb{Z}} \Big|a^{-j\theta} K(a^j, f; X_0, X_1)\Big|^q\Big]^{1/q}, & 1 \leqslant q < \infty, \\ \sup_{j\in\mathbb{Z}} a^{-j\theta} K(a^j, f; X_0, X_1), & q = \infty, \end{cases}$$

in which a is any fixed positive constant.

Definition 3.2 (Discrete $J_{\theta,q}$-functor [5, Page 43, Lemma 3.2.3]) Let $0<\theta<1$, $1\leqslant q\leqslant\infty$ and let (X_0,X_1) be a compatible couple. The interpolation space $(X_0,X_1)_{\theta,q;J}$ consists of function $f\in X_0+X_1$ such that $\|f\|_{(X_0,X_1)_{\theta,q;J}}<\infty$, where

$$\|f\|_{(X_0,X_1)_{\theta,q;J}} := \begin{cases} \inf\limits_{f=\sum_j f_j} \Big(\sum\limits_{j\in\mathbb{Z}} \Big|a^{-j\theta}J(a^j,f_j;X_0,X_1)\Big|^q\Big)^{\frac{1}{q}}, & 1\leqslant q<\infty,\\ \inf\limits_{f=\sum_j f_j} \Big(\sup\limits_{j\in\mathbb{Z}} a^{-j\theta}J(a^j,f_j;X_0,X_1)\Big), & q=\infty, \end{cases}$$

in which a is any fixed positive constant, and the infimum extends over all possible decompositions

$$f=\sum_{j\in\mathbb{Z}} f_j \quad \text{with } f_j\in X_0\cap X_1 \text{ and convergence in } X_0+X_1. \tag{3.23}$$

It is known that $(X_0,X_1)_{\theta,q;K}$ and $(X_0,X_1)_{\theta,q;J}$ are the same vector space with equivalent norms; see [5, Page 44, Theorem 3.3.1]). For simplicity, we denote by $(X_0,X_1)_{\theta,q}$ the common vector space of $(X_0,X_1)_{\theta,q;K}$ and $(X_0,X_1)_{\theta,q;J}$, with the norm $\|\cdot\|_{(X_0,X_1)_{\theta,q}}$.

Lemma 3.1 ([4, Page 301, Theorem 1.12]) Let (X_0,X_1) and (Y_0,Y_1) be compatible couples and let $0<\theta<1$, $1\leqslant q<\infty$ or $0\leqslant\theta\leqslant 1$, $q=\infty$. Let $T: X_0+X_1\to Y_0+Y_1$ be a linear operator such that T maps X_i to Y_i and

$$\|Tf\|_{Y_i}\leqslant M_i\|f\|_{X_i} \quad \forall\, f\in X_i,\ i=0,1.$$

Then T maps $(X_0,X_1)_{\theta,q}$ to $(Y_0,Y_1)_{\theta,q}$ and

$$\|Tf\|_{(Y_0,Y_1)_{\theta,q}}\leqslant M_0^{1-\theta}M_1^{\theta}\|f\|_{(X_0,X_1)_{\theta,q}} \quad \forall\, f\in (X_0,X_1)_{\theta,q}.$$

The main results of this subsection are presented in the following lemma.

Lemma 3.2 For all $1\leqslant p,q\leqslant\infty$ and $0<\theta<1$ there holds

$$B^qL^p(\Omega;\dot H^s(\mathcal{O})) = \big(L^p(\Omega;\dot H^{s_0}(\mathcal{O})),L^p(\Omega;\dot H^{s_1}(\mathcal{O}))\big)_{\theta,q}, \tag{3.24}$$

$$\big(B^\infty L^p(\Omega;\dot H^{s_0}(\mathcal{O}),B^\infty L^p(\Omega;\dot H^{s_1}(\mathcal{O})\big)_{\theta,q} = B^qL^p(\Omega;\dot H^s(\mathcal{O})), \tag{3.25}$$

where $-2\leqslant s_0<s_1\leqslant 2$ and $s=(1-\theta)s_0+\theta s_1$.

Proof. If $2^{j-1}\leqslant k\leqslant 2^j-1$, then $\lambda_k^s\sim k^{\frac{2s}{d}}\sim 2^{\frac{2js}{d}}$, which implies that

$$\|\Pi_j f\|_{\dot H^s}\sim 2^{\frac{js}{d}}\|\Pi_j f\|_{L^2(\mathcal{O})}. \tag{3.26}$$

Hence for any $f\in L^p(\Omega;\dot H^{s_0})+L^p(\Omega;\dot H^{s_1})$, there exists a decomposition $f=f_0+f_1$ with $f_0\in L^p(\Omega;\dot H^{s_0})$ and $f_1\in L^p(\Omega;\dot H^{s_1})$. Since

$$\begin{aligned}
&\|\Pi_j f\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&\leqslant \|\Pi_j f_0\|_{L^p(\Omega;L^2(\mathcal{O}))}+\|\Pi_j f_1\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&= 2^{-\frac{js_0}{d}}\|2^{\frac{js_0}{d}}\Pi_j f_0\|_{L^p(\Omega;L^2(\mathcal{O}))}+2^{-\frac{js_1}{d}}\|2^{\frac{js_1}{d}}\Pi_j f_1\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&\leqslant 2^{-\frac{js_0}{d}}\Big\|\Big(\sum_{j=1}^\infty 2^{\frac{2js_0}{d}}\|\Pi_j f_0\|^2_{L^2(\mathcal{O})}\Big)^{\frac12}\Big\|_{L^p(\Omega)}+2^{-\frac{js_1}{d}}\Big\|\Big(\sum_{j=1}^\infty 2^{\frac{2js_1}{d}}\|\Pi_j f_1\|^2_{L^2(\mathcal{O})}\Big)^{\frac12}\Big\|_{L^p(\Omega)}\\
&\leqslant C2^{-\frac{js_0}{d}}\|f_0\|_{L^p(\Omega;\dot H^{s_0}(\mathcal{O}))}+C2^{-\frac{js_1}{d}}\|f_1\|_{L^p(\Omega;\dot H^{s_1}(\mathcal{O}))},
\end{aligned}$$

it follows that

$$\begin{aligned}
\|\Pi_j f\|_{L^p(\Omega;L^2(\mathcal{O}))} &\leqslant C\inf_{f=f_0+f_1} 2^{-\frac{js_0}{d}}\Big(\|f_0\|_{L^p(\Omega;\dot H^{s_0}(\mathcal{O}))}+2^{\frac{j(s_0-s_1)}{d}}\|f_1\|_{L^p(\Omega;\dot H^{s_1}(\mathcal{O}))}\Big)\\
&\leqslant C2^{-\frac{js_0}{d}}K(2^{\frac{j(s_0-s_1)}{d}},f;X_0,X_1),
\end{aligned}$$

where $X_0 = L^p(\Omega; \dot H^{s_0}(\mathcal{O}))$ and $X_1 = L^p(\Omega; \dot H^{s_1}(\mathcal{O}))$. This further implies that

$$\begin{aligned} 2^{\frac{js}{d}}\|\Pi_j f\|_{L^p(\Omega;L^2(\mathcal{O}))} &\leqslant C2^{\frac{j(s-s_0)}{d}}K(2^{\frac{j(s_0-s_1)}{d}}, f; X_0, X_1) \\ &\leqslant C(2^{\frac{s_0-s_1}{d}})^{-j\theta}K((2^{\frac{(s_0-s_1)}{d}})^j, f; X_0, X_1). \end{aligned}$$

By considering the ℓ^q norm of the inequality above with respect to j and using Definition 3.1, we obtain

$$\|f\|_{B^qL^p(\Omega;\dot H^s(\mathcal{O}))} \leqslant C\|f\|_{(L^p(\Omega;\dot H^{s_0}(\mathcal{O})),L^p(\Omega;\dot H^{s_1}(\mathcal{O})))_{\theta,q}} \quad \forall\, 1\leqslant q\leqslant\infty,$$

which means that

$$(L^p(\Omega; \dot H^{s_0}(\mathcal{O})), L^p(\Omega; \dot H^{s_1}(\mathcal{O})))_{\theta,q} \hookrightarrow B^qL^p(\Omega; \dot H^s(\mathcal{O})) \quad \forall\, 1\leqslant q\leqslant\infty. \tag{3.27}$$

Conversely, since $s_0 < s < s_1$, if $f \in B^qL^p(\Omega; \dot H^s(\mathcal{O}))$ then

$$f \in L^p(\Omega; \dot H^{s_0}(\mathcal{O})) = L^p(\Omega; \dot H^{s_0}(\mathcal{O})) + L^p(\Omega; \dot H^{s_1}(\mathcal{O}))$$

and

$$\begin{aligned} \|f\|_{L^p(\Omega;\dot H^{s_0}(\mathcal{O}))} &\leqslant C\Big\|\Big(\sum_{j=1}^\infty 2^{\frac{2js_0}{d}}\|\Pi_j f\|_{L^2(\mathcal{O})}^2\Big)^{\frac12}\Big\|_{L^p(\Omega)} \\ &\leqslant C\sum_{j=1}^\infty 2^{\frac{j(s_0-s)}{d}}2^{\frac{js}{d}}\|\Pi_j f\|_{L^p(\Omega;L^2(\mathcal{O}))} \\ &\leqslant C\Big(\sum_{j=1}^\infty 2^{\frac{j(s_0-s)}{d}q'}\Big)^{\frac{1}{q'}}\Big(\sum_{j=1}^\infty \|\Pi_j f\|_{L^p(\Omega;\dot H^s(\mathcal{O}))}^q\Big)^{\frac1q} \\ &\leqslant C\|f\|_{B^qL^p(\Omega;\dot H^s(\mathcal{O}))}, \end{aligned}$$

where q' is the constant satisfying $\frac1q + \frac1{q'} = 1$. Substituting this inequality into the expression $J(t, f; X_0, X_1) = \max\{\|f\|_{X_0}, t\|f\|_{X_1}\}$ yields

$$\begin{aligned} &2^{\frac{j(s-s_0)}{d}}J(2^{\frac{j(s_0-s_1)}{d}}, \Pi_j f; L^p(\Omega; \dot H^{s_0}(\mathcal{O})), L^p(\Omega; \dot H^{s_1}(\mathcal{O}))) \\ &= 2^{\frac{j(s-s_0)}{d}}\max\big\{\|\Pi_j f\|_{L^p(\Omega;\dot H^{s_0}(\mathcal{O}))}, 2^{\frac{j(s_0-s_1)}{d}}\|\Pi_j f\|_{L^p(\Omega;\dot H^{s_1}(\mathcal{O}))}\big\} \\ &\leqslant C2^{\frac{js}{d}}\|\Pi_j f\|_{L^p(\Omega;L^2(\mathcal{O}))} \\ &\leqslant C\|\Pi_j f\|_{L^p(\Omega;\dot H^s(\mathcal{O}))}. \end{aligned}$$

In view of the inequality above and Definition 3.2, we have

$$\|f\|_{(L^p(\Omega;\dot H^{s_0}(\mathcal{O})),L^p(\Omega;\dot H^{s_1}(\mathcal{O})))_{\theta,q}} \leqslant C\|f\|_{B^qL^p(\Omega;\dot H^s(\mathcal{O}))} \quad \forall\, 1\leqslant q\leqslant\infty,$$

which means that

$$B^qL^p(\Omega; \dot H^s(\mathcal{O})) \hookrightarrow (L^p(\Omega; \dot H^{s_0}(\mathcal{O})), L^p(\Omega; \dot H^{s_1}(\mathcal{O})))_{\theta,q} \quad \forall\, 1\leqslant q\leqslant\infty. \tag{3.28}$$

The two results in (3.27) and (3.28) imply (3.24). Then (3.25) follows from (3.24) as a result of the reiteration theorem in the real interpolation theory; see [5, Page 50, Theorem 3.5.3]. □

The following inverse inequality will be often used in the error estimation: If $f \in S_M$ and $-2 \leqslant s_0 \leqslant s \leqslant 2$ then

$$\|f\|_{\dot H^s(\mathcal{O})} = \Big(\sum_{k=1}^M \lambda_k^s|(f,\phi_k)|^2\Big)^{\frac12} \leqslant C\Big(\lambda_M^{s-s_0}\sum_{k=1}^M \lambda_k^{s_0}|(f,\phi_k)|^2\Big)^{\frac12} \leqslant CM^{s-s_0}\|f\|_{\dot H^{s_0}(\mathcal{O})}. \tag{3.29}$$

Correspondingly, for a stochastic function $f \in L^p(\Omega; S_M) \hookrightarrow L^p(\Omega; \dot H^s(\mathcal{O}))$ the following result holds:

$$\|f\|_{L^p(\Omega;\dot H^s(\mathcal{O}))} \leqslant CM^{s-s_0}\|f\|_{L^p(\Omega;\dot H^{s_0}(\mathcal{O}))} \quad \text{for } -2\leqslant s_0\leqslant s\leqslant 2. \tag{3.30}$$

By choosing $-2 \leqslant s_2 < s_0 < s_1 \leqslant s \leqslant 2$ and consider the real interpolation between the two results,

$$\|f\|_{L^p(\Omega;\dot H^s(\mathcal{O}))} \leqslant CM^{s-s_1}\|f\|_{L^p(\Omega;\dot H^{s_1}(\mathcal{O}))} \quad \text{and} \quad \|f\|_{L^p(\Omega;\dot H^s(\mathcal{O}))} \leqslant CM^{s-s_2}\|f\|_{L^p(\Omega;\dot H^{s_2}(\mathcal{O}))},$$

we obtain the following inequality for $f \in L^p(\Omega; S_M) \hookrightarrow L^p(\Omega; \dot H^s(\mathcal{O}))$:

$$\|f\|_{L^p(\Omega;\dot H^s(\mathcal{O}))} \leqslant CM^{s-s_0}\|f\|_{B^\infty L^p(\Omega;\dot H^{s_0}(\mathcal{O}))} \quad \text{for } -2 < s_0 < s \leqslant 2. \tag{3.31}$$

3.2 The error equation

By iterating the second relation in (2.20) for $n \geqslant 2$, the numerical solution in (2.20) can be written as

$$U_M^n = e^{-(t_n-\tau_1)A}U_M^1 + \sum_{j=2}^n \int_{t_{j-1}}^{t_j} e^{-(t_n-s)A} I_M f(U_M^{j-1})ds + \int_{t_1}^{t_n} e^{-(t_n-s)A} P_M dW(s). \tag{3.32}$$

The difference between (3.32) and (1.3) yields the following expression for the error of the numerical solution:

$$\begin{aligned} U_M^n - u(t_n) &= e^{-(t_n-\tau_1)A}U_M^1 - e^{-(t_n-\tau_1)A}u(t_1) \\ &\quad + \sum_{j=2}^n \int_{t_{j-1}}^{t_j} \big(e^{-(t_n-s)A} I_M f(U_M^{j-1}) - e^{-(t_n-s)A} f(U_M^{j-1})\big)ds \\ &\quad + \sum_{j=2}^n \int_{t_{j-1}}^{t_j} \big(e^{-(t_n-s)A} f(U_M^{j-1}) - e^{-(t_n-s)A} f(u(t_{j-1}))\big)ds \\ &\quad + \sum_{j=2}^n \int_{t_{j-1}}^{t_j} \big(e^{-(t_n-s)A} f(u(t_{j-1})) - e^{-(t_n-s)A} f(u(s))\big)ds \\ &\quad + \int_{t_1}^{t_n} \big(e^{-(t_n-s)A} P_M - e^{-(t_n-s)A}\big)dW(s) \\ &=: \mathcal{E}_1^n + \mathcal{E}_2^n + \mathcal{E}_3^n + \mathcal{E}_4^n + \mathcal{E}_5^n, \end{aligned} \tag{3.33}$$

which is decomposed into five parts.

By using the first relation in (2.20) (when $n = 1$), the first part on the right-hand side of (3.33) can be further written as

$$\begin{aligned} \mathcal{E}_1^n &= e^{-t_nA}P_Mu^0 - e^{-t_nA}u^0 \\ &\quad - e^{-(t_n-t_1)A}\int_0^{t_1} e^{-(t_1-s)A} f(u(s))\mathrm{d}s - e^{-(t_n-t_1)A}\int_0^{t_1} e^{-(t_1-s)A}\mathrm{d}W(s). \end{aligned} \tag{3.34}$$

The first term in (3.34) can be estimated by using the classical error estimates of spectral method for the heat equation, i.e.,

$$\|e^{-t_nA}P_Mu^0 - e^{-t_nA}u^0\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant CM^{-\alpha}t_n^{-\frac{\alpha-\beta}{2}}\|u^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))}.$$

The second and third terms in (3.34) can be estimated by using the regularity results in Proposition 2.1 and Assumption 2.1 (3) on the noise. Then we obtain

$$\begin{aligned} \|\mathcal{E}_1^n\|_{L^2(\Omega;L^2(\mathcal{O}))} &\leqslant CM^{-\alpha}t_n^{-\frac{\alpha-\beta}{2}}\|u^0\|_{\dot H^\beta(\mathcal{O})} + \int_0^{\tau_1} C(1+s^{\frac{\beta}{2}})\mathrm{d}s + C\tau_1^{\frac{\alpha}{2}} \\ &\leqslant CM^{-\alpha}t_n^{-\frac{\alpha-\beta}{2}} + C(\tau_1 + \tau_1^{1+\frac{\beta}{2}}) + C\tau_1^{\frac{\alpha}{2}} \\ &\leqslant CM^{-\alpha}t_n^{-\frac{\alpha-\beta}{2}} + C(\tau^{\frac{1}{1-\gamma}} + \tau^{\frac{1}{1-\gamma}(1+\frac{\beta}{2})}) + C\tau^{\frac{1}{1-\gamma}\frac{\alpha}{2}}. \end{aligned}$$

If $\gamma \geqslant \max\big(\frac{1}{2}, 1-(1+\frac{\beta}{2})\frac{1}{\alpha}\big)$, then the inequality above reduces to the following desired result:

$$\|\mathcal{E}_1^n\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant CM^{-\alpha}t_n^{-\frac{\alpha-\beta}{2}} + C\tau^{\alpha}. \tag{3.35}$$

The following estimates are known for the standard trigonometric interpolation I_M^* and L^2 projection P_M:

$$\|v - I_M^* v\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant CM^{-s}\|v\|_{L^2(\Omega;\dot{H}^s(\mathcal{O}))} \quad \text{for } v \in \dot{H}^s(\mathcal{O}), \quad \frac{1}{2} < s \leqslant 2, \tag{3.36}$$

$$\|v - P_M v\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant CM^{-s}\|v\|_{L^2(\Omega;\dot{H}^s(\mathcal{O}))} \quad \text{for } v \in \dot{H}^s(\mathcal{O}), \quad 0 \leqslant s \leqslant 2, \tag{3.37}$$

where the error estimates of trigonometric interpolation for periodic functions can be found in [22, Page 209, Theorem 11.8]; the error estimates of trigonometric sine interpolation for functions satisfying the Dirichlet boundary condition on $\mathcal{O} = (0,1)$ follow by extending the function to $[-1,1]$ as a periodic odd function. Since odd extension of a function preserves the continuity of the function and its first-order derivative, it follows that the odd extension maps $\dot{H}^s(0,1)$ to the periodic function space $H^s_{\rm per}[-1,1]$ for $s \in [0,2]$ (by considering the real interpolation between the two endpoint cases $s=0$ and $s=2$).

For the function $g = f(U_M^{j-1}) - f(0)$ which satisfies the zero boundary condition, the following error estimate holds:

$$\|I_M^* g - g\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant CM^{-1}\|g\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))}.$$

Since $e^{-(t-s)A}$ commutes with the projection operator P_M, it follows that

$$\begin{aligned}
\|e^{-(t-s)A}(P_M 1 - 1)\|_{L^2(\Omega;L^2(\mathcal{O}))} &= \|P_M e^{-(t-s)A}1 - e^{-(t-s)A}1\|_{L^2(\Omega;L^2(\mathcal{O}))} \\
&\leqslant CM^{-1}\|e^{-(t-s)A}1\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))} \\
&\leqslant C(t-s)^{-\frac{1}{2}}M^{-1}\|1\|_{L^2(\Omega;L^2(\mathcal{O}))} \\
&\leqslant C(t-s)^{-\frac{1}{2}}M^{-1}.
\end{aligned}$$

Therefore, by using expression $I_M f(U_M^{j-1}) = I_M^* g + f(0)P_M 1$ and the triangle inequality,

$$\begin{aligned}
&\|e^{-(t-s)A}[I_M f(U_M^{j-1}) - f(U_M^{j-1})]\|_{L^2(\Omega;L^2(\mathcal{O}))} \\
&\leqslant \|e^{-(t-s)A}(I_M^* g - g)\|_{L^2(\Omega;L^2(\mathcal{O}))} + |f(0)|\|e^{-(t-s)A}(P_M 1 - 1)\|_{L^2(\Omega;L^2(\mathcal{O}))} \\
&\leqslant CM^{-1}\|f(U_M^{j-1}) - f(0)\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))} + C|f(0)|(t-s)^{-\frac{1}{2}}M^{-1}.
\end{aligned}$$

Then, using (3.36)–(3.37) and the inverse inequalities in (3.30)–(3.31), the second term on the right-hand side of (3.33) can be estimated as follows for $\alpha \in (0,1)$:

$$\begin{aligned}
&\|\mathcal{E}_2^n\|_{L^2(\Omega;L^2(\mathcal{O}))} \\
&\leqslant C\sum_{j=2}^n \int_{t_{j-1}}^{t_j} M^{-1}\big(\|f(U_M^{j-1}) - f(0)\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))} + |f(0)|(t-s)^{-\frac{1}{2}}\big)\mathrm{d}s \\
&\leqslant C\sum_{j=2}^n \int_{t_{j-1}}^{t_j} M^{-1}\big(\|U_M^{j-1}\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))} + (t-s)^{-\frac{1}{2}}\big)\mathrm{d}s \\
&\leqslant C\sum_{j=2}^n \int_{t_{j-1}}^{t_j} M^{-1}\big(\|U_M^{j-1} - P_M u(t_{j-1})\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))} + \|P_M u(t_{j-1})\|_{L^2(\Omega;\dot{H}^1(\mathcal{O}))}\big)\mathrm{d}s + CM^{-1} \\
&\leqslant C\sum_{j=2}^n \int_{t_{j-1}}^{t_j} \big(\|U_M^{j-1} - P_M u(t_{j-1})\|_{L^2(\Omega;L^2(\mathcal{O}))} + M^{-\alpha}\|P_M u(t_{j-1})\|_{B^\infty L^2(\Omega;\dot{H}^\alpha(\mathcal{O}))}\big)\mathrm{d}s + CM^{-1} \\
&\leqslant C\sum_{j=2}^n \int_{t_{j-1}}^{t_j} \big(\|U_M^{j-1} - P_M u(t_{j-1})\|_{L^2(\Omega;L^2(\mathcal{O}))} + M^{-\alpha}\|u(t_{j-1})\|_{B^\infty L^2(\Omega;\dot{H}^\alpha(\mathcal{O}))}\big)\mathrm{d}s + CM^{-1},
\end{aligned} \tag{3.38}$$

where we have used the inverse inequalities in (3.30)–(3.31) for $\alpha \in (0,1)$ and the stability of P_M on the Besov space. The stochastic Besov norm $\|u(t_{j-1})\|_{B^\infty L^2(\Omega;\dot H^\alpha(\mathcal{O}))}$ in the last inequality can be furthermore estimated by the regularity result in (2.17). In the case $\alpha = 1$, for trace class noise, we can simply replace the stochastic Besov norm $\|P_M u(t_{j-1})\|_{B^\infty L^2(\Omega;\dot H^\alpha(\mathcal{O}))}$ by the classical Sobolev norm $\|P_M u(t_{j-1})\|_{L^2(\Omega;\dot H^\alpha(\mathcal{O}))}$ and use the regularity result in Remark 2.2. In both cases, we can furthermore obtain the following result:

$$
\begin{aligned}
\|\mathcal{E}_2^n\|_{L^2(\Omega;L^2(\mathcal{O}))} &\leqslant C\sum_{j=2}^n \tau_j \|U_M^{j-1} - P_M u(t_{j-1})\|_{L^2(\Omega;L^2(\mathcal{O}))} + C\sum_{j=2}^n \tau_j t_{j-1}^{-\frac{\alpha-\beta}{2}} M^{-\alpha} + CM^{-1} \\
&\leqslant C\sum_{j=2}^n \tau_j \|U_M^{j-1} - u(t_{j-1})\|_{L^2(\Omega;L^2(\mathcal{O}))} + CM^{-\alpha}. \qquad (3.39)
\end{aligned}
$$

The third term on the right-hand side of (3.33) can be estimated directly by using the Lipschitz continuity of f, i.e.,

$$
\|\mathcal{E}_3^n\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant C\sum_{j=2}^n \tau_j \|U_M^{j-1} - u(t_{j-1})\|_{L^2(\Omega;L^2(\mathcal{O}))}. \qquad (3.40)
$$

The estimation for the fourth and fifth terms on the right-hand side of (3.33) are the technical parts in the proof, and are presented in the following two subsections, respectively.

3.3 Estimation of $\mathcal{E}_4^n$

Lemma 3.3 *Under Assumption 2.1, the remainder $\mathcal{E}_4^n$ in (3.33) has the following bound:*

$$
\|\mathcal{E}_4^n\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant C\tau^\alpha. \qquad (3.41)
$$

Proof. The estimation of $\mathcal{E}_4^n$ is by introducing an intermediate term $f(e^{-(s-t_{j-1})A}u(t_{j-1}))$ between $f(u(s))$ and $f(u(t_{j-1})$. By this means, we decompose $\mathcal{E}_4^n$ into the following two parts:

$$
\begin{aligned}
\mathcal{E}_4^n &= \sum_{j=2}^n \int_{t_{j-1}}^{t_j} e^{-(t_n-s)A}[f(u(s)) - f(u(t_{j-1})]\mathrm{d}s \\
&= \sum_{j=2}^n \int_{t_{j-1}}^{t_j} e^{-(t_n-s)A}[f(e^{-(s-t_{j-1})A}u(t_{j-1})) - f(u(t_{j-1})]\mathrm{d}s \\
&\quad + \sum_{j=1}^n \int_{t_{j-1}}^{t_j} e^{-(t_n-s)A}[f(u(s)) - f(e^{-(s-t_{j-1})A}u(t_{j-1}))]\mathrm{d}s \\
&=: \mathcal{E}_{4,1}^n + \mathcal{E}_{4,2}^n. \qquad (3.42)
\end{aligned}
$$

The two parts are estimated separately.

Since $A = -\Delta$, the first part on the right-hand side of (3.42) can be furthermore written as

$$
\begin{aligned}
\mathcal{E}_{4,1}^n &= -\sum_{j=2}^n \int_{t_{j-1}}^{t_j} e^{-(t_n-s)A} \int_0^{s-t_{j-1}} f'(e^{-\delta A}u(t_{j-1}))\Delta(e^{-\delta A}u(t_{j-1}))\mathrm{d}\delta\mathrm{d}s \\
&= -\sum_{j=2}^n \int_{t_{j-1}}^{t_j} e^{-(t_n-s)A} \int_0^{s-t_{j-1}} \nabla\cdot\Big(f'(e^{-\delta A}u(t_{j-1}))\nabla\big(e^{-\delta A}u(t_{j-1})\big)\Big)\mathrm{d}\delta\mathrm{d}s \\
&\quad + \sum_{j=2}^n \int_{t_{j-1}}^{t_j} e^{-(t_n-s)A} \int_0^{s-t_{j-1}} f''(e^{-\delta A}u(t_{j-1}))\Big|\nabla\big(e^{-\delta A}u(t_{j-1})\big)\Big|^2\mathrm{d}\delta\mathrm{d}s.
\end{aligned}
$$

Since $\dot H^1(\mathcal{O}) = H_0^1(\mathcal{O})$, it follows that $\dot H^{-1}(\mathcal{O}) = H_0^1(\mathcal{O})'$. As a result, the following result holds for all $\vec{h} \in L^2(\mathcal{O})^d$:

$$\|A^{-\frac{1}{2}}\nabla\cdot\vec{h}\|_{L^2(\mathcal{O})} = \|\nabla\cdot\vec{h}\|_{H^{-1}(\mathcal{O})} = \sup_{g\in H_0^1(\mathcal{O})}\frac{|(\nabla\cdot\vec{h},g)|}{\|g\|_{H_0^1(\mathcal{O})}} = \sup_{g\in H_0^1(\mathcal{O})}\frac{|(\vec{h},\nabla g)|}{\|g\|_{H_0^1(\mathcal{O})}} \leqslant C\|\vec{h}\|_{L^2(\mathcal{O})}.$$

By using this result and item (2) in Assumption 2.1, we have

$$\begin{aligned}
&\|\mathcal{E}_{4,1}^n\|_{L^2(\Omega;L^2(\mathcal{O}))}\\
&\leqslant C\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\int_0^{s-t_{j-1}}\Big\|A^{\frac{1}{2}}e^{-(t_n-s)A}A^{-\frac{1}{2}}\nabla\cdot\Big(f'(e^{-\delta A}u(t_{j-1}))\nabla\big(e^{-\delta A}u(t_{j-1})\big)\Big)\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}\delta\mathrm{d}s\\
&\quad+C\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\int_0^{s-t_{j-1}}\Big\|A^{\frac{\eta}{2}}e^{-(t_n-s)A}A^{-\frac{\eta}{2}}f''(e^{-\delta A}u(t_{j-1}))\big|\nabla\big(e^{-\delta A}u(t_{j-1})\big)\big|^2\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}\delta\mathrm{d}s\\
&\leqslant C\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\int_0^{s-t_{j-1}}(t_n-s)^{-\frac{1}{2}}\Big\|f'(e^{-\delta A}u(t_{j-1}))\nabla\big(e^{-\delta A}u(t_{j-1})\big)\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}\delta\mathrm{d}s\\
&\quad+C\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\int_0^{s-t_{j-1}}(t_n-s)^{-\frac{\eta}{2}}\big\|\nabla\big(e^{-\delta A}u(t_{j-1})\big)\big\|_{L^4(\Omega;L^2(\mathcal{O}))}^2\mathrm{d}\delta\mathrm{d}s.
\end{aligned}$$

The analyticity of the semigoup e^{-tA} implies the following estimates for all $1 \leqslant p \leqslant \infty$:

$$\begin{aligned}
\|e^{-\delta A}u(t)\|_{L^p(\Omega;\dot{H}^1(\mathcal{O}))} &\leqslant C\delta^{-\frac{1}{2}}\|u(t)\|_{L^p(\Omega;L^2(\mathcal{O}))},\\
\|e^{-\delta A}u(t)\|_{L^p(\Omega;\dot{H}^1(\mathcal{O}))} &\leqslant C\|u(t)\|_{L^p(\Omega;\dot{H}^1(\mathcal{O}))}.
\end{aligned}$$

By using Lemma 3.1, Lemma 3.2 and the result (2.17) in Proposition 2.1, we obtain

$$\begin{aligned}
\|e^{-\delta A}u(t)\|_{L^p(\Omega;\dot{H}^1(\mathcal{O}))} &\leqslant C\delta^{\frac{\alpha-1}{2}}\|u(t)\|_{(L^p(\Omega;L^2(\mathcal{O})),L^p(\Omega;\dot{H}^1(\mathcal{O})))_{\alpha,\infty}}\\
&\leqslant C\delta^{\frac{\alpha-1}{2}}\|u(t)\|_{B^\infty L^p(\Omega;\dot{H}^\alpha(\mathcal{O}))}\\
&\leqslant C\delta^{\frac{\alpha-1}{2}}t^{-\frac{\alpha-\beta}{2}}\|u^0\|_{L^p(\Omega;\dot{H}^\beta(\mathcal{O}))}
\end{aligned}$$

for $p \in \{2,4\}$ and $\beta \in (-1,\alpha]$. Therefore, by using the inequality above, we have

$$\begin{aligned}
\|\mathcal{E}_{4,1}^n\|_{L^2(\Omega;L^2(\mathcal{O}))} &\leqslant C\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\int_0^{s-t_{j-1}}(t_n-s)^{-\frac{1}{2}}\delta^{\frac{\alpha-1}{2}}t_{j-1}^{-\frac{\alpha-\beta}{2}}\mathrm{d}\delta\mathrm{d}s\|u^0\|_{L^2(\Omega;\dot{H}^\beta(\mathcal{O}))}\\
&\quad+C\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\int_0^{s-t_{j-1}}(t_n-s)^{-\frac{\eta}{2}}\delta^{\alpha-1}t_{j-1}^{-(\alpha-\beta)}\mathrm{d}\delta\mathrm{d}s\|u^0\|_{L^4(\Omega;\dot{H}^\beta(\mathcal{O}))}^2\\
&\leqslant C\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\Big((t_n-s)^{-\frac{1}{2}}\tau_j^{\frac{\alpha+1}{2}}t_{j-1}^{-\frac{\alpha-\beta}{2}}+(t_n-s)^{-\frac{\eta}{2}}\tau_j^{\alpha}t_{j-1}^{-(\alpha-\beta)}\Big)\mathrm{d}s\\
&\leqslant C\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\Big((t_n-s)^{-\frac{1}{2}}\tau^{\frac{\alpha+1}{2}}s^{\frac{\gamma(\alpha+1)-\alpha+\beta}{2}}+(t_n-s)^{-\frac{\eta}{2}}\tau^{\alpha}s^{\alpha\gamma-\alpha+\beta}\Big)\mathrm{d}s,
\end{aligned}\tag{3.43}$$

where the second to last inequality has used $\tau_j \sim t_j^\gamma\tau$. Since condition (2.22) implies

$$\frac{\gamma(\alpha+1)-\alpha+\beta}{2} > -1 \quad\text{and}\quad \alpha\gamma-\alpha+\beta > -1$$

it follows that

$$\|\mathcal{E}_{4,1}^n\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant C\tau^\alpha. \tag{3.44}$$

The second part on the right-hand side of (3.42) can be further decomposed into three parts by using Taylor's expansion, i.e.,

$$\begin{aligned}
\mathcal{E}_{4,2}^n &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{-(t_n-s)A} f'(e^{-(s-t_{i-1})A}u(t_{i-1}))\big(u(s) - e^{-(s-t_{i-1})A}u(t_{i-1})\big)\mathrm{d}s \\
&\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{-(t_n-s)A} f''(\xi_s)\big|u(s) - e^{-(s-t_{i-1})A}u(t_{i-1})\big|^2 \mathrm{d}s \\
&= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{-(t_n-s)A} f'(e^{-(s-t_{i-1})A}u(t_{i-1})) \int_{t_{i-1}}^{s} e^{-(s-\delta)A} f(u(\delta))\mathrm{d}\delta\mathrm{d}s \\
&\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{-(t_n-s)A} f'(e^{-(s-t_{i-1})A}u(t_{i-1})) \int_{t_{i-1}}^{s} e^{-(s-\delta)A} \mathrm{d}W(\delta)\mathrm{d}s \\
&\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} A^{\frac{\eta}{2}} e^{-(t_n-s)A} A^{-\frac{\eta}{2}} \Big(f''(\xi_s)\big|u(s) - e^{-(s-t_{i-1})A}u(t_{i-1})\big|^2\Big)\mathrm{d}s \\
&= \mathcal{E}_{*,1}^n + \mathcal{E}_{*,2}^n + \mathcal{E}_{*,3}^n. \qquad (3.45)
\end{aligned}$$

The first part on the right-hand side of (3.45) can be estimated directly by using the regularity results in Proposition 2.1, i.e.,

$$\begin{aligned}
\|\mathcal{E}_{*,1}^n\|_{L^2(\Omega;L^2(\mathcal{O}))} &\leqslant C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \|f(u(\delta))\|_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}\delta\mathrm{d}s \\
&\leqslant C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} (1 + \|u(\delta)\|_{L^2(\Omega;L^2(\mathcal{O}))})\mathrm{d}\delta\mathrm{d}s \\
&\leqslant C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} (1 + \delta^{\frac{\beta}{2}})\mathrm{d}\delta\mathrm{d}s \\
&\leqslant C \sum_{i=1}^n \Big(\tau_i + t_i^{1+\frac{\beta}{2}} - t_{i-1}^{1+\frac{\beta}{2}}\Big)\tau + C\tau \\
&\leqslant C\tau.
\end{aligned}$$

The second part on the right-hand side of (3.45) can be estimated by using Itô's isometry and item (3) in Assumption 2.1, as shown in (A.53) (see Appendix), i.e.,

$$\begin{aligned}
&\|\mathcal{E}_{*,2}^n\|_{L^2(\Omega;L^2(\mathcal{O}))}^2 \\
&\leqslant C \sum_{i=1}^n \Big\| \int_{t_{i-1}}^{t_i} e^{-(t_n-s)A} f'(e^{-(s-t_{i-1})A}u(t_{i-1})) \int_{t_{i-1}}^{s} e^{-(s-\delta)A}\mathrm{d}W(\delta)\mathrm{d}s \Big\|_{L^2(\Omega;L^2(\mathcal{O}))}^2 \\
&\leqslant C \sum_{i=1}^n \Big(\int_{t_{i-1}}^{t_i} \Big\| \int_{t_{i-1}}^{s} e^{-(s-\delta)A}\mathrm{d}W(\delta) \Big\|_{L^2(\Omega;L^2(\mathcal{O}))} \mathrm{d}s\Big)^2 \\
&\leqslant C \sum_{i=1}^n \Big(\int_{t_{i-1}}^{t_i} (s-t_{i-1})^{\frac{\alpha}{2}}\mathrm{d}s\Big)^2 \quad \text{(here (A.53) in Appendix is used)} \\
&\leqslant C\tau^{1+\alpha}.
\end{aligned}$$

This proves the following result:

$$\|\mathcal{E}_{*,2}^n\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant C\tau^{\frac{1+\alpha}{2}}.$$

The third part on the right-hand side of (3.45) can be estimated by using the following identity for $s \in [t_{i-1}, t_i]$:

$$u(s) = e^{-(s-t_{i-1})A}u(t_{i-1}) + \int_{t_{i-1}}^{s} e^{-(s-t)A} f(u(t))\mathrm{d}t + \int_{t_{i-1}}^{s} e^{-(s-t)A}\mathrm{d}W(t),$$

which implies that

$$
\begin{aligned}
&\|u(s)-e^{-(s-t_{i-1})A}u(t_{i-1})\|_{L^4(\Omega;L^2(\mathcal{O}))}\\
&\leqslant \Big\|\int_{t_{i-1}}^{s} e^{-(s-t)A}f(u(t))\mathrm{d}t\Big\|_{L^4(\Omega;L^2(\mathcal{O}))}+\Big\|\int_{t_{i-1}}^{s} e^{-(s-t)A}\mathrm{d}W(t)\Big\|_{L^4(\Omega;L^2(\mathcal{O}))}\\
&\leqslant C\int_{t_{i-1}}^{s}(1+\|u(t)\|_{L^4(\Omega;L^2(\mathcal{O}))})\mathrm{d}t+C\Big\|\int_{0}^{s-t_{i-1}} e^{-(s-t_{i-1}-t)A}\mathrm{d}W(t)\Big\|_{L^4(\Omega;L^2(\mathcal{O}))}\\
&\leqslant C\int_{t_{i-1}}^{s}(1+t^{\frac{\beta}{2}})\mathrm{d}t+C\Big\|\int_{0}^{s-t_{i-1}} e^{-(s-t_{i-1}-t)A}\mathrm{d}W(t)\Big\|_{L^4(\Omega;L^2(\mathcal{O}))}\\
&\leqslant C\tau^{\frac{\alpha}{2}},
\end{aligned}
$$

where the second to last inequality uses the regularity results in Proposition 2.1. Then, by using item (2) in Assumption 2.1 on the nonlinearity, we have

$$
\begin{aligned}
&\|\mathcal{E}^n_{*,3}\|_{L^2(\Omega;L^2(\mathcal{O}))}\\
&\leqslant C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{-\frac{\eta}{2}}\Big\|A^{-\frac{\eta}{2}}\Big(f''(\xi_s)|u(s)-e^{-(s-t_{i-1})A}u(t_{i-1})|^2\Big)\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}s\\
&\leqslant C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{-\frac{\eta}{2}}\|u(s)-e^{-(s-t_{i-1})A}u(t_{i-1})\|^2_{L^4(\Omega;L^2(\mathcal{O}))}\mathrm{d}s\\
&\leqslant C\sum_{i=1}^{n}\Big((t_n-t_{i-1})^{1-\frac{\eta}{2}}-(t_n-t_i)^{1-\frac{\eta}{2}}\Big)\tau^{\alpha}\\
&\leqslant C\tau^{\alpha}.
\end{aligned}
$$

By substituting the estimates of $\|\mathcal{E}^n_{*,j}\|_{L^2(\Omega;L^2(\mathcal{O}))}$, $j=1,2,3$, into (3.45), we obtain

$$
\|\mathcal{E}^n_{4,2}\|_{L^2(\Omega;L^2(\mathcal{O}))}\leqslant C\tau^{\alpha}. \tag{3.46}
$$

Estimates (3.44) and (3.46) imply the desired result of Lemma 3.3. □

3.4 Estimation of $\mathcal{E}^n_5$

The last term on the right-hand side of (3.33) can be estimated as follows, by considering two different cases and using the real interpolation method between two Besov spaces.

Case 1: $t_n>M^{-2}$. On the one hand, for any $\delta\in(0,\alpha)$ the following estimate holds:

$$
\begin{aligned}
\|\Pi_j\mathcal{E}^n_5\|^2_{L^2(\Omega;\dot H^{\delta}(\mathcal{O}))}&=\|A^{\frac{\delta}{2}}\Pi_j\mathcal{E}^n_5\|^2_{L^2(\Omega;L^2(\mathcal{O}))}\\
&=\int_{t_1}^{t_n}\|A^{\frac{\delta}{2}}\Pi_j(e^{-(t_n-s)A}-e^{-(t_n-s)A}P_M)\|^2_{L^2(\Omega;\mathcal{L}_2^0)}\mathrm{d}s\\
&\leqslant C\int_0^{t_n-M^{-2}}\sum_{k=2^{j-1}}^{2^j-1}\mu_k M^{-2\alpha}(t_n-s)^{-(\alpha+\delta+1-\alpha)}\|\phi_k\|^2_{\dot H^{-(1-\alpha)}}\mathrm{d}s\\
&\quad+C\int_{t_n-M^{-2}}^{t_n}\sum_{k=2^{j-1}}^{2^j-1}\mu_k(t_n-s)^{-(\delta+1-\alpha)}\|\phi_k\|^2_{\dot H^{-(1-\alpha)}}\mathrm{d}s\\
&\leqslant CM^{-2\alpha}\int_0^{t_n-M^{-2}}(t_n-s)^{-(\delta+1)}\mathrm{d}s\sum_{k=2^{j-1}}^{2^j-1}\mu_k\lambda_k^{\alpha-1}\\
&\quad+C\int_{t_n-M^{-2}}^{t_n}(t_n-s)^{-(\delta+1-\alpha)}\mathrm{d}s\sum_{k=2^{j-1}}^{2^j-1}\mu_k\lambda_k^{\alpha-1}
\end{aligned}
$$

$$\leqslant C\Big(M^{-2\alpha}(M^{2\delta}-t_n^{-\delta})+CM^{-2\alpha+2\delta}\Big)\sum_{k=2^{j-1}}^{2^j-1}\mu_k\lambda_k^{\alpha-1},$$

where

$$\sum_{k=2^{j-1}}^{2^j-1}\mu_k\lambda_k^{\alpha-1}\leqslant C\Big\|\int_0^T e^{-(T-s)A}\mathrm{d}W(s)\Big\|_{B^\infty L^2(\Omega;\dot H^\alpha(\mathcal{O}))}\leqslant C.$$

As a consequence,

$$\|\Pi_j\mathcal{E}_5^n\|^2_{L^2(\Omega;\dot H^\delta(\mathcal{O}))}\leqslant CM^{-2\alpha+2\delta}.$$

On the other hand, by choosing a constant δ_0 such that $\delta_0-\delta>0$, we have

$$\begin{aligned}\|\Pi_j\mathcal{E}_5^n\|^2_{L^2(\Omega;\dot H^{-\delta}(\mathcal{O}))}&=\int_0^{t_n}\sum_{k=2^{j-1}}^{2^j-1}\mu_k\|A^{-\frac{\delta}{2}}(e^{-(t_n-s)A}-e^{-(t_n-s)A}P_M)\phi_k\|^2_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}s\\&\leqslant CM^{-2\alpha-2\delta_0}\int_0^{t_n-M^{-2}}(t_n-s)^{-(\alpha+\delta_0-\delta+1-\alpha)}\mathrm{d}s\sum_{k=2^{j-1}}^{2^j-1}\mu_k\lambda_k^{\alpha-1}\\&\quad+CM^{-2\delta_0}\int_{t_n-M^{-2}}^{t_n}(t_n-s)^{-(\delta_0-\delta+1-\alpha)}\mathrm{d}s\sum_{k=2^{j-1}}^{2^j-1}\mu_k\lambda_k^{\alpha-1}\\&\leqslant C\Big(M^{-2\alpha-2\delta_0}(M^{2(\delta_0-\delta)}-t_n^{-(\delta_0-\delta)})/(\delta_0-\delta)+M^{-2\alpha-2\delta}\Big)\sum_{k=2^{j-1}}^{2^j-1}\mu_k\lambda_k^{\alpha-1}\\&\leqslant CM^{-2\alpha-2\delta}.\end{aligned}$$

Case 2: $t_n\leqslant M^{-2}$. In this case, the integral from M^{-2} to t_n vanishes in the estimates above. The integral from 0 to M^{-2} can be estimated similarly.

Overall, in both cases, the following estimate holds for all $\delta\in(0,\alpha)$ and and $t_n\in[t_1,T]$:

$$\|\Pi_j\mathcal{E}_5^n\|_{B^\infty L^2(\Omega;\dot H^{\pm\delta}(\mathcal{O}))}\leqslant CM^{-\alpha\pm\delta}.$$

By using the real interpolation method and the results in Lemma 3.2, we obtain

$$\|\mathcal{E}_5^n\|_{L^2(\Omega;L^2(\mathcal{O}))}=\|\mathcal{E}_5^n\|_{(B^\infty L^2(\Omega;\dot H^{-\delta}(\mathcal{O})),B^\infty L^2(\Omega;\dot H^{\delta}(\mathcal{O})))_{\frac12,2}}\leqslant CM^{-\alpha}.\tag{3.47}$$

3.5 Completion of the proof

To conclude, by substituting the estimates of $\mathcal{E}_k$, $k=1,2,\cdots,5$, into (3.33), we obtain that

$$\begin{aligned}&\|U_M^n-u(t_n)\|_{L^2(\Omega;L^2(\mathcal{O}))}\\&\leqslant CM^{-\alpha}t_n^{-\frac{\alpha-\beta}{2}}\|u^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))}+\sum_{j=2}^n\tau_j\|U_M^{j-1}-u(t_{j-1})\|_{L^2(\Omega;L^2(\mathcal{O}))}+C\tau^\alpha\\&\leqslant CM^{-\alpha}t_n^{-\frac{\alpha-\beta}{2}}\|u^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))}+\sum_{j=1}^{n-1}\tau_j\|U_M^{j}-u(t_{j})\|_{L^2(\Omega;L^2(\mathcal{O}))}+C\tau^\alpha\quad\text{for }n\geqslant 2.\end{aligned}\tag{3.48}$$

By comparing the first relation of (2.20) with (1.3), we obtain that

$$\begin{aligned}&\tau_1\|U_M^1-u(t_1)\|_{L^2(\Omega;L^2(\mathcal{O}))}\\&\leqslant\tau_1\|e^{-\tau_1A}U_M^0-e^{-\tau_1A}u^0\|_{L^2(\Omega;L^2(\mathcal{O}))}\\&\quad+\tau_1\Big\|\int_0^{\tau_1}e^{-(\tau_1-s)A}f(u(s))\mathrm{d}s+\int_0^{\tau_1}e^{-(\tau_1-s)A}\mathrm{d}W(s)\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}\\&\leqslant C(\tau_1^{1+\frac{\beta}{2}}+\tau_1)\|u^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))}+\tau_1\Big\|\int_0^{\tau_1}(C+Cs^{\frac{\beta}{2}})\mathrm{d}s\Big\|_{L^2(\Omega;L^2(\mathcal{O}))}+C\tau_1^{1+\frac{\alpha}{2}}\end{aligned}$$

$$\leqslant C\tau_1^{1+\frac{\beta}{2}} + C\tau_1 + C\tau_1^{2+\frac{\beta}{2}} + C\tau_1^{1+\frac{\alpha}{2}} \leqslant C\tau^{\frac{1}{1-\gamma}(1+\frac{\beta}{2})} + C\tau^{\frac{1}{1-\gamma}},$$

where we have used the property $\tau_1 = O(\tau^{\frac{1}{1-\gamma}})$. Since condition (2.22) implies $\frac{1}{1-\gamma}(1+\frac{\beta}{2}) \geqslant \alpha$ and $\frac{1}{1-\gamma} > 2 > \alpha$, it follows that

$$\tau_1 \|U_M^1 - u(t_1)\|_{L^2(\Omega;L^2(\mathcal{O}))} \leqslant C\tau^\alpha.$$

Substituting this into (3.48) yields

$$\begin{aligned}
&\|U_M^n - u(t_n)\|_{L^2(\Omega;L^2(\mathcal{O}))} \\
&\leqslant CM^{-\alpha} t_n^{-\frac{\alpha-\beta}{2}} + \sum_{j=2}^{n-1} \tau_j \|U_M^j - u(t_j)\|_{L^2(\Omega;L^2(\mathcal{O}))} + C\tau^\alpha \quad \text{for } n \geqslant 2.
\end{aligned} \tag{3.49}$$

Then, by applying the discrete Gronwall's inequality, we obtain

$$\|U_M^n - u(t_n)\| \leqslant CM^{-\alpha} t_n^{-\frac{\alpha-\beta}{2}} + C\tau^\alpha \quad \text{for } n \geqslant 2.$$

This proves the result of Theorem 2.2. □

4 Numerical experiments

In this section, we present numerical results to support the theoretical analysis. All computations are performed by Matlab with double precision (see [15] for algorithmic implementation on Matlab for stochastic differential equations).

Let $\mathcal{O} = [0, 1]$ and $T = 0.5$. We solve problem (1.1) by the proposed modified exponential Euler scheme with Fourier collocation method in (2.20), with the nonlinear drift function

$$f(u) = \sqrt{1 + u^2},$$

which satisfies items (1)–(2) in Assumption 2.1. The following two deterministic initial values are tested

$$u_1^0(x) = \sin(\pi x) \quad \text{and} \quad u_2^0(x) = \delta\Big(x - \frac{1}{2}\Big),$$

where $u_1^0 \in H_0^1(\mathcal{O}) \cap C^\infty(\mathcal{O})$ and $u_2^0 \in \dot{H}^{-\frac{1}{2}-\varepsilon}(\mathcal{O})$ is the Dirac delta function, where $\varepsilon > 0$ can be an arbitrary small number.

The implementation of the numerical method is simple, i.e., the nonlinear term can be calculated by

$$\int_{t_{n-1}}^{t_n} e^{-(t_n-s)A} I_M f(U_M^{n-1}) \mathrm{d}s = \left(\frac{1 - e^{-\tau_n A}}{A}\right) I_M f(U_M^{n-1}),$$

where I_M can be calculated by using FFT with $O(M \ln M)$ operations at every time level.

The noise term can be calculated by

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} e^{-(t_n-s)A} P_M \mathrm{d}W(s) &= \sum_{k=1}^M \sqrt{\mu_k}\phi_k \int_{t_{n-1}}^{t_n} e^{-(t_n-s)\lambda_k} \mathrm{d}W_k(s) \\
&= \sum_{k=1}^M \sqrt{\mu_k}\phi_k \left(\frac{1 - e^{-2\tau_n\lambda_k}}{2\lambda_k}\right)^{\frac{1}{2}} \xi_k^n
\end{aligned}$$

with independent and standard normally distributed random variables ξ_k^n for $1 \leqslant k \leqslant M$ and $1 \leqslant n \leqslant N$. If the noises ξ_k^n are generated with a fine mesh in time, then the following identity can be used to calculate the numerical solution with a coarse mesh in time (with stepsize $t_{n+m} - t_n$):

$$\int_{t_n}^{t_{n+m}} e^{-(t_{n+m}-s)\lambda_k} \mathrm{d}W_k(s) = \sum_{j=1}^m e^{-(t_{n+m}-t_{n+j})\lambda_k} \int_{t_{n+j-1}}^{t_{n+j}} e^{-(t_{n+j}-s)\lambda_k} \mathrm{d}W_k(s)$$

$$= \sum_{j=1}^{m} e^{-(t_{n+m}-t_{n+j})\lambda_k} \left(\frac{1-e^{-2\tau_{n+j}\lambda_k}}{2\lambda_k}\right)^{\frac{1}{2}} \xi_k^{n+j} \quad \forall m \geqslant 1.$$

This allows us to test the errors and convergence orders by using a reference solution with a very fine mesh in time.

To test the spatial convergence orders, we fix a sufficiently small time stepsize $\tau = 2^{-10}$ and calculate the error by

$$E_1(M) = \left(\frac{1}{I}\sum_{i=1}^{I} \|U_{\tau,M}^N(\omega_i) - U_{\tau,2M}^N(\omega_i)\|_{L^2(\mathcal{O})}^2\right)^{\frac{1}{2}}$$

for $M = 16, 32, 64, 128$, i.e., the expectations of errors over $I = 1000$ samples at $t = T$, and then present them in Tables 1–2 for different initial data.

Table 1: Spatial discretization error $E_1(M)$ with initial data u_1^0 and $\gamma = 0.7$

$\mu_k \backslash M$	16	32	64	128	Order
$\mu_k \equiv 1$	3.866e-2	2.786e-2	1.980e-2	1.402e-2	$\approx 0.50\,(0.50)$
$\mu_k = 1/k^{0.5}$	1.787e-2	1.076e-2	6.469e-3	3.859e-3	$\approx 0.75\,(0.75)$
$\mu_k = 1/k^{0.8}$	1.126e-2	6.188e-3	3.326e-3	1.780e-3	$\approx 0.90\,(0.90)$
$\mu_k = 1/k$	8.243e-3	4.238e-3	2.126e-3	1.071e-3	$\approx 0.99\,(1.00)$
$\mu_k = 1/k^{1.1}$	7.156e-3	3.506e-3	1.710e-3	8.311e-4	$\approx 1.04\,(1.00)$

Table 2: Spatial discretization error $E_1(M)$ with initial data u_2^0 and $\gamma = 0.7$

$\mu_k \backslash M$	16	32	64	128	Order
$\mu_k \equiv 1$	3.858e-2	2.776e-2	1.971e-2	1.403e-2	$\approx 0.49\,(0.50)$
$\mu_k = 1/k^{0.5}$	1.787e-2	1.080e-2	6.443e-3	3.853e-3	$\approx 0.74\,(0.75)$
$\mu_k = 1/k^{0.8}$	1.127e-2	6.164e-3	3.310e-3	1.788e-3	$\approx 0.89\,(0.90)$
$\mu_k = 1/k$	8.261e-3	4.205e-3	2.139e-3	1.072e-3	$\approx 1.00\,(1.00)$
$\mu_k = 1/k^{1.1}$	7.130e-3	3.511e-3	1.705e-3	8.293e-4	$\approx 1.04\,(1.00)$

In the case $\mu_k = 1/k^\delta$ $(0 \leqslant \delta < 1)$, the noise satisfies Assumption 2.1 (3) with $\alpha = \frac{1+\delta}{2}$. This order of convergence in space is well illustrated by the numerical results in Tables 1–2. From Tables 2 we see that the modified exponential Euler method with Fourier collocation method in space is robust with respect to the regularity of the initial data, including measure-valued functions such as the Diract delta function.

To test the temporal convergence orders, we choose $M = N$ and calculate the error by

$$E_2(\tau) = \left(\frac{1}{I}\sum_{i=1}^{I} \|U_{\tau,N}^N(\omega_i) - U_{\tau/2,N_2}^{N_2}(\omega_i)\|_{L^2(\mathcal{O})}^2\right)^{\frac{1}{2}},$$

where N_2 is the number of the time levels for time stepsize $\tau/2$. By Theorem 2.2, the spatial convergence order equals the temporal convergence order. The numerical results are given in Tables 3–4, where the observed temporal convergence orders are consistent with the theoretical result proved in Theorem 2.2.

Table 3: Temporal discretization error $E_2(\tau)$ with initial data u_1^0 and $\gamma = 0.7$

$\mu_k \backslash \tau$	1/16	1/32	1/64	1/128	Order
$\mu_k \equiv 1$	2.401e-2	1.665e-2	1.192e-2	8.476e-3	$\approx 0.49\,(0.50)$
$\mu_k = 1/k^{0.5}$	8.661e-3	5.055e-3	3.040e-3	1.824e-3	$\approx 0.74\,(0.75)$
$\mu_k = 1/k^{0.8}$	4.739e-3	2.509e-3	1.362e-3	7.315e-4	$\approx 0.90\,(0.90)$
$\mu_k = 1/k$	3.187e-3	1.594e-3	8.085e-4	4.047e-4	$\approx 1.00\,(1.00)$
$\mu_k = 1/k^{1.1}$	2.671e-3	1.284e-3	6.283e-4	3.051e-4	$\approx 1.04\,(1.00)$

Table 4: Temporal discretization error $E_2(\tau)$ with initial data u_2^0 and $\gamma = 0.7$

$\mu_k \backslash \tau$	1/16	1/32	1/64	1/128	Order
$\mu_k \equiv 1$	2.399e-2	1.668e-2	1.187e-2	8.478e-3	$\approx 0.49\,(0.50)$
$\mu_k = 1/k^{0.5}$	8.659e-3	5.056e-3	3.054e-3	1.824e-3	$\approx 0.74\,(0.75)$
$\mu_k = 1/k^{0.8}$	4.728e-3	2.501e-3	1.361e-3	7.319e-4	$\approx 0.90\,(0.90)$
$\mu_k = 1/k$	3.202e-3	1.587e-3	8.072e-4	4.042e-4	$\approx 1.00\,(1.00)$
$\mu_k = 1/k^{1.1}$	2.664e-3	1.294e-3	6.379e-4	3.085e-4	$\approx 1.05\,(1.00)$

5 Conclusions

We have considered a modified exponential Euler method for the semilinear stochastic heat equation, with Fourier Galerkin and Fourier collocation method in space. Some new techniques are introduced to the error analysis, including the stochastic Besov spaces and its interpolation properties to characterize the noises, and a class of locally refined variable stepsizes to resolve the singularity of the solution at $t = 0$. By using these new techniques, we have proved that the method has αth-order convergence for initial data in $L^4(\Omega; H^\beta(\mathcal{O}))$ with $\beta \in (-1, \alpha]$, for a class of noises characterized by a parameter $\alpha \in (0, 1]$, which includes trace-class noises (with $\alpha = 1$) and one-dimensional space-time white noises (with $\alpha = \frac{1}{2}$). The numerical results also support the theoretical analysis.

In the numerical schemes of (2.19) and (2.20), we have used variable stepsizes and modified the exponential integrator at the initial time level to address the singularity of the solution at $t = 0$. This is needed when $\beta < 0$ because the initial data u^0 may not be a pointwisely defined function and therefore the term $f(u^0)$ in the standard exponential integrator may not be pointwisely well-defined. In the case $0 \leqslant \beta \leqslant \alpha$, the variable stepsize and the modification of the initial step may not be necessary. However, since the estimate of $\mathcal{E}_{4,1}^n$ in (3.43) involves $t_{j-1}^{-(\alpha-\beta)/2}$, the estimation of this term at the initial step needs to be changed to a different way in the case $0 \leqslant \beta < \alpha$.

In estimates (3.38)–(3.39), we have used the additional assumption that in the case $\alpha = 1$ the noise is trace class; see in Assumption 2.1 (3). This is only needed for the Fourier sine collocation method in (2.20) with trigonometric interpolation. Theorem 2.1 (for the spectral Galerkin method) still holds without requiring the noise to be trace class in the case $\alpha = 1$. This is because that $e^{-(t_n-s)A}$ commutes with P_M and therefore (3.38) can be estimated in the following different way:

$$\|\mathcal{E}_2^n\|_{L^2(\Omega;L^2(\mathcal{O}))} = \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \big(P_M e^{-(t_n-s)A} f(U_M^{j-1}) - e^{-(t_n-s)A} f(U_M^{j-1})\big) ds$$

$$
\begin{aligned}
&\leqslant C\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} M^{-1}\|e^{-(t_n-s)A}f(U_M^{j-1})\|_{L^2(\Omega;\dot H^1(\mathcal{O}))}\mathrm{d}s\\
&\leqslant C\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} M^{-1}(t_n-s)^{-\frac12}\|f(U_M^{j-1})\|_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}s\\
&\leqslant C\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} M^{-1}(t_n-s)^{-\frac12}(\|f(0)\|_{L^2(\Omega;L^2(\mathcal{O}))}+\|U_M^{j-1}\|_{L^2(\Omega;L^2(\mathcal{O}))})\mathrm{d}s
\end{aligned}
$$

where we have used the smoothing property of the analytic semigroup e^{-tA}, i.e.,

$$
\|e^{-(t_n-s)A}g\|_{\dot H^1(\mathcal{O})}\leqslant C(t_n-s)^{-\frac12}\|g\|_{L^2(\mathcal{O})}\quad\text{for } g\in L^2(\mathcal{O}).
$$

From (2.19a) we see that

$$
\begin{aligned}
\|U_M^1\|_{L^2(\Omega;L^2(\mathcal{O}))}=\|e^{-\tau_1A}P_Mu^0\|_{L^2(\Omega;L^2(\mathcal{O}))}&=\|A^{-\frac{\beta}{2}}e^{-\tau_1A}P_Mu^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))}\\
&\leqslant C(1+t_1^{\frac{\beta}{2}})\|u^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))}. \qquad (5.50)
\end{aligned}
$$

From (2.19b) we can obtain the following expression of U_M^n similarly as (3.32) (with I_M replaced by P_M therein):

$$
U_M^n=e^{-(t_n-\tau_1)A}U_M^1+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} e^{-(t_n-s)A}P_Mf(U_M^{j-1})ds+\int_{t_1}^{t_n} e^{-(t_n-s)A}P_MdW(s),
$$

which implies that

$$
\begin{aligned}
&\|U_M^n\|_{L^2(\Omega;L^2(\mathcal{O}))}\\
&\leqslant \|e^{-(t_n-\tau_1)A}U_M^1\|_{L^2(\Omega;L^2(\mathcal{O}))}+\sum_{j=2}^{n}\tau_j\|f(U_M^{j-1})\|_{L^2(\Omega;L^2(\mathcal{O}))}+t_n^{\frac{\alpha}{2}}\quad\text{(here (2.10) is used)}\\
&\leqslant \|A^{-\frac{\beta}{2}}e^{-t_nA}P_Mu^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))}+\sum_{j=2}^{n}\tau_j\left(\|U_M^{j-1}\|_{L^2(\Omega;L^2(\mathcal{O}))}+\|f(0)\|_{L^2(\Omega;L^2(\mathcal{O}))}\right)+t_n^{\frac{\alpha}{2}}\\
&\leqslant (1+t_n^{\frac{\beta}{2}})\|u^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))}+\sum_{j=2}^{n}\tau_j\|U_M^{j-1}\|_{L^2(\Omega;L^2(\mathcal{O}))}\qquad\text{for } 2\leqslant n\leqslant N,
\end{aligned}
$$

where we have used the Lipschitz continuity of f in the second to last inequality. Applying the discrete Gronwall inequality to this equation, together with equation (5.50), we can derive

$$
\|U_M^n\|_{L^2(\Omega;L^2(\mathcal{O}))}\leqslant C(1+t_n^{\frac{\beta}{2}})\quad\text{for } 1\leqslant n\leqslant N.
$$

Therefore,

$$
\begin{aligned}
\|\mathcal{E}_2^n\|_{L^2(\Omega;L^2(\mathcal{O}))}&\leqslant C\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} M^{-1}(t_n-s)^{-\frac12}(1+t_{j-1}^{\frac{\beta}{2}})\mathrm{d}s\\
&\leqslant C\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} M^{-1}(t_n-s)^{-\frac12}(1+s^{\frac{\beta}{2}})\mathrm{d}s\\
&\leqslant CM^{-1}. \qquad (5.51)
\end{aligned}
$$

For the Fourier sine collocation method in (2.20), if the noise is not trace class in the case $\alpha=1$, Theorem 2.2 can still proved by using the inverse inequality (proof is omitted)

$$
\|P_Mu(t_{j-1})\|_{L^2(\Omega;\dot H^1(\mathcal{O}))}\leqslant C(\ln M)^{\frac12}\|P_Mu(t_{j-1})\|_{B^\infty L^2(\Omega;\dot H^1(\mathcal{O}))}.
$$

This loses a logarithmic order of convergence in the case $\alpha=1$.

Appendix: Proof of Proposition 2.1

A.1 Existence and uniqueness

We prove the existence and uniqueness of mild solutions by using the Banach fixed point theorem.

For $v \in X = \big\{ v \in L^1\big(0,T; L^2(\Omega; L^2(\mathcal{O}))\big) : \sup_{t\in(0,T]} (1+t^{\frac{\beta}{2}})^{-1} \|v(t)\|_{L^2(\Omega;L^2(\mathcal{O}))} < \infty \big\}$ we define a nonlinear operator $M : X \to X$ by

$$Mv(t) = e^{-tA}u^0 + \int_0^t e^{-(t-s)A} f(v(s))\mathrm{d}s + \int_0^t e^{-(t-s)A}\mathrm{d}W(s),$$

which is well-defined as

$$\begin{aligned}\|Mv(t)\|_{L^2(\Omega;L^2(\mathcal{O}))} &\leqslant C(1+t^{\frac{\beta}{2}})\|u^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))} + C\int_0^t (1+\|v(s)\|_{L^2(\Omega;L^2(\mathcal{O}))})\mathrm{d}s + Ct^{\frac{\alpha}{2}} \\ &\leqslant C(1+t^{\frac{\beta}{2}})\|u^0\|_{L^2(\Omega;\dot H^\beta(\mathcal{O}))} + C\int_0^t (1+s^{\frac{\beta}{2}})\|v\|_X \mathrm{d}s + Ct^{\frac{\alpha}{2}}.\end{aligned}$$

We consider the space X_λ, which is defined as the vector space X with the equivalent norm

$$\|v\|_{X_\lambda} := \sup_{t\in(0,T]} e^{-\lambda t}(1+t^{\frac{\beta}{2}})^{-1}\|v(t)\|_{L^2(\Omega;L^2(\mathcal{O}))},$$

where $\lambda \geqslant 1$ is a fixed constant to be determined later. Therefore, $v \in X_\lambda$ if and only if $v \in X$. If $v_1, v_2 \in X_\lambda$ then

$$\begin{aligned}&e^{-\lambda t}(1+t^{\frac{\beta}{2}})^{-1}\|Mv_1(t) - Mv_2(t)\|_{L^2(\Omega;L^2(\mathcal{O}))} \\ &\leqslant C\int_0^t e^{-\lambda t}\|f(v_1(s)) - f(v_2(s))\|_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}s \\ &\leqslant C\int_0^t e^{-\lambda(t-s)}(1+s^{\frac{\beta}{2}})e^{-\lambda s}(1+s^{\frac{\beta}{2}})^{-1}\|v_1(s)-v_2(s)\|_{L^2(\Omega;L^2(\mathcal{O}))}\mathrm{d}s \\ &\leqslant C\|v_1-v_2\|_{X_\lambda}\int_0^t e^{-\lambda(t-s)}(1+s^{\frac{\beta}{2}})\mathrm{d}s \\ &\leqslant C\|v_1-v_2\|_{X_\lambda}\int_0^{\lambda t} e^{-\lambda t+\delta}(\lambda^{-1}+\lambda^{-\frac{\beta}{2}-1}\delta^{\frac{\beta}{2}})\mathrm{d}\delta \\ &\leqslant C\|v_1-v_2\|_{X_\lambda}\Big(\frac{1-e^{-\lambda t}}{\lambda} + \lambda^{-\frac{\beta}{2}-1}\int_0^1 e^{-\lambda t+\delta}\delta^{\frac{\beta}{2}}\mathrm{d}\delta + \lambda^{-\frac{\beta}{2}-1}\int_1^{\max\{\lambda t,1\}} e^{-\lambda t+\delta}\delta^{\frac{\beta}{2}}\mathrm{d}\delta\Big) \\ &\leqslant C\|v_1-v_2\|_{X_\lambda}\Big(\lambda^{-1} + \lambda^{-\frac{\beta}{2}-1}\int_0^1 \delta^{\frac{\beta}{2}}\mathrm{d}\delta + \lambda^{-\frac{\beta}{2}-1}\int_1^{\max\{\lambda t,1\}} e^{-\lambda t+\delta}\delta^{\frac{\beta}{2}}\mathrm{d}\delta\Big).\end{aligned}$$

Since

$$\begin{aligned}\lambda^{-\frac{\beta}{2}-1}\textstyle\int_1^{\max\{\lambda t,1\}} e^{-\lambda t+\delta}\delta^{\frac{\beta}{2}}\mathrm{d}\delta &\leqslant \begin{cases}\lambda^{-\frac{\beta}{2}-1}\int_1^{\max\{\lambda t,1\}} e^{-\lambda t+\delta}\mathrm{d}\delta & \text{if } \beta\in(-1,0] \\ \lambda^{-\frac{\beta}{2}-1}\int_1^{\max\{\lambda t,1\}} e^{-\lambda t+\delta}(\lambda t)^{\frac{\beta}{2}}\mathrm{d}\delta & \text{if } \beta\in(0,\alpha]\end{cases} \\ &\leqslant \begin{cases}\lambda^{-\frac{\beta}{2}-1} & \text{if } \beta\in(-1,0] \\ \lambda^{-1} & \text{if } \beta\in(0,\alpha],\end{cases}\end{aligned}$$

it follows that

$$\begin{aligned}e^{-\lambda t}(1+t^{\frac{\beta}{2}})^{-1}\|Mv_1(t)-Mv_2(t)\|_{L^2(\Omega;L^2(\mathcal{O}))} &\leqslant C\lambda^{-\frac{\min(\beta,0)}{2}-1}\|v_1-v_2\|_{X_\lambda} \\ &\leqslant C\lambda^{-\frac{1}{2}}\|v_1-v_2\|_{X_\lambda} \quad \text{for } \beta\in(-1,\alpha].\end{aligned}$$

Therefore, M is a contraction map on X_λ when λ is sufficiently large. This and the Banach fixed point theorem imply that there exists a unique fixed point of M on $X_\lambda = X$. This fixed point of M is denoted by u, which is the mild solution of problem (1.1).

A.2 Regularity

By using the expression of the mild solution in (1.3) and the property of the noise in (2.10), we have

$$\|u(t)\|_{L^p(\Omega;L^2(\mathcal{O}))} \leqslant \|A^{-\frac{\beta}{2}}e^{-tA}A^{\frac{\beta}{2}}u^0\|_{L^p(\Omega;L^2(\mathcal{O}))} + \big\|\textstyle\int_0^t e^{-(t-s)A}f(u(s))ds\big\|_{L^p(\Omega;L^2(\mathcal{O}))} + \big\|\textstyle\int_0^t e^{-(t-s)A}dW(s)\big\|_{L^p(\Omega;L^2(\mathcal{O}))}$$

Since e^{-tA} is an analytic semigroup, it follows that

$$\|A^{-\frac{\beta}{2}}e^{-tA}A^{\frac{\beta}{2}}u^0\|_{L^p(\Omega;L^2(\mathcal{O}))} \leqslant \begin{cases} Ct^{\frac{\beta}{2}}\|A^{\frac{\beta}{2}}u^0\|_{L^p(\Omega;L^2(\mathcal{O}))} & \text{if } \beta\in(-1,0], \\ C\|A^{\frac{\beta}{2}}u^0\|_{L^p(\Omega;L^2(\mathcal{O}))} & \text{if } \beta\in(0,\alpha]. \end{cases}$$

And with the help of (2.8) and (2.9) it follows that

$$\big\|\textstyle\int_0^t e^{-(t-s)A}dW(s)\big\|_{L^p(\Omega;L^2(\mathcal{O}))} \leqslant C\big\|\textstyle\int_0^t e^{-(t-s)A}dW(s)\big\|_{L^2(\Omega;L^2(\mathcal{O}))}$$

holds for $p\geqslant 2$. Therefore,

$$\begin{aligned}\|u(t)\|_{L^p(\Omega;L^2(\mathcal{O}))} &\leqslant C(1+t^{\frac{\beta}{2}})\|u^0\|_{L^p(\Omega;\dot H^\beta(\mathcal{O}))} + \textstyle\int_0^t\|f(u(s))\|_{L^p(\Omega;L^2(\mathcal{O}))}\mathrm{d}s + Ct^{\frac{\alpha}{2}} \\ &\leqslant C(1+t^{\frac{\beta}{2}}) + \textstyle\int_0^t(\|f(0)\|_{L^p(\Omega;L^2(\mathcal{O}))} + C\|u(s)\|_{L^p(\Omega;L^2(\mathcal{O}))})\mathrm{d}s + Ct^{\frac{\alpha}{2}} \\ &\leqslant C(1+t^{\frac{\beta}{2}}) + \textstyle\int_0^t C\|u(s)\|_{L^p(\Omega;L^2(\mathcal{O}))}\mathrm{d}s.\end{aligned}$$

Then applying Gronwall's inequality (see [14, Lemma 7.1.1]) yields (2.16).

By applying the projection operator Π_j to (1.3) and considering the result in the $L^p(\Omega;\dot H^\alpha(\mathcal{O}))$ norm, we have

$$\begin{aligned}&\|\Pi_j u(t)\|_{L^p(\Omega;\dot H^\alpha(\mathcal{O}))} = \|A^{\frac{\alpha}{2}}\Pi_j u(t)\|_{L^p(\Omega;L^2(\mathcal{O}))} \\ &\leqslant \|A^{\frac{\alpha-\beta}{2}}e^{-tA}A^{\frac{\beta}{2}}\Pi_j u^0\|_{L^p(\Omega;L^2(\mathcal{O}))} \\ &\quad + \Big\|\int_0^t A^{\frac{\alpha}{2}}e^{-(t-s)A}\Pi_j f(u(s))\mathrm{d}s\Big\|_{L^p(\Omega;L^2(\mathcal{O}))} + \Big\|\Pi_j\int_0^t A^{\frac{\alpha}{2}}e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{L^p(\Omega;L^2(\mathcal{O}))} \\ &\leqslant Ct^{-\frac{\alpha-\beta}{2}}\|A^{\frac{\beta}{2}}\Pi_j u^0\|_{L^p(\Omega;L^2(\mathcal{O}))} \\ &\quad + C\int_0^t(t-s)^{-\frac{\alpha}{2}}\|f(u(s))\|_{L^p(\Omega;L^2(\mathcal{O}))}\mathrm{d}s + \Big\|\Pi_j\int_0^t e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{L^p(\Omega;\dot H^\alpha(\mathcal{O}))} \\ &\leqslant Ct^{-\frac{\alpha-\beta}{2}}\|u^0\|_{L^p(\Omega;\dot H^\beta(\mathcal{O}))} \\ &\quad + C\int_0^t(t-s)^{-\frac{\alpha}{2}}(1+s^{\frac{\beta}{2}})\mathrm{d}s + \Big\|\Pi_j\int_0^t e^{-(t-s)A}\mathrm{d}W(s)\Big\|_{L^p(\Omega;\dot H^\alpha(\mathcal{O}))} \\ &\qquad \text{(here we use the Lipschitz continuity of } f \text{ and (2.16), which is already proved)} \\ &\leqslant Ct^{-\frac{\alpha-\beta}{2}} + C(t^{1-\frac{\alpha}{2}} + t^{1-\frac{\alpha-\beta}{2}}) + C,\end{aligned}$$

where the last inequality uses assumptions (2.10)–(2.11). Then, by taking maximum in the above inequality among all $j\geqslant 1$, we obtain (2.17).

Next we prove that $u\in C\big((\varepsilon,T];L^p(\Omega;L^2(\mathcal{O}))\big)$. Obviously, by (2.12) for $0<t_2<t_1\leqslant T$ there hold

$$\begin{aligned}&\Big\|\int_0^{t_2}(e^{-(t_1-s)A}-e^{-(t_2-s)A})\mathrm{d}W(s)\Big\|^2_{L^p(\Omega;L^2(\mathcal{O}))} \\ &\leqslant C\int_0^{t_2}\sum_{k=1}^\infty \mu_k\|(e^{-(t_1-s)A}-e^{-(t_2-s)A})\phi_k\|^2_{L^2(\mathcal{O})}\mathrm{d}s \\ &\leqslant C\sum_{k=1}^\infty \mu_k(e^{-t_1\lambda_k}-e^{-t_2\lambda_k})^2\frac{e^{2t_2\lambda_k}-1}{2\lambda_k} \qquad \text{(A.52)}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{k=1}^{\infty}\frac{\mu_k}{\lambda_k}(1-e^{-2(t_1-t_2)\lambda_k})\\
&\leqslant C\big\|\textstyle\int_0^{t_1-t_2}e^{-(t_1-t_2-s)A}\mathrm{d}W(s)\big\|^2_{L^2(\Omega;L^2(\mathcal{O}))}\\
&\leqslant C(t_1-t_2)^{\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|\textstyle\int_{t_2}^{t_1}e^{-(t_1-s)A}\mathrm{d}W(s)\big\|^2_{L^p(\Omega;L^2(\mathcal{O}))} &\leqslant C\textstyle\int_0^{t_1-t_2}\displaystyle\sum_{k=1}^{\infty}\mu_k\|e^{-(t_1-t_2-\sigma)A}\phi_k\|^2_{L^2(\mathcal{O})}\mathrm{d}\sigma\\
&\sim \big\|\textstyle\int_0^{t_1-t_2}e^{-(t_1-t_2-s)A}\mathrm{d}W(s)\big\|^2_{L^2(\Omega;L^2(\mathcal{O}))}\\
&\sim (t_1-t_2)^{\alpha},
\end{aligned}
\tag{A.53}
$$

where the last inequality is due to item (3) in Assumption 2.1. Combining these estimates with (2.16), we derive for $0<\varepsilon\leqslant t_2<t_1\leqslant T$ and $0<\delta<1$ that

$$
\begin{aligned}
&\|u(t_1)-u(t_2)\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&\leqslant \|e^{-t_1A}u^0-e^{-t_2A}u^0\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&\quad+\big\|\textstyle\int_0^{t_2}(e^{-(t_1-s)A}-e^{-(t_2-s)A})f(u(s))\mathrm{d}s+\int_{t_2}^{t_1}e^{-(t_1-s)A}f(u(s))\mathrm{d}s\big\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&\quad+\big\|\textstyle\int_0^{t_2}(e^{-(t_1-s)A}-e^{-(t_2-s)A})\mathrm{d}W(s)+\int_{t_2}^{t_1}e^{-(t_1-s)A}\mathrm{d}W(s)\big\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&\leqslant \|A^{-\frac{\beta}{2}+\delta}e^{-t_2A}A^{-\delta}(e^{-(t_1-t_2)A}-I)A^{\frac{\beta}{2}}u^0\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&\quad+\textstyle\int_0^{t_2}\|A^{\delta}e^{-(t_2-s)A}A^{-\delta}(e^{-(t_1-t_2)A}-I)f(u(s))\|_{L^p(\Omega;L^2(\mathcal{O}))}\mathrm{d}s\\
&\quad+C\textstyle\int_{t_2}^{t_1}(1+s^{\frac{\beta}{2}})\mathrm{d}s\\
&\quad+\big\|\textstyle\int_0^{t_2}(e^{-(t_1-s)A}-e^{-(t_2-s)A})\mathrm{d}W(s)\big\|_{L^p(\Omega;L^2(\mathcal{O}))}+\big\|\int_{t_2}^{t_1}e^{-(t_1-s)A}\mathrm{d}W(s)\big\|_{L^p(\Omega;L^2(\mathcal{O}))}\\
&\leqslant C(t_2^{\frac{\beta}{2}-\delta}+1)(t_1-t_2)^{\delta}\|u^0\|_{L^p(\Omega;\dot{H}^{\beta}(\mathcal{O}))}\\
&\quad+\textstyle\int_0^{t_2}(t_2-s)^{-\delta}(t_1-t_2)^{\delta}(1+s^{\frac{\beta}{2}})\mathrm{d}s+C(t_1-t_2+t_1^{1+\frac{\beta}{2}}-t_2^{1+\frac{\beta}{2}})\\
&\quad+C(t_1-t_2)^{\frac{\alpha}{2}}\\
&\leqslant C(\varepsilon^{\frac{\beta}{2}-\delta}+1)(t_1-t_2)^{\delta}+C(t_1-t_2)^{1+\frac{\min\{0,\beta\}}{2}}+C(t_1-t_2)^{\frac{\alpha}{2}}.
\end{aligned}
$$

This means $u\in C^{\delta}\big([\varepsilon,T];L^p(\Omega;L^2(\mathcal{O}))\big)$ for $\delta\in\big(0,\min\{1+\frac{\min\{0,\beta\}}{2},\frac{\alpha}{2}\}\big)$. The last two terms in the inequality above indicate that the second and third terms in expression (1.3) are in $C([0,T];L^2(\Omega;L^2(\mathcal{O})))$. Provided $\bar{\beta}=\min\{0,\beta\}$, the first term in expression (1.3) is clearly in $C([0,T];L^2(\Omega;\dot{H}^{\bar{\beta}}(\mathcal{O})))$ because e^{-tA} is a strongly continuous semigroup on $\dot{H}^{\bar{\beta}}(\mathcal{O})$. As a result, the mild solution u is in $C([0,T];L^2(\Omega;\dot{H}^{\bar{\beta}}(\mathcal{O})))$. This completes the proof of Proposition 2.1.

□

References

[1] Rikard Anton, David Cohen, Stig Larsson, and Xiaojie Wang. Full discretization of semilinear stochastic wave equations driven by multiplicative noise. *SIAM J. Numer. Anal.*, 54(2):1093–1119, 2016.

[2] Rikard Anton, David Cohen, and Lluis Quer-Sardanyons. A fully discrete approximation of the one-dimensional stochastic heat equation. *IMA J. Numer. Anal.*, 40(1):247–284, 2020.

[3] Lehel Banjai, Gabriel Lord, and Jeta Molla. Strong convergence of a Verlet integrator for the semilinear stochastic wave equation. *SIAM J. Numer. Anal.*, 59(4):1976–2003, 2021.

[4] Colin Bennett and Robert Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[5] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[6] Charles-Edouard Bréhier, Jianbo Cui, and Jialin Hong. Strong convergence rates of semidiscrete splitting approximations for the stochastic Allen-Cahn equation. *IMA J. Numer. Anal.*, 39(4):2096–2134, 2019.

[7] Yanzhao Cao, Jialin Hong, and Zhihui Liu. Finite element approximations for second-order stochastic differential equation driven by fractional Brownian motion. *IMA J. Numer. Anal.*, 38(1):184–197, 2018.

[8] Yanzhao Cao and Li Yin. Spectral Galerkin method for stochastic wave equations driven by space-time white noise. *Commun. Pure Appl. Anal.*, 6(3):607–617, 2007.

[9] David Cohen and Annika Lang. Numerical approximation and simulation of the stochastic wave equation on the sphere. *Preprint, arXiv:2102.04224*, 2021.

[10] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[11] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[12] István Gyöngy. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I. *Potential Anal.*, 9(1):1–25, 1998.

[13] István Gyöngy. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. *Potential Anal.*, 11(1):1–37, 1999.

[14] Daniel Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[15] Desmond J. Higham and Peter E. Kloeden. *An introduction to the numerical simulation of stochastic differential equations*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, [2021] ©2021.

[16] Desmond J. Higham, Xuerong Mao, and Andrew M. Stuart. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal.*, 40(3):1041–1063, 2002.

[17] Martin Hutzenthaler and Arnulf Jentzen. Convergence of the stochastic Euler scheme for locally Lipschitz coefficients. *Found. Comput. Math.*, 11(6):657–706, 2011.

[18] Martin Hutzenthaler and Arnulf Jentzen. On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with nonglobally monotone coefficients. *Ann. Probab.*, 48(1):53–93, 2020.

[19] Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Ann. Appl. Probab.*, 22(4):1611–1641, 2012.

[20] Arnulf Jentzen and Peter E. Kloeden. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 465(2102):649–667, 2009.

[21] Mihály Kovács, Stig Larsson, and Fardin Saedpanah. Finite element approximation of the linear stochastic wave equation with additive noise. *SIAM J. Numer. Anal.*, 48(2):408–427, 2010.

[22] Rainer Kress. *Linear integral equations*, volume 82 of *Applied Mathematical Sciences*. Springer, New York, third edition, 2014.

[23] Raphael Kruse. *Strong and weak approximation of semilinear stochastic evolution equations*, volume 2093 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.

[24] Annika Lang, Andreas Petersson, and Andreas Thalhammer. Mean-square stability analysis of approximations of stochastic differential equations in infinite dimensions. *BIT*, 57(4):963–990, 2017.

[25] Buyang Li and Shu Ma. A high-order exponential integrator for nonlinear parabolic equations with nonsmooth initial data. *J. Sci. Comput.*, 87(1):Paper No. 23, 16, 2021.

[26] Gabriel J. Lord, Catherine E. Powell, and Tony Shardlow. *An introduction to computational stochastic PDEs*. Cambridge Texts in Applied Mathematics. Cambridge University Press, New York, 2014.

[27] Paul Malliavin. *Integration and probability*, volume 157 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With the collaboration of Hélène Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, With a foreword by Mark Pinsky.

[28] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[29] Jean Daniel Mukam and Antoine Tambue. Strong convergence of a stochastic Rosenbrock-type scheme for the finite element discretization of semilinear SPDEs driven by multiplicative and additive noise. *Stochastic Process. Appl.*, 130(8):4968–5005, 2020.

[30] Jean Daniel Mukam and Antoine Tambue. Strong convergence of the linear implicit Euler method for the finite element discretization of semilinear non-autonomous SPDEs driven by multiplicative or additive noise. *Appl. Numer. Math.*, 147:222–253, 2020.

[31] Claudia Prévôt and Michael Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[32] Ruisheng Qi and Xiaojie Wang. Error estimates of finite element method for semilinear stochastic strongly damped wave equation. *IMA J. Numer. Anal.*, 39(3):1594–1626, 2019.

[33] Elias M. Stein and Rami Shakarchi. *Functional analysis*, volume 4 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2011. Introduction to further topics in analysis.

[34] Xiaojie Wang. Weak error estimates of the exponential Euler scheme for semi-linear SPDEs without Malliavin calculus. *Discrete Contin. Dyn. Syst.*, 36(1):481–497, 2016.

[35] Xiaojie Wang. Strong convergence rates of the linear implicit Euler method for the finite element discretization of SPDEs with additive noise. *IMA J. Numer. Anal.*, 37(2):965–984, 2017.

[36] Xiaojie Wang. An efficient explicit full-discrete scheme for strong approximation of stochastic Allen-Cahn equation. *Stochastic Process. Appl.*, 130(10):6271–6299, 2020.

[37] Xiaojie Wang, Siqing Gan, and Jingtian Tang. Higher order strong approximations of semilinear stochastic wave equation with additive space-time white noise. *SIAM J. Sci. Comput.*, 36(6):A2611–A2632, 2014.

[38] Xiaojie Wang and Ruisheng Qi. A note on an accelerated exponential Euler method for parabolic SPDEs with additive noise. *Appl. Math. Lett.*, 46:31–37, 2015.

[39] Yubin Yan. Galerkin finite element methods for stochastic parabolic partial differential equations. *SIAM J. Numer. Anal.*, 43(4):1363–1384, 2005.