A NEW PERFECTLY MATCHED LAYER METHOD FOR THE HELMHOLTZ EQUATION IN NONCONVEX DOMAINS

BUYANG LI*, YONGLIN LI*†, AND WEIYING ZHENG‡

Abstract. A new coupled perfectly matched layer (PML) method is proposed for the Helmholtz equation in the whole space with inhomogeneity concentrated on a nonconvex domain. Rigorous analysis is presented for the stability and convergence of the proposed coupled PML method, which shows that the PML solution converges to the solution of the original Helmholtz problem exponentially with respect to the product of the wave number and the width of the layer. An iterative algorithm and a continuous interior penalty finite element method (CIP-FEM) are also proposed for solving the system of equations associated to the coupled PML. Numerical experiments are presented to illustrate the convergence and performance of the proposed coupled PML method as well as the iterative algorithm and the CIP-FEM.

Key words. Helmholtz equation, nonconvex, perfectly matched layer, exponential convergence, finite element method

AMS subject classifications. 65N12, 65N15, 65N30, 78A40

1. Introduction.

We consider the acoustic scattering problem in $\mathbb{R}^n$, $n \in \{2,3\}$, described by the Helmholtz equation under the radiation boundary condition, i.e.,

$$\Delta u + k^2 u = f \qquad \text{in } \mathbb{R}^n, \tag{1.1}$$

$$\left|\frac{\partial u}{\partial r} - \mathbf{i}ku\right| = o\big(r^{\frac{1-n}{2}}\big) \quad \text{as } r = |x| \to \infty, \tag{1.2}$$

where k is the wave number and f is a given function. Moreover, $k = k_0$ and $f = 0$ outside a bounded, Lipschitz and nontrapping domain Ω, where k_0 is a positive constant. The unique solvability and various stability estimates for the Helmholtz problem (1.1)–(1.2) have been studied in the literatures [21, 9, 56, 57, 26, 11, 62, 31, 58, 32, etc.].

The Helmholtz problem (1.1)–(1.2) is often solved approximately by using the perfectly matched layer (PML) method, which was originally proposed in [5] and then developed in [16, 18, 48, 49, 13, 7, 6, 17, 15, 42, 64, 24, 10, 28, etc.]. In the existing PML methods, one can choose a rectangular or circular domain to cover the region Ω and construct an absorbing layer outside the rectangular or circular domain, denoted by Ω_d, as shown in Figure 1.1 (left). The fundamental analysis indicates that the rectangular or circular PML converges exponentially to the radiation solution when the width of the layer Ω_d or the PML parameter tends to infinity, see, e.g., [4, 12, 13, 42, 51, 6, 7, 14, 15]. Then one can solve the original problem approximately in a bounded domain, with zero boundary condition at the exterior boundary of Ω_d. This approach generally works well in approximating the solution to the Helmholtz equation in the bounded domain Ω. Especially, the wave-number-explicit convergence analyses for PML are obtained in [14, 51, 10, 28] recently.

However, in some special cases, for example when Ω is a nonconvex slender region (such as an L-shape domain), using a convex rectangular or circular PML would require much more computational cost than solving the equation in a small neighborhood of Ω, as shown in Figure 1.1 (right). To resolve this issue, Laurens [50] proposed a new PML method through a diffeomorphism defined on an absorbing pseudo-Riemannian manifold. Such PML techniques only require one to solve equations in a small neighborhood of the nonconvex domain Ω. The convergence of the approximate solutions given by such PML for a nonconvex domain Ω, as well as the dependence on the wave number k and the

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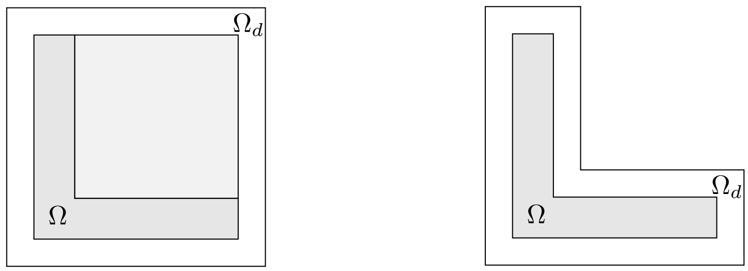

FIG. 1.1. *Left: PML in convex domain. Right: PML in nonconvex domain.*

width d of the PML, is not known so far. Recently, many authors have considered radial complex scalings based on the pole condition for the Helmholtz scattering problem with a star-shaped interior domain or scatterer; see [67, 35, 36, 63]. Such a PML approach requires a parameterization of the boundary piecewise. In the exterior domain, some new finite element approaches such as Hardy space infinite elements [35] are proposed. The convergence analyses of the approximations through domain truncation and finite element discretization for the resonance problems are reported in [36]. For the scattering problem, the similar related results remain open.

The objective of this article is to construct a new coupled PML method for the Helmholtz equation on a nonconvex domain, which admits rigorous analysis for the exponential convergence with respect to k, d, and the PML parameter. The key idea is to divide the nonconvex domain into several disjoint convex subdomains and to set up a PML for each subdomain. Some auxiliary solutions are solved in these subdomains and coupled with the original solution u through some interface conditions and an impedance boundary condition. Since the subdomains are convex, most of the popular PML methods, such as the uniaxial PML, can be applied, and therefore, the usual finite element methods (FEM) can be used. Since the standard finite element method for Helmholtz problem with large wave number suffers from the pollution effect, see [44, 45, 3, 22], we adopt the CIP-FEM [66, 25, 51, 52] to reduce the pollution effect (which arises when k is large) and propose an iterative algorithm for solving the coupled PML system. For other methods to reduce the pollution error, we refer to [56, 27, 1, 55, 33, 43, 30, 40, 41, 38, 39, 47, 8, 60, etc.]

The proposed new coupled PML method can also be applied to the multiple scattering problems in [54]. Under the well-separated assumption, i.e., the minimal distance among the scatterers is much larger than the diameters of the scatterers, two coupled methods using the multiple-DtN and PML techniques were proposed in [34] and [46], respectively. In [46], the PML solution for each scatterer is solved in the corresponding subdomain and can be extended to other subdomains by using the wave propagation operator, which is defined as the integrals of the Green's function over the subdomains, resulting in expensive costs. In contrast, the new coupled PML method in this paper does not require the well-separated assumption and avoids computing the DtN operator or the integrals of Green's function over the subdomains, but only requires computing some integrals on the boundary. In particular, the stability and exponential convergence of the new coupled PML method are proved without requiring the well-separated assumption (i.e., there is no restriction on the distance among the scatterers). In Section 5 we present some numerical tests to illustrate the effectiveness of the new PML method for a multiple scattering problem.

The outline of this article is as follows. The construction of the PML based on a domain decomposition and the derivation of the coupled PML system are presented in Section 2. The convergence analysis for the proposed coupled PML method is presented in Section 3. An iterative algorithm and a CIP-FEM for solving the system of equations associated to the coupled PML system are proposed in Section 4. Finally, numerical experiments are presented in Section 5 to illustrate the convergence and performance of the proposed coupled PML method and iterative CIP-FEM.

2. Construction of the coupled PML system.

2.1. Basic notations. For any domain $G \subset \mathbb{R}^n$ and part of its boundary $\Sigma \subset \partial G$, we denote by $(\cdot,\cdot)_G$ and $\langle \cdot,\cdot \rangle_\Sigma$ the inner products on the complex-valued Hilbert spaces $L^2(G)$ and $L^2(\Sigma)$, respectively. Moreover, the $H^{\frac{1}{2}}$-norm defined on the boundary Σ is given by

$$\|w\|_{H^{\frac{1}{2}}(\Sigma)} := \left(\|w\|^2_{L^2(\Sigma)} + |w|^2_{H^{\frac{1}{2}}(\Sigma)} \right)^{\frac{1}{2}}, \tag{2.1}$$

with

$$|w|^2_{H^{\frac{1}{2}}(\Sigma)} := \int_\Sigma \int_\Sigma \frac{|w(x) - w(x')|^2}{|x - x'|^n} ds(x) ds(x'). \tag{2.2}$$

The energy norm on G is defined by

$$|||w|||_G := \left(\|\nabla w\|^2_{L^2(G)} + k^2 \|w\|^2_{L^2(G)} \right)^{\frac{1}{2}}. \tag{2.3}$$

For any disjoint domains G_1 and G_2, the piecewise Sobolev space is defined by

$$H^m(G_1 \cup G_2) := \{ v : v|_{G_1} \in H^m(G_1),\ \ v|_{G_2} \in H^m(G_2) \} \quad \text{for } m \geq 1,$$

with the norm

$$\|\cdot\|_{H^m(G_1 \cup G_2)} = \|\cdot\|_{H^m(G_1)} + \|\cdot\|_{H^m(G_2)}.$$

Throughout the paper, we denote by C a generic positive constant which is independent of k, f, the PML parameters σ_0 and d. The notation $A \lesssim B$ or $B \gtrsim A$ stands for the statement "$A \leq CB$ for some constant C"; similarly, $A \approx B$ means "$A \lesssim B$ and $A \gtrsim B$". Moreover, we let $C_p(a, b, \cdots)$ be a generic positive constant which has at most polynomial growth in the variables a, b, and so on. The constants C and C_p may vary with different occurrences.

2.2. Stability estimates for the original Helmholtz problem. It is known that the solution to the Helmholtz problem (1.1)–(1.2) satisfies the following stability estimate:

$$\|u\|_{H^1(\Omega)} + \|ku\|_{L^2(\Omega)} \leq C_{\text{stab}} \|f\|_{L^2(\Omega)}. \tag{2.4}$$

In general, the stability constant C_{stab} depends on the wave number k and the diameter of Ω. It is known that for problems with homogeneous medium, or nontrapping medium in general, the stability constant C_{stab} is independent of k, see [9, 56, 57, 65] and [29, 32]. For more general $k(x)$, the C_{stab} may grows super-algebraically as k increases, see [62, 58, 31, 32].

For the convenience of theoretical analysis of PML, in the rest of this article, we assume that $k = k_0$ in $\mathbb{R}^n$ and therefore the stability constant C_{stab} in (2.4) is independent of k. The results in this article can be directly extended to the case of general $k(x)$ if the stability constant C_{stab} in (2.4) grows at most polynomially with respect to k.

2.3. Domain decomposition into convex subdomains. For a given nonconvex bounded domain $\Omega \subset \mathbb{R}^n$, we divide it into several disjoint convex subdomains Ω_j, $j = 1, \cdots, m$, such that $\overline{\Omega} = \cup_{j=1}^m \overline{\Omega_j}$, as illustrated in Figure 2.1. Let $\widehat{\Omega}_j$ be a neighborhood of Ω_j in $\mathbb{R}^n \setminus \overline{\Omega_j}$ with the thickness $d > 0$, i.e., we denote by $x = (x_1, \cdots, x_n)^{\mathrm{T}}$ and define

$$\widehat{\Omega}_j = \left\{ x \in \mathbb{R}^n \setminus \overline{\Omega_j} : \exists\, y \in \partial\Omega_j \text{ such that } |x_i - y_i| < d,\ i = 1, \cdots, n. \right\}.$$

The practical computational domains would be $B_j = \overline{\Omega_j} \cup \widehat{\Omega}_j$, with PML filled in $\widehat{\Omega}_j$, $j = 1, 2, \cdots, m$. The boundaries of these domains are denoted by $\Gamma = \partial\Omega$, $\Gamma_j = \partial\Omega_j$ and $\widehat{\Gamma}_j = \partial B_j$.

Since $f = 0$ outside Ω, the solution satisfies the homogeneous Helmholtz equation in the exterior domain $\mathbb{R}^n \setminus \overline{\Omega}$, i.e.,

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.5}$$

It is known that the solution u of the above homogeneous equation has the following

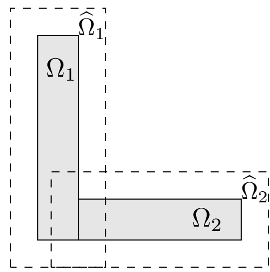

FIG. 2.1. *A nonconvex domain Ω partitioned into convex subdomains Ω_j, $j = 1, 2$.*

boundary integral representation (see, e.g., [61, Theorem 3.1.6]):

$$u(x) = \int_\Gamma u(y)\partial_{\mathbf{n}(y)} G(x,y)\,\mathrm{d}s(y) - \int_\Gamma \partial_{\mathbf{n}} u(y) G(x,y)\,\mathrm{d}s(y) \quad \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega}, \tag{2.6}$$

where $\mathbf{n}$ denotes the unit outward normal on Γ, $\partial_{\mathbf{n}(y)}$ denotes the outward normal derivative with respect to the variable y, and $G(x,y)$ is the fundamental solution to the Helmholtz equation, given by

$$G(x,y) = \begin{cases} \frac{\mathbf{i}}{4} H_0^{(1)}(k\,|x-y|) & \text{for } \mathbb{R}^2, \\ \dfrac{e^{\mathbf{i}k|x-y|}}{4\pi\,|x-y|} & \text{for } \mathbb{R}^3, \end{cases} \tag{2.7}$$

which satisfies the following equation:

$$\Delta_y G(x,y) + k^2 G(x,y) = -\delta(x-y).$$

In view of (2.6), we define some new functions u_j, $j = 1, \ldots, m$, by

$$u_j(x) = \int_{\Gamma_j} u(y)\partial_{\mathbf{n}_j(y)} G(x,y)\,\mathrm{d}s(y) - \int_{\Gamma_j} \partial_{\mathbf{n}_j} u(y) G(x,y)\,\mathrm{d}s(y) \quad \text{for } x \in \mathbb{R}^n \setminus \Gamma_j, \tag{2.8}$$

where $\mathbf{n}_j$ denotes the unit outward normal on Γ_j, and $\partial_{\mathbf{n}_j(y)}$ denotes the outward normal derivative with respect to the variable y. Since the integral of $u(y)\partial_{\mathbf{n}_j(y)}G(x,y)$ and $\partial_{\mathbf{n}_j} u(y) G(x,y)$ from two sides of Γ_j would cancel (the normal vectors from the two sides have opposite directions), summing up (2.8) for $j = 1, \ldots, m$ yields

$$u(x) = \sum_{j=1}^m u_j(x) \quad \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.9}$$

For a function φ, define

$$\varphi^\pm(x) = \lim_{h \to 0^+} \varphi(x \pm h\mathbf{n}_j(x)) \quad \text{and} \quad \partial_{\mathbf{n}_j}\varphi^\pm(x) = (\partial_{\mathbf{n}_j}\varphi)^\pm(x), \quad x \in \Gamma_j.$$

Let $[\varphi] := \varphi^- - \varphi^+$ denote the jump of φ on Γ_j. According to [59, Theorem 3.1.1], the boundary integral representation (2.8) implies that u_j is the solution to the following interface problem:

$$\begin{aligned} \Delta u_j + k^2 u_j &= 0 && \text{in } \mathbb{R}^n \setminus \Gamma_j, \\ [u_j] = -u, \ \ [\partial_{\mathbf{n}_j} u_j] &= -\partial_{\mathbf{n}_j} u && \text{on } \Gamma_j, \\ |\partial_{\mathbf{n}} u_j - \mathbf{i}k u_j| &= o\big(|x|^{\frac{1-n}{2}}\big) && \text{as } |x| \to \infty. \end{aligned}$$

Moreover, taking normal derivative of (2.9) yields $\partial_{\mathbf{n}} u - \mathbf{i}ku = \sum_{j=1}^m (\partial_{\mathbf{n}} - \mathbf{i}k) u_j|_{\mathbb{R}^n \setminus \overline{\Omega_j}}$ on Γ. Therefore, the original Helmholtz problem (1.1)–(1.2) is equivalent to the following system:

$$\Delta u_j + k^2 u_j = 0 \quad \text{in } \mathbb{R}^n \setminus \Gamma_j, \tag{2.10a}$$

$$[u_j] = -u, \ \ [\partial_{\mathbf{n}_j} u_j] = -\partial_{\mathbf{n}_j} u \quad \text{on } \Gamma_j, \tag{2.10b}$$

$$|\partial_{\mathbf{n}} u_j - \mathbf{i}k u_j| = o\big(|x|^{\frac{1-n}{2}}\big) \quad \text{as } |x| \to \infty, \tag{2.10c}$$

$$\Delta u + k^2 u = f \quad \text{in } \Omega, \tag{2.10d}$$

$$\partial_{\mathbf{n}} u - \mathbf{i}ku = \sum_{j=1}^{m} (\partial_{\mathbf{n}} - \mathbf{i}k) u_j|_{\mathbb{R}^n \setminus \overline{\Omega_j}} \quad \text{on } \Gamma. \tag{2.10e}$$

The equivalence between (1.1)–(1.2) and (2.10) can be seen as follows. We have shown that if u is the solution to (1.1)–(1.2) then u and the function u_j defined by (2.8) satisfy the equations in (2.10). Conversely, if u and u_j are the solutions to (2.10), the function $w = \sum_{j=1}^m u_j$ would satisfy the equations:

$$\Delta w + k^2 w = 0 \quad \text{in } \mathbb{R}^n \setminus \Gamma, \tag{2.11a}$$

$$[w] = -u, \ \ [\partial_{\mathbf{n}} w] = -\partial_{\mathbf{n}} u \quad \text{on } \Gamma, \tag{2.11b}$$

$$|\partial_{\mathbf{n}} w - \mathbf{i}kw| = o\big(|x|^{\frac{1-n}{2}}\big) \quad \text{as } |x| \to \infty. \tag{2.11c}$$

Combining (2.11b) and (2.10e) implies that $[\partial_{\mathbf{n}} w - \mathbf{i}kw] = -(\partial_{\mathbf{n}} u - \mathbf{i}ku) = -(\partial_{\mathbf{n}} - \mathbf{i}k)w|_{\mathbb{R}^n \setminus \overline{\Omega}}$ on Γ. This means that

$$(\partial_{\mathbf{n}} - \mathbf{i}k) w|_{\Omega} = 0 \quad \text{on } \Gamma. \tag{2.12}$$

Since the Helmholtz equation (2.11a) with impedance boundary condition (2.12) has unique solution (see, e.g., [57, 11]), it follows that $w|_\Omega = 0$. As a result of this and the interface condition (2.11b), we have $u = w|_{\mathbb{R}^n \setminus \overline{\Omega}}$ and $\partial_n u = \partial_n w|_{\mathbb{R}^n \setminus \overline{\Omega}}$ on the interior side of Γ. If we define $u|_{\mathbb{R}^n \setminus \overline{\Omega}} := w|_{\mathbb{R}^n \setminus \overline{\Omega}}$, then $[u] = [\partial_{\mathbf{n}} u] = 0$ on Γ and $\Delta u + k^2 u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$, which implies that u is the solution to the original Helmholtz equation (1.1)–(1.2).

In the equivalent formulation (2.10), the equations of u_j are defined in an unbounded domain with radiation boundary condition. Since each Ω_j is a convex domain, PML can be set up in the domain $\widehat{\Omega}_j$ to approximate the solution u_j in Ω. This is presented in the next several subsections.

2.4. Uniaxial PML method. For simplicity, in the rest of this paper, we assume that all the subdomains Ω_j are rectangles or cuboids whose sides are parallel to the main coordinate axes. We remark that such an additional assumption is for the convenience of the presentation in our theoretical analysis. Indeed, the PML can be set up in a local Cartesian coordinate system with the origin at the centre of the subdomain and the axes parallel to the sides of the subdomain.

Let $\widetilde{x}^j := F_j(x) = x + \mathbf{i}\sigma_j(x)$ be a transformation with a function $\sigma_j \in C^1 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following conditions:

$$(x-y)\cdot \operatorname{Im} \widetilde{x}^j = (x-y)\cdot \sigma_j(x) > 0 \quad \text{for } x \in \widehat{\Omega}_j, \ y \in \Gamma_j, \tag{2.13a}$$

$$\sigma_j(x) = 0 \quad \text{for } x \in \overline{\Omega_j}, \tag{2.13b}$$

$$\sigma_0 d \le |\sigma_j(x)| \le \beta \sigma_0 d \quad \text{for } x \in \widehat{\Gamma}_j, \tag{2.13c}$$

where $\sigma_0 > 0$ is a given constant and $\beta \ge 1$ is a constant depending only on n. Denote the centre of Ω_j by $O_j = (O_{j,1}, \cdots, O_{j,n})^{\mathrm{T}}$ and the diameter of Ω_j in the i-th dimension by $L_{j,i}$. Then according to the definition of PML in Section 2.3, the layer is given by

$$\widehat{\Omega}_j = \left\{ x \in \mathbb{R}^n \setminus \overline{\Omega_j} : |x_i - O_{j,i}| \le d + L_{j,i}/2, \ i = 1, \cdots, n. \right\}.$$

In order to use the results in [14, 7] on the inf-sup conditions and uniquenesses of the PML problems, we let $\sigma_j(x)$ be defined as follows:

$$\sigma_j(x) = (\sigma_{j,1}(x_1), \cdots, \sigma_{j,n}(x_n))^{\mathrm{T}} \quad \text{with} \quad \sigma_{j,i}(x_i) = \int_{O_{j,i}}^{x_i} \tilde{\sigma}_{j,i}(t)\,\mathrm{d}t, \tag{2.14}$$

where $\tilde{\sigma}_{j,i}(t) \in C(\mathbb{R})$ satisfies $\tilde{\sigma}_{j,i} \ge 0$, $\tilde{\sigma}_{j,i}(O_{j,i} + t) = \tilde{\sigma}_{j,i}(O_{j,i} - t)$ and

$$\tilde{\sigma}_{j,i}(t) = 0 \text{ for } |t - O_{j,i}| \le \frac{L_{j,i}}{2} \quad \text{and} \quad \tilde{\sigma}_{j,i}(t) = \bar{\sigma}_{j,i} \text{ for } |t - O_{j,i}| \ge \bar{d} + \frac{L_{j,i}}{2},$$

where $\bar{d} \in (0, d)$ is a constant and $\bar{\sigma}_{j,i} > 0$ is given by σ_0. More precisely, $\bar{\sigma}_{j,i}$ satisfies

$$\int_{O_{j,i}+\frac{L_{j,i}}{2}}^{O_{j,i}+\frac{L_{j,i}}{2}+\bar{d}} \tilde{\sigma}_{j,i}(t)\,\mathrm{d}t + \bar{\sigma}_{j,i}(d-\bar{d}) = \int_{O_{j,i}}^{O_{j,i}+\frac{L_{j,i}}{2}+d} \tilde{\sigma}_{j,i}(t)\,\mathrm{d}t = \sigma_0 d.$$

Let $\beta = \sqrt{n}$. It's easy to verify that $\sigma_j(x)$ defined by (2.14) satisfies all the conditions in (2.13). Notice that $\sigma_{j,i}(x_i)$ depends only on x_i, such a construction of PML is called the uniaxial PML method (see, e.g., [13, 14, 7]).

Condition (2.13a) guarantees

$$(2.15)\qquad \begin{aligned}(\widetilde{x}_1^j - y_1)^2 + (\widetilde{x}_2^j - y_2)^2 + (\widetilde{x}_3^j - y_3)^2 &= |x-y|^2 - |\sigma_j(x)|^2 + 2(x-y)\cdot\sigma_j(x)\mathbf{i} \\ &\in \mathbb{C}\setminus(-\infty,0] \quad \text{for } x \in \widehat{\Omega}_j.\end{aligned}$$

Since the square root function $\sqrt{\cdot} : \mathbb{C}\setminus(-\infty,0] \to \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is analytic, it follows that the complex distance function

$$\rho(z,y) = \sqrt{(z_1-y_1)^2 + (z_2-y_2)^2 + (z_3-y_3)^2}$$

is well defined and analytic for z in some neighborhood of $\widetilde{x}^j$. This implies that the function

$$(2.16)\qquad u_j(\widetilde{x}^j) := \int_{\Gamma_j} u(y)\partial_{\mathbf{n}_j(y)}\widetilde{G}_j(x,y)\,\mathrm{d}s(y) - \int_{\Gamma_j} \partial_{\mathbf{n}_j} u(y)\widetilde{G}_j(x,y)\,\mathrm{d}s(y),$$

where

$$(2.17)\qquad \widetilde{G}_j(x,y) := \begin{cases} \frac{\mathbf{i}}{4}H_0^{(1)}(k\rho(\widetilde{x}^j,y)) & \text{for } \mathbb{R}^2, \\ \dfrac{e^{\mathbf{i}k\rho(\widetilde{x}^j,y)}}{4\pi\rho(\widetilde{x}^j,y)} & \text{for } \mathbb{R}^3, \end{cases}$$

is analytic in a small neighborhood of $\widetilde{x}^j$ and then satisfies the Helmholtz equation, i.e.

$$\Delta_{\widetilde{x}^j} u_j(\widetilde{x}^j) + k^2 u_j(\widetilde{x}^j) = 0.$$

By using the chain rule (cf. [49, Theorem 2.5]), we find that the function $\widetilde{u}_j(x) := u_j(\widetilde{x}^j)$ satisfies the following PML equation

$$(2.18)\qquad \operatorname{div}(A_j\nabla\widetilde{u}_j) + k^2 J_j\widetilde{u}_j = 0,$$

where

$$(2.19)\qquad A_j = J_j H_j^{\mathrm{T}} H_j, \quad H_j = \left(I + \mathbf{i}(D\sigma_j)^{\mathrm{T}}\right)^{-1} = (DF_j)^{-\mathrm{T}} \quad \text{and} \quad J_j = \det(DF_j).$$

In particular, $A_j^{\mathrm{T}} = A_j$ is symmetric; condition (2.13b) implies that $A_j = I$, $J_j = 1$ in $\overline{\Omega_j}$, and

$$(2.20)\qquad \widetilde{u}_j = u_j \quad \text{in } \overline{\Omega_j}.$$

Since Ω_j is convex, by using [49, Corollary 3.2 and Lemma 4.2], A_j is elliptic and J_j is bounded. More precisely, the following coercivity and continuity hold for any domain $G \subset \mathbb{R}^n$ and $\varphi, \psi \in H^1(G)$:

$$(2.21)\qquad \operatorname{Re}(A_j\nabla\varphi,\nabla\varphi)_G \ge C_p(\sigma_0)^{-1}\|\nabla\varphi\|^2_{L^2(G)},$$

$$(2.22)\qquad \left|(A_j\nabla\varphi,\nabla\psi)_G - k^2(J_j\varphi,\psi)_G\right| \le C_p(\sigma_0)\,|||\varphi|||_G\,|||\psi|||_G.$$

In the next subsection we show that $\widetilde{u}_j$ decays exponentially with respect to $k\sigma_0 d$ and therefore close to zero on $\widehat{\Gamma}_j$. As a result, we can approximate $\widetilde{u}_j$ by solving (2.18) with zero boundary condition.

2.5. Exponential decay in the PML. We denote

$$\gamma := \min_{1\le j\le m} \frac{d}{\sqrt{\sum_{i=1}^n (L_{j,i}+d)^2}} \quad \text{and}$$

$$\lambda := \max_{\substack{1\le i\le n\\1\le j\le m}} \|\partial_{x_i}\widetilde{x}_i^j\|_{L^\infty(\widehat{\Gamma}_j)} \eqsim 1 + \max_{\substack{1\le i\le n\\1\le j\le m}} \|\tilde{\sigma}_{j,i}\|_{L^\infty(\widehat{\Gamma}_j)}.$$

The following estimates for the modified Green function $\widetilde{G}_j(x,y)$ hold:

LEMMA 2.1. *Let* (2.13a)–(2.13c) *be satisfied and*

$$\gamma k \sigma_0 d \geq 1. \tag{2.23}$$

Then there exists a positive constant C depending only on the constant β in (2.13c) *such that for any $x \in \widehat{\Gamma}_j$, $y \in \overline{\Omega}_j$ and $1 \leq i, l \leq n$, there hold:*

$$\left|\widetilde{G}_j(x,y)\right| \leq C(\gamma\sigma_0 d)^{-1} e^{-\gamma k \sigma_0 d}, \tag{2.24}$$

$$\left|\partial_{y_i}\widetilde{G}_j(x,y)\right| \leq Ck\gamma^{-1}(\gamma\sigma_0 d)^{-1} e^{-\gamma k \sigma_0 d}, \tag{2.25}$$

$$\left|\partial_{x_l}\widetilde{G}_j(x,y)\right| \leq C\lambda k\gamma^{-1}(\gamma\sigma_0 d)^{-1} e^{-\gamma k \sigma_0 d}, \tag{2.26}$$

$$\left|\partial_{x_l}\partial_{y_i}\widetilde{G}_j(x,y)\right| \leq C\lambda k^2\gamma^{-2}(\gamma\sigma_0 d)^{-1} e^{-\gamma k \sigma_0 d}. \tag{2.27}$$

Proof. We consider only the case of $n = 3$ and refer to [13, Lemma 3.3] (which considered a rectangular PML) for $n = 2$. By using (2.15) and (2.13a), it is easy to verify that

$$\operatorname{Im}\rho(\widetilde{x}^j, y) \geq \frac{(x-y)\cdot \operatorname{Im}\widetilde{x}^j}{|x-y|} = \frac{(x-y)\cdot\sigma_j(x)}{|x-y|}.$$

(i) Since $x \in \widehat{\Gamma}_j$ and $y \in \overline{\Omega}_j$, from (2.13a) and (2.13c) we derive the following inequality:

$$(x-y)\cdot\sigma_j(x) = |\sigma_j(x)|\,|x-y|\cos\langle x-y, \sigma_j(x)\rangle \geq \sigma_0 d\cdot \operatorname{dist}(x,\Gamma_j) \geq \sigma_0 d^2,$$

which implies that

$$\left|\rho(\widetilde{x}^j, y)\right| \geq \operatorname{Im}\rho(\widetilde{x}^j, y) \geq \frac{\sigma_0 d^2}{\sqrt{\sum_{i=1}^n (L_{j,i}+d)^2}} \geq \gamma\sigma_0 d.$$

Substituting this into (2.17) yields (2.24).

(ii) Some straightforward calculations yield

$$\partial_{y_i}\widetilde{G}_j(x,y) = \left(\mathbf{i}k - \rho^{-1}\right)\widetilde{G}_j(x,y)\partial_{y_i}\rho(\widetilde{x}^j, y) \quad \text{and} \quad \partial_{y_i}\rho = \frac{y_i - \widetilde{x}_i^j}{\rho}.$$

If $|x-y| \geq 2\,|\sigma_j(x)|$, then from (2.15) we derive that

$$|\rho| \geq \left|\operatorname{Re}\rho^2\right|^{1/2} = \left(|x-y|^2 - |\sigma_j(x)|^2\right)^{1/2} \geq \frac{\sqrt{3}}{2}\,|x-y|\,,$$

and therefore

$$|\partial_{y_i}\rho| = \left|\frac{\widetilde{x}_i^j - y_i}{\rho}\right| \leq \frac{2\left(|x-y|^2 + |\sigma_j(x)|^2\right)^{1/2}}{\sqrt{3}\,|x-y|} \leq \sqrt{\frac{5}{3}}.$$

Else if $|x-y| < 2\,|\sigma_j(x)|$, then by using $|\rho| \geq \operatorname{Im}\rho \geq \gamma\sigma_0 d$ and (2.13c), we have

$$|\partial_{y_i}\rho| = \left|\frac{\widetilde{x}_i^j - y_i}{\rho}\right| \leq \frac{\left(|x-y|^2 + |\sigma_j(x)|^2\right)^{1/2}}{\gamma\sigma_0 d} \leq \frac{\sqrt{5}\,|\sigma_j(x)|}{\gamma\sigma_0 d} \leq \sqrt{5}\beta\gamma^{-1}.$$

In this case, from (2.23) and $\gamma^{-1} \geq 1$ we obtain

$$\left|\partial_{y_i}\widetilde{G}_j(x,y)\right| \lesssim \left(k + (\gamma\sigma_0 d)^{-1}\right)\gamma^{-1}(\gamma\sigma_0 d)^{-1}e^{-\gamma k\sigma_0 d} \lesssim k\gamma^{-1}(\gamma\sigma_0 d)^{-1}e^{-\gamma k \sigma_0 d},$$

where in the second inequality we have used $(\gamma\sigma_0 d)^{-1} \leq k$.

(iii) Similarly to (ii), by noting $\partial_{x_l}\widetilde{x}_i^j = (1 + \mathbf{i}\tilde{\sigma}_{j,i})\delta_{l,i}$, where $\delta_{l,i}$ is the Kronecker delta function, and using

$$\partial_{x_l}\rho = \frac{(\widetilde{x}_l^j - y_l)(1 + \mathbf{i}\tilde{\sigma}_{j,l})}{\rho} \quad \text{and} \quad \partial_{x_l}\widetilde{G}_j(x,y) = \left(\mathbf{i}k - \rho^{-1}\right)\widetilde{G}_j(x,y)\partial_{x_l}\rho(\widetilde{x}^j, y),$$

we can prove $|\partial_{x_l}\rho| \lesssim \lambda\beta\gamma^{-1}$ and then obtain (2.26).

(iv) Note that

$$\partial_{x_l}\partial_{y_i}\widetilde{G}_j(x,y) = (\mathbf{i}k - \rho^{-1})^2\widetilde{G}_j\partial_{y_i}\rho\partial_{x_l}\rho + \widetilde{G}_j\left(\rho^{-2}\partial_{x_l}\rho\partial_{y_i}\rho + (\mathbf{i}k - \rho^{-1})\partial_{x_l}\partial_{y_i}\rho\right)$$

and

$$|\partial_{x_l}\partial_{y_i}\rho| = \left|\rho^{-1}\partial_{x_l}\widetilde{x}_i^j - \rho^{-1}\partial_{y_i}\rho\partial_{x_l}\rho\right| \lesssim (\gamma\sigma_0 d)^{-1}\left(\lambda + \lambda\gamma^{-2}\right) \lesssim \lambda(\gamma^3\sigma_0 d)^{-1},$$

which imply that

$$\begin{aligned}\left|\partial_{x_l}\partial_{y_i}\widetilde{G}_j(x,y)\right| &\lesssim \left(\lambda k^2\gamma^{-2} + \lambda(\gamma\sigma_0 d)^{-2}\gamma^{-2} + \lambda k(\gamma^3\sigma_0 d)^{-1}\right)\left|\widetilde{G}_j\right| \\ &\lesssim \lambda k^2\gamma^{-2}(\gamma\sigma_0 d)^{-1}e^{-\gamma k\sigma_0 d}.\end{aligned}$$

The proof of this lemma is completed. ∎

Next we present the exponential decaying estimate of $\widetilde{u}_j(x)$ defined in (2.16).

LEMMA 2.2. *Let (2.13a)–(2.13c) and (2.23) be satisfied. Then there exists a positive constant C depending only on β and Ω such that*

$$|\widetilde{u}_j(x)| \le CC_{\mathrm{stab}}k^2\gamma^{-1}(\gamma\sigma_0 d)^{-1}e^{-\gamma k\sigma_0 d}\|f\|_{L^2(\Omega)}, \quad x \in \widehat{\Gamma}_j, \tag{2.28}$$

where C_{stab} is from the stability estimate (2.4).

Proof. From (2.16) we see that

$$|\widetilde{u}_j(x)| \le \|u\|_{L^2(\Gamma_j)}\left\|\partial_{\mathbf{n}_j}\widetilde{G}_j(x,\cdot)\right\|_{L^2(\Gamma_j)} + \left\|\partial_{\mathbf{n}_j}u\right\|_{H^{-1/2}(\Gamma_j)}\left\|\widetilde{G}_j(x,\cdot)\right\|_{H^{1/2}(\Gamma_j)}.$$

Since

$$\left|\widetilde{G}_j(x,y) - \widetilde{G}_j(x,y')\right| \le \left\|\nabla_y\widetilde{G}_j(x,y)\right\|_{L^\infty(\Gamma_j)}|y-y'|$$

the following inequalities hold in view of the notation in (2.1)–(2.2):

$$\int_{\Gamma_j}\int_{\Gamma_j}\frac{1}{|y-y'|^{n-2}}\,\mathrm{d}s(y)\,\mathrm{d}s(y') \le C(\Omega)^2$$

and therefore

$$\left|\widetilde{G}_j(x,\cdot)\right|_{H^{1/2}(\Gamma_j)} \le C(\Omega)\left\|\nabla_y\widetilde{G}_j(x,y)\right\|_{L^\infty(\Gamma_j)},$$

where $C(\Omega) > 0$ denotes some constant depending only on Ω. By using (2.24)–(2.25) and (2.23) we obtain

$$\begin{aligned}|\widetilde{u}_j(x)| &\lesssim \max_{x\in\widehat{\Gamma}_j, y\in\Gamma_j}\left\{\left|\nabla_y\widetilde{G}_j(x,y)\right|, \left|\widetilde{G}_j(x,y)\right|\right\}\left(\|u\|_{H^1(\Omega_j)} + \|\Delta u\|_{L^2(\Omega_j)}\right) \\ &\lesssim k\gamma^{-1}(\gamma\sigma_0 d)^{-1}e^{-\gamma k\sigma_0 d}\left(\|u\|_{H^1(\Omega_j)} + \left\|f - k^2u\right\|_{L^2(\Omega_j)}\right) \\ &\lesssim C_{\mathrm{stab}}k^2\gamma^{-1}(\gamma\sigma_0 d)^{-1}e^{-\gamma k\sigma_0 d}\|f\|_{L^2(\Omega)},\end{aligned}$$

where we have used the stability estimate (2.4) in the last inequality. The proof is completed. ∎

Remark 2.3. For example, we consider a two-dimensional narrow nonconvex domain whose subdomains are all rectangles with length L and width W satisfying $L \gg W$. We choose the PML width d such that $d \lesssim L$, then $\gamma \eqsim dL^{-1}$ and the PML condition (2.23) requires $k\sigma_0 d^2 \gtrsim L$. One possible choice for the PML parameters is $\sigma_0 \eqsim 1$ and $d \eqsim (L/k)^{1/2}$. Since the degrees of freedom in the discrete system is $N \approx L(d+W)/h^2$, where h denotes the mesh size, which is generally chosen to be about $1/k$, it follows that $N \approx (Lk)^{3/2} + (LW)k^2$ for the coupled PML. However, for the standard PML method, there holds $N \approx (L/h)^2 \approx (Lk)^2$. Therefore, the proposed PML method has less degrees of freedom when Lk is large and $L \gg W$.

In the rest of this paper, for simplicity, we denote by

$$L := \max_{1\le i\le n,\, 1\le j\le m} L_{j,i}. \tag{2.29}$$

It is easy to see that $\frac{d}{\sqrt{n}(L+d)} \le \gamma \le \frac{d}{L+d}$.

2.6. A system of equations for coupled PML. Before presenting the coupled PML system, we define some linear operators to be used in the subsequent analysis.

First, we define the single and double layer potentials (see, e.g., [59, 61]) as

$$S_j\varphi(x) = \int_{\Gamma_j} \varphi(y)G(x,y)\,\mathrm{d}s(y) \quad \text{and} \quad D_j\psi(x) = \int_{\Gamma_j} \psi(y)\partial_{\mathbf{n}_j(y)}G(x,y)\,\mathrm{d}s(y).$$

Let $T_j : H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j)$ be the DtN operator for Helmholtz problem [19], namely, for any $\varphi \in H^{1/2}(\Gamma_j)$, let $T_j\varphi = \partial_{\mathbf{n}_j} w$ on Γ_j, where $w \in H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{\Omega_j})$ solves

$$\begin{aligned} \Delta w + k^2 w &= 0 && \text{in } \mathbb{R}^n \setminus \overline{\Omega_j}, \\ w &= \varphi && \text{on } \Gamma_j, \\ \left|\frac{\partial w}{\partial r} - \mathbf{i}kw\right| &= o\big(r^{\frac{1-n}{2}}\big) && \text{as } r = |x| \to \infty. \end{aligned} \tag{2.30}$$

By the Green's formula (see, e.g., [20, Theorem 2.5]), we have

$$w = D_j w - S_j \partial_{\mathbf{n}_j} w = (D_j - S_j T_j)\varphi \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega_j}. \tag{2.31}$$

Define the extension operator as

$$E_j := (D_j - S_j T_j) : H^{1/2}(\Gamma_j) \to H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{\Omega_j}). \tag{2.32}$$

From (2.31), there hold

$$\partial_{\mathbf{n}_j} E_j \varphi = T_j \varphi \quad \text{and} \quad E_j\varphi = \varphi \quad \text{on } \Gamma_j. \tag{2.33}$$

Moreover, noting from (2.10a) and (2.10c), we get $T_j u_j^+ = \partial_{\mathbf{n}_j} u_j^+$ and $u_j|_{\mathbb{R}^n\setminus\overline{\Omega_j}} = E_j u_j^+$.

Next, we define

$$\widetilde{E}_j\varphi(x) := \int_{\Gamma_j} \varphi(y)\partial_{\mathbf{n}_j(y)}\widetilde{G}_j(x,y)\,\mathrm{d}s(y) - \int_{\Gamma_j} T_j\varphi(y)\widetilde{G}_j(x,y)\,\mathrm{d}s(y), \quad x \in \mathbb{R}^n \setminus \overline{\Omega_j}. \tag{2.34}$$

Obviously, for any $\varphi \in H^{1/2}(\Gamma_j)$, we have

$$\widetilde{E}_j\varphi = E_j\varphi,\ \partial_{\mathbf{n}_j}\widetilde{E}_j\varphi = \partial_{\mathbf{n}_j}E_j\varphi \quad \text{on } \Gamma_j; \quad \text{and} \quad \widetilde{u}_j = \widetilde{E}_j u_j^+ \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega_j}. \tag{2.35}$$

Let $\widehat{T}_j : H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j)$ be the DtN operator for the PML problem [13], namely, for any $\varphi \in H^{1/2}(\Gamma_j)$, let $\widehat{T}_j\varphi = \partial_{\mathbf{n}_j} w$ on Γ_j, where w solves the PML problem in the layer:

$$\begin{aligned} \operatorname{div}(A_j\nabla w) + k^2 J_j w &= 0 && \text{in } \widehat{\Omega}_j, \\ w &= \varphi && \text{on } \Gamma_j, \\ w &= 0 && \text{on } \widehat{\Gamma}_j. \end{aligned} \tag{2.36}$$

Define the extension operator with respect to $\widehat{T}_j$ as

$$\widehat{E}_j := (D_j - S_j\widehat{T}_j) : H^{1/2}(\Gamma_j) \to H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{\Omega_j}). \tag{2.37}$$

Now we give the coupled PML system by using these extension operators. From (2.10), (2.18), (2.28), and noting $u_j^+ = \widetilde{u}_j^+$ on Γ_j, we see that the solution u to Helmholtz equation (1.1)–(1.2) and the PML functions $\widetilde{u}_j$ defined in (2.16) satisfy the following coupled system of four equations and an inequality:

$$\operatorname{div}(A_j\nabla\widetilde{u}_j) + k^2 J_j\widetilde{u}_j = 0 \qquad \text{in } \mathbb{R}^n \setminus \Gamma_j, \tag{2.38a}$$

$$[\widetilde{u}_j] = -u, \ \ \big[\partial_{\mathbf{n}_j}\widetilde{u}_j\big] = -\partial_{\mathbf{n}_j} u \qquad \text{on } \Gamma_j, \tag{2.38b}$$

$$\widetilde{u}_j \text{ is bounded} \qquad \text{as } |x| \to \infty, \tag{2.38c}$$

$$\Delta u + k^2 u = f \qquad \text{in } \Omega, \tag{2.38d}$$

$$\partial_{\mathbf{n}} u - \mathbf{i}ku = \sum_{j=1}^{m} (\partial_{\mathbf{n}} - \mathbf{i}k)\, E_j\widetilde{u}_j^+ \qquad \text{on } \Gamma. \tag{2.38e}$$

The motivation of (2.38) is as follows: First, (2.38a)–(2.38c) follow directly from (2.10), (2.18) and (2.28). From (2.38a)–(2.38c) we see that $\widetilde{u}_j$ is uniquely determined by the values u and $\partial_{\mathbf{n}_j} u$ on Γ_j. Therefore, it suffices to couple $\widetilde{u}_j$ with the equation of u to have a closed system. An impedance type of boundary conditions such as (2.38e) would lead to good stability estimates. In particular, the boundary condition in (2.38e) is due to the fact that $u=\sum_{j=1}^m u_j$ in $\mathbb{R}^n\backslash\overline{\Omega}$ and $u_j=E_ju_j^+=E_j\widetilde{u}_j^+$ (see the text below (2.33) and note that $u_j^+=\widetilde{u}_j^+$ on Γ_j), which implies that $u=\sum_{j=1}^m E_j\widetilde{u}_j^+$ outside $\overline{\Omega}$. Therefore, applying operator $\partial_{\mathbf{n}}-\mathbf{i}k$ to the relation $u=\sum_{j=1}^m E_j\widetilde{u}_j^+$ yields (2.38e). In addition, it should be mentioned that, we cannot use $\widetilde{u}_j$ instead of $E_j\widetilde{u}_j^+$ in the right-hand side of (2.38e) since $\widetilde{u}_j|_{\Gamma\backslash\Gamma_j}\neq u_j|_{\Gamma\backslash\Gamma_j}$.

In view of Lemma 2.2, the solution $\widetilde{u}_j$ is close to zero on the outer boundary $\widehat{\Gamma}_j$ of the PML region. Therefore, we can truncate the exterior domain to a bounded one and set homogeneous Dirichlet boundary condition on the truncation boundary $\widehat{\Gamma}_j$. This leads to the following system of equations for the coupled PML:

$$
\begin{aligned}
&\text{(2.39a)} && \operatorname{div}(A_j\nabla v_j)+k^2J_jv_j=0 && \text{in } B_j\setminus\Gamma_j,\\
&\text{(2.39b)} && [v_j]=-v,\ \ [\partial_{\mathbf{n}_j}v_j]=-\partial_{\mathbf{n}_j}v && \text{on } \Gamma_j,\\
&\text{(2.39c)} && v_j=0 && \text{on } \widehat{\Gamma}_j,\\
&\text{(2.39d)} && \Delta v+k^2v=f && \text{in } \Omega,\\
&\text{(2.39e)} && \partial_{\mathbf{n}}v-\mathbf{i}kv=\sum_{j=1}^m(\partial_{\mathbf{n}}-\mathbf{i}k)\widehat{E}_jv_j^+ && \text{on } \Gamma,
\end{aligned}
$$

where we have replaced (2.38c) by (2.39c) and E_j in (2.38e) by $\widehat{E}_j$ in (2.39e).

Remark 2.4. Some important explanations for the coupled systems are as follows.

(i) From (2.39a)–(2.39c) we see that v_j solves an elliptic interface problem and is uniquely determined by the values v and $\partial_{\mathbf{n}_j}v$ on Γ_j. Moreover, v_j is the PML approximation of $\widetilde{u}_j|_{B_j\backslash\Gamma_j}$ if v is an approximation of u.

(ii) The PML approximation of u in Ω is v, which is coupled with v_j through the interface conditions (2.39b) and boundary condition (2.39e). Let $D=\cup_{j=1}^m B_j$. If we define

$$\hat{u}=\begin{cases}v & \text{in } \Omega,\\ \sum_{j=1}^m v_j & \text{in } D\setminus\Omega,\end{cases}$$

where v_j is extended by zero in $D\setminus B_j$, then $\hat{u}$ is the PML approximation of u with Dirichlet boundary condition $\hat{u}=0$ on ∂D. In view of this, the PML in the global domain is actually with nonconvex shape.

(iii) The motivation of (2.39e) is as follows: First, we cannot use v_j in the right-hand side of (2.39e) as $v_j|_{\Gamma\backslash\Gamma_j}$ is not an approximation of $u_j|_{\Gamma\backslash\Gamma_j}=E_j\widetilde{u}_j^+|_{\Gamma\backslash\Gamma_j}$. Second, since the error between operators T_j and $\widehat{T}_j$ is exponentially small (see [13]), $\widehat{E}_jv_j^+$ is an approximation of $E_jv_j^+$, and also an approximation of $u_j|_{\mathbb{R}^n\backslash\overline{\Omega_j}}=E_j\widetilde{u}_j^+$ if v_j is the PML approximation of $\widetilde{u}_j$ in $B_j\setminus\overline{\Omega_j}$. Third, in practice, $\widehat{E}_j$ is easier to implement than E_j, because the latter requires computing the DtN operator T_j.

(iv) Noting that $v_j|_{\widehat{\Omega}_j}$ satisfies (2.36) with $\varphi=v_j^+$, we have

$$\partial_{\mathbf{n}_j}v_j^+=\widehat{T}_jv_j^+\quad\text{on } \Gamma_j,\ \ \text{hence,}\ \ \widehat{E}_jv_j^+=(D_j-S_j\widehat{T}_j)v_j^+=(D_j-S_j\partial_{\mathbf{n}_j})v_j^+,$$

which means that (2.39e) can be simply obtained by evaluating two integrals on the boundary Γ_j. The coupled PML system (2.39) can be solved by an interface penalty FEM presented in Section 4.

To end this section, we give a stability estimate for v_j when $v\in H^1(\Omega)$ is given.

LEMMA 2.5. *For given $v\in H^1(\Omega)$, the problem (2.39a)–(2.39c) is well-defined and*

$$\text{(2.40)}\qquad |||v_j|||_{\Omega_j\cup\widehat{\Omega}_j}\leq C_p(\sigma_0)k^{3/2}\Big(\|v\|_{H^{1/2}(\Gamma_j)}+\big\|\partial_{\mathbf{n}_j}v\big\|_{H^{-1/2}(\Gamma_j)}\Big),$$

where $|||\cdot|||^2_{\Omega_j\cup\widehat{\Omega}_j}:=|||\cdot|||^2_{\Omega_j}+|||\cdot|||^2_{\widehat{\Omega}_j}$.

Proof. Let $\Phi_1 \in H^1(\Omega_j \cup \widehat{\Omega}_j)$ solve the elliptic interface problem

$$(2.41)\qquad \begin{aligned} -\operatorname{div}(A_j \nabla \Phi_1) + \Phi_1 &= 0 && \text{in } B_j \setminus \Gamma_j, \\ [\Phi_1] = -v, \quad \left[\partial_{\mathbf{n}_j} \Phi_1\right] &= -\partial_{\mathbf{n}_j} v && \text{on } \Gamma_j, \\ \Phi_1 &= 0 && \text{on } \widehat{\Gamma}_j, \end{aligned}$$

and let $\Phi_2 \in H^1(B_j)$ solve the PML problem

$$(2.42)\qquad \begin{aligned} -\operatorname{div}(A_j \nabla \Phi_2) - k^2 J_j \Phi_2 &= (1 + k^2 J_j)\Phi_1 && \text{in } B_j, \\ \Phi_2 &= 0 && \text{on } \widehat{\Gamma}_j, \end{aligned}$$

respectively. It's easy to see that $v_j = \Phi_1 + \Phi_2$ solves (2.39a)–(2.39c). By the proof of [53, Theorem 2.1] and utilizing the coercivity in (2.21), we know that problem (2.41) has a unique solution and satisfies the following stability estimate:

$$(2.43)\qquad |||\Phi_1|||_{\Omega_j \cup \widehat{\Omega}_j} \lesssim C_p(\sigma_0)\big(\|v\|_{H^{1/2}(\Gamma_j)} + \left\|\partial_{\mathbf{n}_j} v\right\|_{H^{-1/2}(\Gamma_j)}\big).$$

On the other hand, from [14, §3.1 and eq. (3.4)], and noting that B_j is convex, problem (2.42) has a unique solution and satisfies the stability estimate

$$|||\Phi_2|||_{B_j} \lesssim C_p(\sigma_0) k^{1/2} \left\|(1 + k^2 J_j)\Phi_1\right\|_{L^2(B_j)} \lesssim C_p(\sigma_0) k^{3/2}\, |||\Phi_1|||_{\Omega_j \cup \widehat{\Omega}_j},$$

which together with (2.43) gives (2.40) and concludes the proof of this lemma. ∎

3. Convergence analysis for the truncated PML problem.

In this section, we prove the exponential convergence of v to u with respect to k, σ_0 and d, where u is the solution to problem (2.10) and v is the solution to the truncated PML problem (2.39). The well-posedness for the PML system (2.39) is derived as a consequence of the truncation error analysis.

3.1. Exponentially decaying estimates of the PML extension.

Firstly, we show the continuity of the DtN operator T_j with explicit dependence on k.

Lemma 3.1. *There exists a constant C (which depends only on Γ_j) such that*

$$(3.1)\qquad \|T_j \varphi\|_{H^{-1/2}(\Gamma_j)} \le Ck \|\varphi\|_{H^{1/2}(\Gamma_j)} \quad \forall \varphi \in H^{1/2}(\Gamma_j).$$

Proof. Since $T_j \varphi = \partial_{\mathbf{n}_j} w$ on Γ_j, where w is the solution to the exterior Helmholtz problem with the boundary condition $w = \varphi$ on Γ_j and Sommerfeld radiation boundary condition at infinity, the well-known stability estimate for w (see, e.g., [9]) yields

$$|||w|||_{B_R \setminus \overline{\Omega_j}} \le C(R)\, \|\varphi\|_{H^{1/2}(\Gamma_j)},$$

where $B_R \supset \Omega_j$ denotes the ball with some radius R. Therefore,

$$\|T_j \varphi\|_{H^{-1/2}(\Gamma_j)} = \left\|\partial_{\mathbf{n}_j} w\right\|_{H^{-1/2}(\Gamma_j)} \lesssim \|\nabla w\|_{L^2(B_R \setminus \overline{\Omega_j})} + \|\Delta w\|_{L^2(B_R \setminus \overline{\Omega_j})} \lesssim k\, |||w|||_{B_R \setminus \overline{\Omega_j}},$$

which implies (3.1) and concludes the proof of this lemma. ∎

Then, the following estimate for $\widetilde{E}_j$ defined in (2.34) holds:

Lemma 3.2. *Let (2.13a)–(2.13c) and (2.23) be satisfied. For any $\varphi \in H^{1/2}(\Gamma_j)$, there exists a positive constant C independent of k, σ_0 and d, but depends on Ω, such that*

$$\left\|\widetilde{E}_j \varphi\right\|_{H^{1/2}(\widehat{\Gamma}_j)} \le C\lambda k^3 \gamma^{-2} (\gamma\sigma_0 d)^{-1} (1 + d\gamma^{-1})^{n-1} e^{-\gamma k \sigma_0 d} \|\varphi\|_{H^{1/2}(\Gamma_j)}.$$

Proof. Similarly to the proof of Lemma 2.2, from (2.24)–(2.25) and (3.1), when $x \in \widehat{\Gamma}_j$, we have

$$\begin{aligned} \left|\widetilde{E}_j \varphi(x)\right| &\le \|\varphi\|_{L^2(\Gamma_j)} \left\|\partial_{\mathbf{n}_j} \widetilde{G}_j(x,\cdot)\right\|_{L^2(\Gamma_j)} + \|T_j \varphi\|_{H^{-1/2}(\Gamma_j)} \left\|\widetilde{G}_j(x,\cdot)\right\|_{H^{1/2}(\Gamma_j)} \\ &\lesssim \max_{x \in \widehat{\Gamma}_j, y \in \Gamma_j} \left\{\left|\nabla_y \widetilde{G}_j(x,y)\right|, \left|\widetilde{G}_j(x,y)\right|\right\} k \|\varphi\|_{H^{1/2}(\Gamma_j)} \\ &\lesssim k^2 \gamma^{-1} (\gamma\sigma_0 d)^{-1} e^{-\gamma k \sigma_0 d} \|\varphi\|_{H^{1/2}(\Gamma_j)}. \end{aligned}$$

Then we get

$$\|\widetilde{E}_j\varphi\|_{L^2(\widehat{\Gamma}_j)} \lesssim |\hat{\Gamma}_j|^{\frac{1}{2}}\|\widetilde{E}_j\varphi\|_{L^\infty(\widehat{\Gamma}_j)} \lesssim k^2\gamma^{-1}(\gamma\sigma_0 d)^{-1}(L+d)^{\frac{n-1}{2}}e^{-\gamma k\sigma_0 d}\|\varphi\|_{H^{1/2}(\Gamma_j)}.$$

To estimate $|\widetilde{E}_j\varphi|_{H^{1/2}(\widehat{\Gamma}_j)}$, we start by noting

$$\left|\widetilde{E}_j\varphi(x) - \widetilde{E}_j\varphi(x')\right| \le \left\|\nabla\widetilde{E}_j\varphi\right\|_{L^\infty(\widehat{\Gamma}_j)}|x-x'|.$$

Similarly, from (2.26)–(2.27) and (3.1), when $x \in \widehat{\Gamma}_j$, we get

$$\begin{aligned}
&\left|\nabla\widetilde{E}_j\varphi(x)\right| \\
&\lesssim \|\varphi\|_{L^2(\Gamma_j)}\max_{x\in\widehat{\Gamma}_j, y\in\Gamma_j}\left|\nabla_x\nabla_y\widetilde{G}_j(x,y)\right| + \left\|T_j\varphi\right\|_{H^{-1/2}(\Gamma_j)}\max_{x\in\widehat{\Gamma}_j}\left\|\nabla_x\widetilde{G}_j(x,\cdot)\right\|_{H^{1/2}(\Gamma_j)} \\
&\lesssim \max_{x\in\widehat{\Gamma}_j, y\in\Gamma_j}\left\{\left|\nabla_x\nabla_y\widetilde{G}_j(x,y)\right|, \left|\nabla_x\widetilde{G}_j(x,y)\right|\right\}k\|\varphi\|_{H^{1/2}(\Gamma_j)} \\
&\lesssim \lambda k^3\gamma^{-2}(\gamma\sigma_0 d)^{-1}e^{-\gamma k\sigma_0 d}\|\varphi\|_{H^{1/2}(\Gamma_j)},
\end{aligned}$$

which implies that

$$\left|\widetilde{E}_j\varphi\right|_{H^{1/2}(\widehat{\Gamma}_j)} \lesssim |\hat{\Gamma}_j|\left\|\nabla\widetilde{E}_j\varphi\right\|_{L^\infty(\widehat{\Gamma}_j)} \lesssim \lambda k^3\gamma^{-2}(\gamma\sigma_0 d)^{-1}(L+d)^{n-1}e^{-\gamma k\sigma_0 d}\|\varphi\|_{H^{1/2}(\Gamma_j)}.$$

This completes the proof of the lemma by noting (2.1) and $L+d \approx d\gamma^{-1}$. ∎

3.2. Stability estimates for the PML equation in the layer. In this subsection, we consider the following Dirichlet PML equation in the layer $\widehat{\Omega}_j$:

$$\begin{aligned}
\operatorname{div}(A_j\nabla w) + k^2 J_j w &= 0 \quad \text{in } \widehat{\Omega}_j, \\
w &= 0 \quad \text{on } \Gamma_j, \\
w &= g \quad \text{on } \widehat{\Gamma}_j.
\end{aligned} \tag{3.2}$$

From [14, §3.1], the inf-sup condition in $H_0^1(\widehat{\Omega}_j)$ holds

$$\sup_{\varphi\in H_0^1(\widehat{\Omega}_j)}\frac{\left|(A_j\nabla\psi,\nabla\varphi)_{\widehat{\Omega}_j} - k^2(J_j\psi,\varphi)_{\widehat{\Omega}_j}\right|}{|||\varphi|||_{\widehat{\Omega}_j}} \ge \mu\,|||\psi|||_{\widehat{\Omega}_j} \quad \forall\,\psi \in H_0^1(\widehat{\Omega}_j).$$

where $\mu^{-1} \le C_p(\sigma_0,\gamma^{-1})k^{3/2}$. Moreover, by following the proof in [7, Theorem 5.7], the PML problem (3.2) in the layer has a unique solution and satisfies the stability estimates. Since the proof is quite similar, we omit it.

Lemma 3.3. *Let $g \in H^{1/2}(\hat{\Gamma}_j)$ and w be the solution to (3.2), for sufficiently large $\sigma_0 d$, there holds*

$$|||w|||_{\widehat{\Omega}_j} + k^{-1}\left\|\partial_{\mathbf{n}_j}w\right\|_{H^{-1/2}(\Gamma_j)} \le C_p(k,\sigma_0,\gamma^{-1})\|g\|_{H^{1/2}(\widehat{\Gamma}_j)}. \tag{3.3}$$

3.3. Convergence of the PML problem. In this subsection, we give the convergence analysis for the PML problem (2.39). First, we derive the PML truncation error equation and divide it into two subproblems. Then, the stability estimates of these subproblems are obtained.

3.3.1. PML truncation error. Let $\eta = u - v$ in Ω, $\eta_j = u_j - v_j$ in Ω_j, and $\widetilde{\eta}_j = E_j(u_j^+ - v_j^+)$ in $\mathbb{R}^n\setminus\overline{\Omega_j}$. By combining (2.10) and (2.39), and noting that $u_j|_{\mathbb{R}^n\setminus\overline{\Omega_j}} = E_j u_j^+$, we obtain the following system of equations for η_j and η:

$$\Delta\eta_j + k^2\eta_j = 0 \qquad \text{in } \Omega_j, \tag{3.4a}$$

$$\Delta\widetilde{\eta}_j + k^2\widetilde{\eta}_j = 0 \qquad \text{in } \mathbb{R}^n\setminus\overline{\Omega_j}, \tag{3.4b}$$

$$\eta_j - \widetilde{\eta}_j = -\eta, \quad \partial_{\mathbf{n}_j}\eta_j - \partial_{\mathbf{n}_j}\widetilde{\eta}_j = -\partial_{\mathbf{n}_j}\eta + \partial_{\mathbf{n}_j}\xi_j \qquad \text{on } \Gamma_j, \tag{3.4c}$$

$$|\partial_{\mathbf{n}}\widetilde{\eta}_j - \mathbf{i}k\widetilde{\eta}_j| = o\big(|x|^{\frac{1-n}{2}}\big) \qquad \text{for } |x|\to\infty, \tag{3.4d}$$

$$\Delta\eta + k^2\eta = 0 \qquad \text{in } \Omega, \tag{3.4e}$$

(3.4f)
$$\partial_{\mathbf{n}}\eta - \mathbf{i}k\eta = \sum_{j=1}^{m}(\partial_{\mathbf{n}} - \mathbf{i}k)(\widetilde{\eta}_j + \zeta_j) \quad \text{on } \Gamma,$$

where $\xi_j = (\widetilde{E}_j v_j^+ - v_j)|_{\widehat{\Omega}_j}$ and $\zeta_j = (E_j - \widehat{E}_j)v_j^+$. Obviously, $\xi_j = 0$ on Γ_j and $\xi_j = \widetilde{E}_j v_j^+$ on $\widehat{\Gamma}_j$. Therefore, ξ_j is the solution to the PML equation (3.2) in the layer with $g = \widetilde{E}_j v_j^+$. From Lemma 3.3 and Lemma 3.2, and noting $\gamma^{-1} \le k\sigma_0 d$, we know that

(3.5)
$$|||\xi_j|||_{\widehat{\Omega}_j} + \big\|\partial_{\mathbf{n}_j}\xi_j\big\|_{H^{-1/2}(\Gamma_j)} \lesssim C_p(k,\sigma_0,d)e^{-\gamma k\sigma_0 d}\big\|v_j^+\big\|_{H^{1/2}(\Gamma_j)}.$$

On the other hand, from (2.32) and (2.37) we see that $\zeta_j = (E_j - \widehat{E}_j)v_j^+ = -S_j(T_j - \widehat{T}_j)v_j^+$. From (2.33), (2.35) and Remark 2.4 (iv), we get

$$(T_j - \widehat{T}_j)v_j^+ = \partial_{\mathbf{n}_j}E_j v_j^+ - \partial_{\mathbf{n}_j}(v_j|_{\widehat{\Omega}_j}) = \partial_{\mathbf{n}_j}\xi_j \quad \text{and} \quad \zeta_j = -S_j\partial_{\mathbf{n}_j}\xi_j.$$

By using the trace theorem and (3.5), and the fact that $\Delta\zeta_j + k^2\zeta_j = 0$ in $\mathbb{R}^n \setminus \overline{\Omega_j}$ and the operator $S_j : H^{-1/2}(\Gamma_j) \to H^1_{\text{loc}}(\mathbb{R}^n \setminus \overline{\Omega_j})$ is continuous (see [61, Theorem 3.1.16]), it follows that

(3.6)
$$\begin{aligned}\|\partial_{\mathbf{n}}\zeta_j\|_{H^{-1/2}(\Gamma)} + \|\zeta_j\|_{H^{1/2}(\Gamma)} &\lesssim k^2\|\zeta_j\|_{H^1(B\setminus\overline{\Omega})} \lesssim C_p(k)\big\|\partial_{\mathbf{n}_j}\xi_j\big\|_{H^{-1/2}(\Gamma_j)}\\ &\lesssim C_p(k,\sigma_0,d)e^{-\gamma k\sigma_0 d}\big\|v_j^+\big\|_{H^{1/2}(\Gamma_j)},\end{aligned}$$

where B denotes a sufficiently large ball which contains $\overline{\Omega}$.

To estimate η, we divide (3.4) into two subproblems. First, we denote

$$w = \eta_j + \sum_{i\neq j}\widetilde{\eta}_i \quad \text{in } \Omega_j \quad \text{and} \quad \widetilde{w} = \sum_{j=1}^{m}\widetilde{\eta}_j \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.$$

From (3.4c), we have

(3.7)
$$[w] = 0 \quad \text{and} \quad \big[\partial_{\mathbf{n}_j}w\big] = \partial_{\mathbf{n}_j}\xi_j + \partial_{\mathbf{n}_{j'}}\xi_{j'} \quad \text{on } \Gamma_j \cap \Gamma_{j'},$$

(3.8)
$$w - \widetilde{w} = -\eta \quad \text{and} \quad \partial_{\mathbf{n}_j}w - \partial_{\mathbf{n}_j}\widetilde{w} = -\partial_{\mathbf{n}}\eta + \partial_{\mathbf{n}}\xi_j \quad \text{on } \Gamma_j \cap \Gamma.$$

Hence, from (3.8) and (3.4f), we get

$$\begin{aligned}(\partial_{\mathbf{n}}w - \mathbf{i}kw) - (\partial_{\mathbf{n}}\widetilde{w} - \mathbf{i}k\widetilde{w}) &= -(\partial_{\mathbf{n}} - \mathbf{i}k)\eta + \partial_{\mathbf{n}}\xi_j\\ &= -(\partial_{\mathbf{n}} - \mathbf{i}k)\widetilde{w} - \sum_{i=1}^{m}(\partial_{\mathbf{n}} - \mathbf{i}k)\zeta_i + \partial_{\mathbf{n}}\xi_j \quad \text{on } \Gamma_j \cap \Gamma,\end{aligned}$$

which yields

$$\partial_{\mathbf{n}}w - \mathbf{i}kw = \partial_{\mathbf{n}}\xi_j - \sum_{i=1}^{m}(\partial_{\mathbf{n}} - \mathbf{i}k)\zeta_i \quad \text{on } \Gamma_j \cap \Gamma.$$

Therefore, by using (3.7), w is the solution to the interior Helmholtz problem:

(3.9)
$$\begin{aligned}\Delta w + k^2 w &= 0 && \text{in } \Omega_j,\ j = 1,\cdots,m,\\ [w] = 0,\ \big[\partial_{\mathbf{n}_j}w\big] &= \partial_{\mathbf{n}_j}\xi_j + \partial_{\mathbf{n}_{j'}}\xi_{j'} && \text{on } \Gamma_j \cap \Gamma_{j'},\\ \partial_{\mathbf{n}}w - \mathbf{i}kw &= \partial_{\mathbf{n}}\xi_j - \sum_{i=1}^{m}(\partial_{\mathbf{n}} - \mathbf{i}k)\zeta_i && \text{on } \Gamma_j \cap \Gamma.\end{aligned}$$

Second, we extend η by defining $\widetilde{\eta} = \widetilde{w}$, from (3.4e), (3.8) and the definition (2.32), it can be shown that η and $\widetilde{\eta}$ are the solutions to the full-space transmission problem:

(3.10)
$$\begin{aligned}\Delta\eta + k^2\eta &= 0 && \text{in } \Omega,\\ \Delta\widetilde{\eta} + k^2\widetilde{\eta} &= 0 && \text{in } \mathbb{R}^n \setminus \overline{\Omega},\\ \eta - \widetilde{\eta} = -w,\ \partial_{\mathbf{n}}\eta - \partial_{\mathbf{n}}\widetilde{\eta} &= \partial_{\mathbf{n}}\xi_j - \partial_{\mathbf{n}}w && \text{on } \Gamma_j \cap \Gamma,\\ |\partial_{\mathbf{n}}\widetilde{\eta} - \mathbf{i}k\widetilde{\eta}| &= o\big(|x|^{\frac{1-n}{2}}\big) && \text{as } |x| \to \infty.\end{aligned}$$

3.3.2. Estimate for w. Denote the sesquilinear form by

$$b_\Omega(\psi,\varphi) := (\nabla\psi,\nabla\varphi)_\Omega - k^2(\psi,\varphi)_\Omega - \mathbf{i}k\,\langle\psi,\varphi\rangle_\Gamma \quad \forall\,\psi,\,\varphi\in H^1(\Omega). \tag{3.11}$$

The weak formulation of (3.9) reads as: find $w\in H^1(\Omega)$ such that

$$b_\Omega(w,\varphi) = \sum_{j=1}^{m}\left\langle \partial_{\mathbf{n}_j}\xi_j,\varphi\right\rangle_{\Gamma_j} - \sum_{j=1}^{m}\langle(\partial_{\mathbf{n}}-\mathbf{i}k)\zeta_j,\varphi\rangle_\Gamma \quad \forall\,\varphi\in H^1(\Omega). \tag{3.12}$$

For given ξ_j and ζ_j, problem (3.12) has a unique solution and satisfies the inf-sup condition (see, e.g., [57, 11])

$$\inf_{0\neq\psi\in H^1(\Omega)}\ \sup_{0\neq\varphi\in H^1(\Omega)} \frac{|b_\Omega(\psi,\varphi)|}{|||\psi|||_\Omega\,|||\varphi|||_\Omega} \geq C_p(k)^{-1}. \tag{3.13}$$

By using the trace theorem, we obtain

$$\begin{aligned} C_p(k)^{-1}\,|||w|||_\Omega &\leq \sup_{0\neq\varphi\in H^1(\Omega)}\frac{|b_\Omega(w,\varphi)|}{|||\varphi|||_\Omega} \\ &\lesssim \sum_{j=1}^{m}\left\|\partial_{\mathbf{n}_j}\xi_j\right\|_{H^{-1/2}(\Gamma_j)} + \sum_{j=1}^{m}\|\partial_{\mathbf{n}}\zeta_j\|_{H^{-1/2}(\Gamma)} + k\sum_{j=1}^{m}\|\zeta_j\|_{L^2(\Gamma)}, \end{aligned}$$

which together with (3.5) and (3.6) gives

$$|||w|||_\Omega \lesssim C_p(k,\sigma_0,d)e^{-\gamma k\sigma_0 d}\sum_{j=1}^{m}\left\|v_j^+\right\|_{H^{1/2}(\Gamma_j)}. \tag{3.14}$$

Furthermore, integration by parts results in

$$\begin{aligned} \left\|\partial_{\mathbf{n}_j}w^-\right\|_{H^{-1/2}(\Gamma_j)} &\lesssim \|\Delta w\|_{L^2(\Omega_j)} + \|\nabla w\|_{L^2(\Omega_j)} = k^2\|w\|_{L^2(\Omega_j)} + \|\nabla w\|_{L^2(\Omega_j)} \\ &\lesssim k\,|||w|||_\Omega \lesssim C_p(k,\sigma_0,d)e^{-\gamma k\sigma_0 d}\sum_{j=1}^{m}\left\|v_j^+\right\|_{H^{1/2}(\Gamma_j)}. \end{aligned} \tag{3.15}$$

3.3.3. Estimate for η. In view of the last three equations of (3.10), $\widetilde{\eta}$ satisfies the exterior Helmholtz problem with Dirichlet data $\widetilde{\eta} = \eta + w$ on Γ, by applying the DtN operator T on Γ (see, e.g., [58, 19, 9]), we deduce $\partial_{\mathbf{n}}\widetilde{\eta} = T(\eta+w)$ on Γ. Then combining with the first and third equations of (3.10) yields

$$\begin{aligned} \Delta\eta + k^2\eta &= 0 && \text{in } \Omega, \\ \partial_{\mathbf{n}}\eta - T(\eta+w) &= \partial_{\mathbf{n}}\xi_j - \partial_{\mathbf{n}}w && \text{on } \Gamma_j\cap\Gamma. \end{aligned}$$

Since T is linear, $\eta\in H^1(\Omega)$ is the weak solution to

$$c_\Omega(\eta,\varphi) = \sum_{j=1}^{m}\langle Tw + \partial_{\mathbf{n}}\xi_j - \partial_{\mathbf{n}}w,\varphi\rangle_{\Gamma_j\cap\Gamma} \quad \forall\varphi\in H^1(\Omega),$$

where

$$c_\Omega(\psi,\varphi) := (\nabla\psi,\nabla\varphi)_\Omega - k^2(\psi,\varphi)_\Omega - \langle T\psi,\varphi\rangle_\Gamma\,. \tag{3.16}$$

Using the interface condition (3.9), we can get

$$c_\Omega(\eta,\varphi) = \langle Tw,\varphi\rangle_\Gamma + \sum_{j=1}^{m}\left\langle\partial_{\mathbf{n}_j}\xi_j,\varphi\right\rangle_{\Gamma_j} - \sum_{j=1}^{m}\left\langle\partial_{\mathbf{n}_j}w^-,\varphi\right\rangle_{\Gamma_j}$$

for all $\varphi \in H^1(\Omega)$. By applying the inf-sup condition of c_Ω (see, e.g., [9]), the continuity of T (see, e.g., [19]) and the trace theorem, the following stability for η holds:

$$
\begin{aligned}
C_p(k)^{-1} \|\|\eta\|\|_\Omega &\lesssim \|Tw\|_{H^{-1/2}(\Gamma)} + \sum_{j=1}^m \left\|\partial_{\mathbf{n}}\xi_j - \partial_{\mathbf{n}} w^-\right\|_{H^{-1/2}(\Gamma_j)} \\
&\lesssim C_p(k) \|\|w\|\|_\Omega + \sum_{j=1}^m \left\|\partial_{\mathbf{n}_j}\xi_j\right\|_{H^{-1/2}(\Gamma_j)} + \sum_{j=1}^m \left\|\partial_{\mathbf{n}_j} w^-\right\|_{H^{-1/2}(\Gamma_j)}.
\end{aligned} \tag{3.17}
$$

Finally, we have the following convergence theorem.

THEOREM 3.4. *Let u and v denote the solutions to (2.38) and (2.39), respectively. There exists a positive constant Λ_0 such that if $\gamma k \sigma_0 d \geq \Lambda_0$, then*

$$
\|\|u - v\|\|_\Omega \leq C_p(k, \sigma_0, d) e^{-\gamma k \sigma_0 d} \|f\|_{L^2(\Omega)}. \tag{3.18}
$$

Proof. By combining (3.5) and (3.14)–(3.17), we get

$$
\|\|u - v\|\|_\Omega \lesssim C_p(k, \sigma_0, d) e^{-\gamma k \sigma_0 d} \sum_{j=1}^m \left\|v_j^+\right\|_{H^{1/2}(\Gamma_j)}.
$$

Using the trace theorem and Lemma 2.5, we obtain

$$
\begin{aligned}
\left\|v_j^+\right\|_{H^{1/2}(\Gamma_j)} \lesssim \|v_j\|_{H^1(\widehat{\Omega}_j)} &\lesssim C_p(k)\big(\|v\|_{H^{1/2}(\Gamma_j)} + \left\|\partial_{\mathbf{n}_j} v\right\|_{H^{-1/2}(\Gamma_j)}\big) \\
&\lesssim C_p(k)\big(\|v\|_{H^1(\Omega_j)} + \|\Delta v\|_{L^2(\Omega_j)}\big) \\
&\lesssim C_p(k)\big(\|v\|_{H^1(\Omega_j)} + \left\|k^2 v\right\|_{L^2(\Omega_j)} + \|f\|_{L^2(\Omega_j)}\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\|u - v\|\|_\Omega &\leq C_p(k, \sigma_0, d) e^{-\gamma k \sigma_0 d} \Big(\|v\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\Big) \\
&\leq C_p(k, \sigma_0, d) e^{-\gamma k \sigma_0 d} \Big(\|u - v\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\Big) \\
&\leq C_p(k, \sigma_0, d) e^{-\gamma k \sigma_0 d} \Big(\|\|u - v\|\|_\Omega + (1 + C_{\text{stab}})\|f\|_{L^2(\Omega)}\Big),
\end{aligned}
$$

where we used the stability estimate (2.4) in the last inequality. Then (3.18) follows by the assertion $C_p(k, \sigma_0, d) e^{-\gamma k \sigma_0 d} \leq 1/2$ if $\gamma k \sigma_0 d$ is large enough. ∎

Furthermore, we can obtain the well-posedness of the PML solution v.

COROLLARY 3.5. *Under the conditions of Theorem 3.4, there holds*

$$
\|v\|_{H^1(\Omega)} \lesssim (1 + C_{\text{stab}})\|f\|_{L^2(\Omega)}, \tag{3.19}
$$

and hence the PML system of equations (2.39) is well-posed.

Proof. The stability estimate (3.19) is a direct consequence of (3.18) and the stability estimate (2.4). The uniquenesses of the solutions v and v_j to the PML system (2.39) follow from the stability estimates (3.19) and (2.40). It suffices to prove the existence of solutions.

First, for any given $v \in H^1(\Omega)$, the solution v_j to (2.39a)–(2.39c), denoted by $v_j(v)$, exists uniquely according to Lemma 2.5.

Second, similar to the derivations of (3.4), (3.9) and (3.10), by defining $\widetilde{v}_j = E_j v_j^+$ in $\mathbb{R}^n \setminus \overline{\Omega_j}$ and letting

$$
\chi = v_j + \sum_{i \neq j} \widetilde{v}_i \quad \text{in } \Omega_j \quad \text{and} \quad \widetilde{\chi} = \sum_{j=1}^m \widetilde{v}_j \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega},
$$

and extending v by $\widetilde{v} = \widetilde{\chi}$, we arrive at

$$
b_\Omega(\chi, \varphi) = -\sum_{j=1}^m \left\langle \partial_{\mathbf{n}_j}\xi_j, \varphi \right\rangle_{\Gamma_j} + \sum_{j=1}^m \langle (\partial_{\mathbf{n}} - \mathbf{i}k)\zeta_j, \varphi \rangle_\Gamma \quad \forall\, \varphi \in H^1(\Omega),
$$

and

$$c_\Omega(v,\varphi) = -(f,\varphi)_\Omega + \langle T\chi,\varphi\rangle_\Gamma - \sum_{j=1}^m \left\langle \partial_{\mathbf{n}_j}\xi_j,\varphi\right\rangle_{\Gamma_j} - \sum_{j=1}^m \left\langle \partial_{\mathbf{n}_j}\chi^-,\varphi\right\rangle_{\Gamma_j} \quad \forall\,\varphi\in H^1(\Omega),$$

where the sesquilinear forms b_Ω and c_Ω are defined in (3.11) and (3.16), respectively. The functions $\xi_j = (\widetilde{E}_j v_j^+ - v_j)|_{\widehat{\Omega}_j}$ and $\zeta_j = (E_j - \widehat{E}_j)v_j^+$, denoted by $\xi_j = \xi_j(v)$ and $\zeta_j = \zeta_j(v)$, are both uniquely determined by the given function v. By the Riesz representation theorem, there exist some bounded linear operators such that

$$(P_1\chi,\varphi) = b_\Omega(\chi,\varphi), \quad (P_2 v,\varphi) = c_\Omega(v,\varphi), \quad (Kf,\varphi) = -(f,\varphi),$$

$$(\mathcal{E}_1 v,\varphi) = -\sum_{j=1}^m \left\langle \partial_{\mathbf{n}_j}\xi_j(v),\varphi\right\rangle_{\Gamma_j}, \quad (\mathcal{E}_2 v,\varphi) = \sum_{j=1}^m \left\langle (\partial_{\mathbf{n}} - \mathbf{i}k)\zeta_j(v),\varphi\right\rangle_{\Gamma},$$

$$(K_1\chi,\varphi) = \langle T\chi,\varphi\rangle_\Gamma, \quad (K_2\chi,\varphi) = -\sum_{j=1}^m \left\langle \partial_{\mathbf{n}_j}\chi^-,\varphi\right\rangle_{\Gamma_j},$$

for any $\chi, v, \varphi \in H^1(\Omega)$ and $f\in L^2(\Omega)$. Then the variational problems of χ and v above are equivalent to finding $v,\chi\in H^1(\Omega)$ such that

$$P_1\chi = (\mathcal{E}_1+\mathcal{E}_2)v \quad \text{and} \quad P_2 v = Kf + (K_1+K_2)\chi + \mathcal{E}_1 v.$$

Since both the sesquilinear forms b_Ω and c_Ω satisfy the inf-sup condition, P_1 and P_2 are invertible operators. Therefore, we write

$$v = P_2^{-1}Kf + P_2^{-1}[(K_1+K_2)P_1^{-1}(\mathcal{E}_1+\mathcal{E}_2) + \mathcal{E}_1]v.$$

By the previous analyses in (3.5) and (3.6) and the stability estimate for v^j in Lemma 2.5, we see that $\mathcal{E}_1$ and $\mathcal{E}_2$ are both exponentially small with respect to v. More precisely, by denoting $M = P_2^{-1}[(K_1+K_2)P_1^{-1}(\mathcal{E}_1+\mathcal{E}_2)+\mathcal{E}_1]$, if $\gamma k\sigma_0 d$ is large enough, then

$$\|M\| \le C_p(k,\sigma_0,d)e^{-\gamma k\sigma_0 d} \le \varepsilon, \quad \text{with some constant } \varepsilon < 1,$$

where in the first inequality we have used the fact that the upper bounds of the norms of P_1^{-1}, P_2^{-1}, K_1 and K_2 are all $C_p(k)$. Then the Neumann series

$$\sum_{i=0}^\infty M^i$$

converges and has the limit $(I-M)^{-1}$, where I denotes the identity operator. It is easy to verify that $v = (I-M)^{-1}P_2^{-1}Kf$ and $v_j = v_j(v)$ solve the system (2.39). This proves the existence of solutions to (2.39). ∎

4. FEM for the coupled PML system.

In addition to the theoretical analysis of the stability and convergence of PML, we also present an iterative algorithm and a continuous interior penalty finite element method (CIP-FEM) in this section for the practical computation using the newly proposed coupled PML method.

4.1. Variational formulation.

Recalling the decomposition $B_j = \Omega_j \cup \Gamma_j \cup \widehat{\Omega}_j$, we define the piecewise H^1 spaces

$$V_j := \left\{ v\in L^2(B_j) : v|_{\Omega_j}\in H^1(\Omega_j),\, v|_{\widehat{\Omega}_j}\in H^1(\widehat{\Omega}_j),\, v|_{\widehat{\Gamma}_j} = 0\right\}, \quad j=1,\dots,m.$$

Denote the average of $\varphi\in V_j$ on Γ_j by $\{\varphi\} = \frac{1}{2}(\varphi^+ + \varphi^-)$. For any $\varphi_j\in V_j$ and $\varphi\in H^1(\Omega)$, applying integration by parts, we find that the solutions v_j and v to (2.39) satisfy the following equations:

$$\begin{aligned} 0 &= (A_j\nabla v_j,\nabla\varphi_j)_{\Omega_j\cup\widehat{\Omega}_j} - k^2(J_j v_j,\varphi_j)_{B_j} - \int_{\Gamma_j}\left[(\partial_{\mathbf{n}_j}v_j)\overline{\varphi}_j\right] \\ &= (A_j\nabla v_j,\nabla\varphi_j)_{\Omega_j\cup\widehat{\Omega}_j} - k^2(J_j v_j,\varphi_j)_{B_j} + \left\langle \partial_{\mathbf{n}_j}v,\{\varphi_j\}\right\rangle_{\Gamma_j} - \left\langle \left\{\partial_{\mathbf{n}_j}v_j\right\},[\varphi_j]\right\rangle_{\Gamma_j}, \end{aligned} \tag{4.1}$$

and

$$(\nabla v, \nabla \varphi)_\Omega - k^2 (v,\varphi)_\Omega - \mathbf{i}k \langle v, \varphi\rangle_\Gamma - \sum_{j=1}^m \left\langle (\partial_{\mathbf{n}} - \mathbf{i}k)\widehat{E}_j v_j^+, \varphi \right\rangle_\Gamma = -(f,\varphi)_\Omega. \tag{4.2}$$

Similar to the proof of Corollary 3.5, the above two equations together with the interface condition $[v_j] + v = 0$ on Γ_j yield a weakly coercive formulation, that is, by decoupling the solutions v with v_j, there exists a sesquilinear form $\tilde{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ such that

$$\tilde{a}(v,\varphi) = -(f,\varphi)_\Omega \quad \forall \varphi \in H^1(\Omega),$$

where $\tilde{a}(v,\varphi) := (P_2 v - P_2 M v, \varphi)$. Since M is an exponentially small perturbation operator and P_2 is weakly coercive, $\tilde{a}$ satisfies the weakly coercivity

$$\tilde{a}(\varphi,\varphi) \geq \alpha_0 \|\varphi\|^2_{H^1(\Omega)} - \alpha_1 \|\varphi\|^2_{L^2(\Omega)}, \quad \text{with constants } \alpha_0 > 0 \text{ and } \alpha_1 > 0,$$

which is useful in the convergence analysis of finite element discretization (cf. [36]).

Let $\mathcal{T}_h$ be a triangulation of $\overline{D} = \cup_{j=1}^m \overline{B_j}$. For simplicity, we assume that the triangulation $\mathcal{T}_h$ fits all the interfaces and boundaries. For any $K \in \mathcal{T}_h$, we define $h_K := \operatorname{diam}(K)$ and $h_e := \operatorname{diam}(e)$ for any edge $e \subset \partial K$. Denote $h = \max_{K\in\mathcal{T}_h} h_K$. Analogous to (4.1), we define the sesquilinear form for the interface problem (2.39a)–(2.39c) as follows

$$\begin{aligned} a_j(\psi,\varphi) := (A_j \nabla\psi, \nabla\varphi)_{\Omega_j\cup\widehat{\Omega}_j} - k^2 (J_j\psi,\varphi)_{B_j} - \Big(\left\langle \{\partial_{\mathbf{n}_j}\psi\}, [\varphi] \right\rangle_{\Gamma_j} + \beta_j \left\langle [\psi], \{\partial_{\mathbf{n}_j}\varphi\} \right\rangle_{\Gamma_j} \Big) \\ + \sum_{e\subset\Gamma_j} \gamma_j h_e^{-1} \langle [\psi], [\varphi] \rangle_e, \end{aligned}$$

where β_j and γ_j are the interface penalty parameters. Furthermore, we define the sesquilinear form for the Helmholtz problem with impedance boundary condtion (2.39d)–(2.39e) as follows:

$$a(\psi,\varphi) := (\nabla\psi,\nabla\varphi)_\Omega - k^2(\psi,\varphi)_\Omega - \mathbf{i}k\langle\psi,\varphi\rangle_\Gamma .$$

Since $[v_j] = -v$ and $[\partial_{\mathbf{n}_j} v_j] = -\partial_{\mathbf{n}_j} v$ on Γ_j, combining (4.1)–(4.2), the variational formulation with interface penalty for (2.39) reads: find $v_j \in V_j$ and $v \in H^1(\Omega)$ such that

$$\begin{cases} a_j(v_j,\varphi_j) = F_j(v,\varphi_j) & \forall \varphi_j \in V_j, \\ a(v,\varphi) = F(v_1,\cdots,v_m,\varphi) & \forall \varphi \in H^1(\Omega), \end{cases} \tag{4.3}$$

where the right hand sides are given by

$$F_j(v,\varphi) = -\left\langle \partial_{\mathbf{n}_j} v, \{\varphi\} \right\rangle_{\Gamma_j} + \beta_j \left\langle v, \{\partial_{\mathbf{n}_j}\varphi\} \right\rangle_{\Gamma_j} - \sum_{e\subset\Gamma_j} \gamma_j h_e^{-1} \langle v, [\varphi]\rangle_e, \tag{4.4}$$

$$F(v_1,\cdots,v_m,\varphi) = -(f,\varphi)_\Omega + \sum_{j=1}^m \left\langle (\partial_{\mathbf{n}} - \mathbf{i}k)\widehat{E}_j v_j^+, \varphi \right\rangle_\Gamma, \tag{4.5}$$

Remark 4.1. Some comments for the variational problem (4.3) are as follows.

(i) The term $\beta_j \langle [\psi], \{\partial_{\mathbf{n}_j}\varphi\}\rangle_{\Gamma_j}$ is the symmetrizing term. In general, β_j can be chosen as 0, ± 1.

(ii) The penalty term $\gamma_j h_e^{-1} \langle [\psi], [\varphi]\rangle_e$ on the interface Γ_j is also called a stabilization term, and the penalty parameter γ_j satisfies $\gamma_j \gtrsim 1$. The idea of using an interface penalty is inspired by the discontinuous Galerkin method. (see, e.g., [2, 53]).

(iii) In view of the definition of $\widehat{E}_j v_j^+(x)$ in Remark 2.4 (iv), the right-hand side of (4.5) actually contains two integrals which contain singularity when x is on Γ_j and are regular when x is away from Γ_j. To avoid evaluating singular integrals, we can consider the equation satisfied by $w_j = \widehat{E}_j v_j^+$:

$$\begin{aligned} \Delta w_j + k^2 w_j &= 0 && \text{in } B_j \setminus \Gamma_j, \\ [w_j] = -v_j^+, \ \ [\partial_{\mathbf{n}_j} w_j] &= -\partial_{\mathbf{n}_j} v_j^+ && \text{on } \Gamma_j, \\ (\partial_{\mathbf{n}} - \mathbf{i}k) w_j &= (\partial_{\mathbf{n}} - \mathbf{i}k)\widehat{E}_j v_j^+ && \text{on } \widehat{\Gamma}_j. \end{aligned} \tag{4.6}$$

$\widehat{E}_j v_j^+(x)$ can be obtained by solving (4.6) when $x \in B_j \supset \Gamma_j$, and by evaluating the two integrals when $x \in \Gamma \backslash B_j$ (in this case there is no singularity). In fact, the domain B_j in (4.6) can be replaced by any neighborhood of Γ_j.

4.2. The iterative FEM. The linear finite element spaces are defined as follows:

$$V_{j,h} := \{v_h \in V_j : v_h|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h,\ \ K \subset B_j\},$$

$$V_h := \left\{v_h \in H^1(\Omega) : v_h|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h,\ \ K \subset \Omega\right\},$$

where $\mathcal{P}_1(K)$ denotes the set of all first order polynomials on K. Then the FEM for the problem (4.3) reads: find $v_{j,h} \in V_{j,h}$ and $v_h \in V_h$ such that

$$\begin{cases} a_j(v_{j,h}, \varphi_{j,h}) = F_j(v_h, \varphi_{j,h}) & \forall\, \varphi_{j,h} \in V_{j,h}, \\ a(v_h, \varphi_h) = F_h(v_{1,h}, \cdots, v_{m,h}, \varphi_h) & \forall\, \varphi_h \in V_h, \end{cases} \tag{4.7}$$

where

$$F_h(v_{1,h}, \cdots, v_{m,h}, \varphi_h) = -(f, \varphi_h) + \sum_{j=1}^{m} \left\langle (\partial_{\mathbf{n}} - \mathbf{i}k) w_{j,h}, \varphi \right\rangle_\Gamma. \tag{4.8}$$

Here $w_{j,h} = \widehat{E}_j v_{j,h}^+$ on $\Gamma \setminus B_j$ and $w_{j,h}|_{B_j}$ is the FE approximation of (4.6) in B_j.

In practice, the coupled system (4.7) can be solved by iterative methods. For example, given an initial value $v_h^0 \in V_h$, find $v_{j,h}^l \in V_{j,h}$ and $v_h^l \in V_h$ for $l = 1, 2, \cdots$, such that

$$\begin{cases} a_j(v_{j,h}^l, \varphi_{j,h}) = F_j(v_h^{l-1}, \varphi_{j,h}) & \forall\, \varphi_{j,h} \in V_{j,h}, \\ a(v_h^l, \varphi_h) = F_h(v_{1,h}^l, \cdots, v_{j,h}^l, \varphi_h) & \forall\, \varphi_h \in V_h. \end{cases} \tag{4.9}$$

The rigorous proof of the convergence of (4.7) and (4.9) remains open and deserves further investigation in future work. The numerical experiments in the next section show that the iterative algorithm (4.9) converges well.

4.3. The CIP-FEM. It is known that the standard FEM will generate pollution errors in solving the Helmholtz equation with large wave number k, see [3, 56, etc.]. Reducing pollution errors requires the mesh size in the standard FEM to satisfy $k^3h^2 \lesssim 1$ in practical computations, which significantly increases the computational costs when k is large. To reduce the pollution error, we introduce a CIP-FEM for solving the coupled PML system. The CIP-FEM was first proposed by Douglas and Dupont in [23] for second order elliptic and parabolic PDEs, and it was applied to the the Helmholtz problem by Wu et al. in [65, 66, 25, 51, 52]. The CIP-FEM has shown great potential in solving the Helmholtz problem with large wave number, since it only requires probably the mesh size to satisfy $kh \lesssim 1$ in practical computation.

Let $\mathcal{E}_h^I$ denote the set of all interior edges (or faces in 3D) of the triangulation $\mathcal{T}_h$ in D. The sesquilinear forms of the CIP-FEM are given by

$$a_{j,h}(\psi, \varphi) := a_j(\psi, \varphi) + \sum_{e \in \mathcal{E}_h^I, e \not\subset \Gamma_j} \gamma_e h_e \left\langle [\partial_{\mathbf{n}} \psi], [\partial_{\mathbf{n}} \varphi] \right\rangle_e,$$

$$a_h(\psi, \varphi) := a(\psi, \varphi) + \sum_{e \in \mathcal{E}_h^I, e \subset \Omega} \gamma_e h_e \left\langle [\partial_{\mathbf{n}} \psi], [\partial_{\mathbf{n}} \varphi] \right\rangle_e,$$

where the penalty parameters γ_e are numbers with nonpositive imaginary parts and the jumps on every $e \subset \partial K_1 \cap \partial K_2 \in \mathcal{E}_h^I$ are defined as

$$[\partial_{\mathbf{n}} \psi]\,|_e = \nabla \psi|_{K_1} \cdot \mathbf{n}_{K_1} + \nabla \psi|_{K_2} \cdot \mathbf{n}_{K_2}.$$

The CIP-FEM for the problem (4.3) can be written as: find $v_{j,h} \in V_{j,h}$ and $v_h \in V_h$ such that

$$\begin{cases} a_{j,h}(v_{j,h}, \varphi_{j,h}) = F_j(v_h, \varphi_{j,h}) & \forall\, \varphi_{j,h} \in V_{j,h}, \\ a_h(v_h, \varphi_h) = F_h(v_{1,h}, \cdots, v_{m,h}, \varphi_h) & \forall\, \varphi_h \in V_h. \end{cases} \tag{4.10}$$

Remark 4.2.

(i) If $\gamma_e \equiv 0$, the CIP-FEM becomes standard FEM. If we consider the scattering problem with time dependence $e^{\mathbf{i}\omega t}$, that is, the sign before $\mathbf{i}$ in (1.2) is positive, then the penalty parameters γ_e should be complex numbers with nonnegative imaginary parts.

(ii) If v_j and v are the exact solutions to (2.39), then $[\partial_{\mathbf{n}} v_j] = 0$ on $e \not\subset \Gamma_j$ and $[\partial_{\mathbf{n}} v] = 0$ on $e \subset \Omega$. In this case $a_{j,h}(v_j, \varphi_{j,h}) = a_j(v_j, \varphi_{j,h})$ and $a_h(v, \varphi_h) = a(v, \varphi_h)$, and therefore, the CIP-FEM in (4.7) is consistent with the variational formulation in (4.3).

(iii) Similarly as (4.9), we can also solve (4.10) by an iterative method.

(iv) In the extreme case that $\Omega \subset \mathbb{R}^2$ is a slender L-shape domain with large length L and small width W, we can choose $\sigma_0 = O(L/k)$ and $d \approx W = O(1)$ so that condition (2.23) is satisfied. Then the degrees of freedom for the coupled PML method is about $O(LWh^{-2})$, while the degrees of freedom for the standard PML method is about $O(L^2h^{-2})$.

5. Numerical experiments. In this section, we present some numerical experiments to demonstrate the convergence and performance of the proposed coupled PML method for the Helmholtz problem (1.1)–(1.2) in an L-shape domain. All the computations are performed by MATLAB.

We first construct an analytical solution to the Helmholtz problem (1.1)–(1.2) in the whole space. As shown in Figure 5.1 (left), Ω_0 is the domain consisting of three disjoint circles of radius $R = 0.25$, and Ω is an L-shape domain containing Ω_0. The source term is defined by $f = -1$ in Ω_0 and $f = 0$ in $\mathbb{R}^2 \backslash \Omega_0$. The corresponding exact solution (see [51]) of the Helmholtz problem (1.1)–(1.2) is given by

$$u(x) = \sum_{l=1}^{3} u_l(x) \text{ with } u_l(x) = \begin{cases} \dfrac{\mathbf{i}\pi R}{2k} H_1^{(1)}(kR) J_0(k\,|x - x_l|) - \dfrac{1}{k^2} & \text{if } |x - x_l| \le R, \\ \dfrac{\mathbf{i}\pi R}{2k} J_1(kR) H_0^{(1)}(k\,|x - x_l|) & \text{otherwise,} \end{cases} \tag{5.1}$$

where x_l $(l = 1, 2, 3)$ denote the centres of the three circles of Ω_0, respectively.

Example 5.1. In the first example, we compare the numerical solutions given by the proposed coupled PML method and classical rectangular PML method by using the iterative FEM (4.9) described in Section 4.2 and the standard FEM, respectively. An L-shape domain Ω is considered, which is very thin in one direction, with length $L = 30$ and width $W = 1$.

The wave number is $k = 10$. The PML thickness and PML parameter are chosen to be $d = 1$ and $\sigma_0 = 8$, respectively. Clearly, the PML thickness is much smaller than the diameter of Ω, and therefore each subsystem of the coupled PML system contains much smaller degrees of freedom than the classical rectangular PML. The interface penalty parameters in the FEM are chosen to be $\beta_j = 1$ and $\gamma_j = 10$.

By comparing the numerical solutions with the exact solution in (5.1), we present the relative H^1-errors of the finite element solutions given by the coupled PML method and rectangular PML method in Figure 5.2 (left), where the horizontal axis represents the degrees of freedom (DOF), and for the coupled PML method refers to the maximum of all the DOFs for all the linear subsystems produced by (4.9). Since each subsystem is solved independently of the others, the maximum of DOFs actually measures the peak memory cost in the entire computation, if parallel method is not considered. The numerical results in Figure 5.2 show that, compared to the classical rectangular PML method, the coupled PML method can achieve the same accuracy with much fewer DOF. In particular, the peak of the memory cost for the coupled PML method is only about 15% of the classical rectangular PML method in order to achieve the accuracy with 10% relative error. In this way, the elapsed time for solving the finite element solutions with the coupled PML and the rectangular PML are almost the same.

Example 5.2. In the second example, we demonstrate the effectiveness of the proposed CIP-FEM compared with the standard FEM, and the convergence of the iterative method (4.9). An L-shape domain Ω with length $L = 6$ and width $W = 1$ is considered.

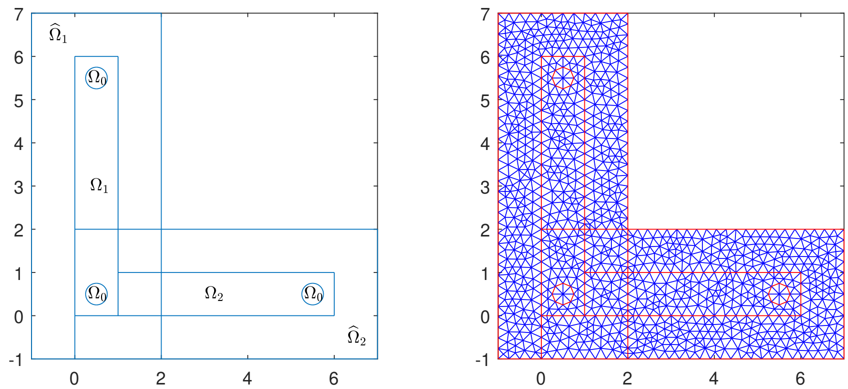

Fig. 5.1. *Left figure: The construction of PML. Right figure: The triangulation.*

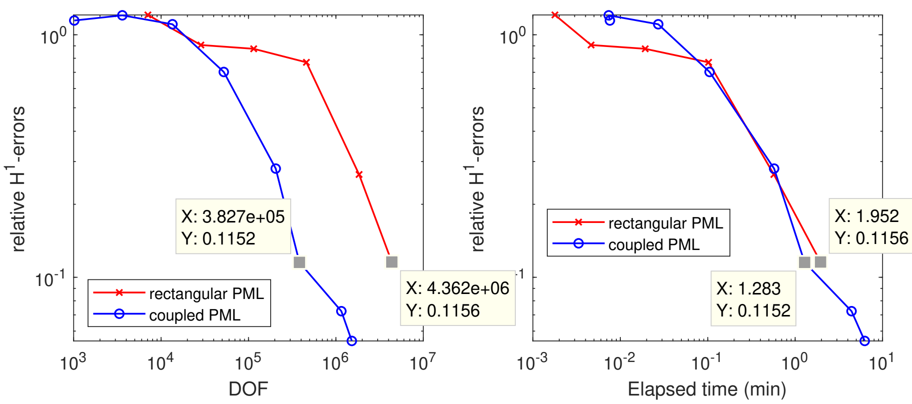

Fig. 5.2. *Relative H^1 errors and elapsed time of the numerical solutions given by the coupled PML method and classical rectangular PML method.*

The PML thickness and PML parameter are chosen to be $d = 1$ and $\sigma_0 = 2$, respectively. The interface penalty parameters are $\beta_j = 1$ and $\gamma_j = 10$, and the interior penalty parameters are given by

$$\gamma_e = \gamma_r + \gamma_i \mathbf{i} \quad \text{with} \quad \gamma_r = -\frac{\sqrt{3}}{24} - \frac{\sqrt{3}}{1728}(kh)^2 \quad \text{and} \quad \gamma_i = -0.01, \tag{5.2}$$

where γ_r is obtained by a dispersion analysis for 2D problem on equilateral triangulations [37]. The triangulation is produced by an algorithm in which most elements are approximate equilateral triangles. This can help to increase the effectiveness of the penalty parameters in reducing the pollution error. The imaginary part γ_i of the penalty parameter is used to enhance the stability of CIP-FEM, see [65, 66].

By comparing the numerical solutions with the exact solution in (5.1), we present the relative H^1-norm errors of the numerical solutions and Lagrange interpolations in Figure 5.3 for different wave numbers and mesh sizes. We let the relative error be "1" when the iteration is divengence. It is shown that for small k, the errors of the CIP-FEM, as well as the FEM, are about $O(h)$ and fit the interpolation errors well as h decreases. This indicates that the coupled PML with either CIP-FEM or FEM is effective in approximating the exact solution for small k. For large k, the errors of the FEM decay more slowly than those of Lagrange interpolation. This behaviour shows clearly the effect of pollution errors of FEM. The CIP-FEM behaves similarly but the pollution range is much smaller than that of FEM, which implies that CIP-FEM has greatly reduced the pollution error.

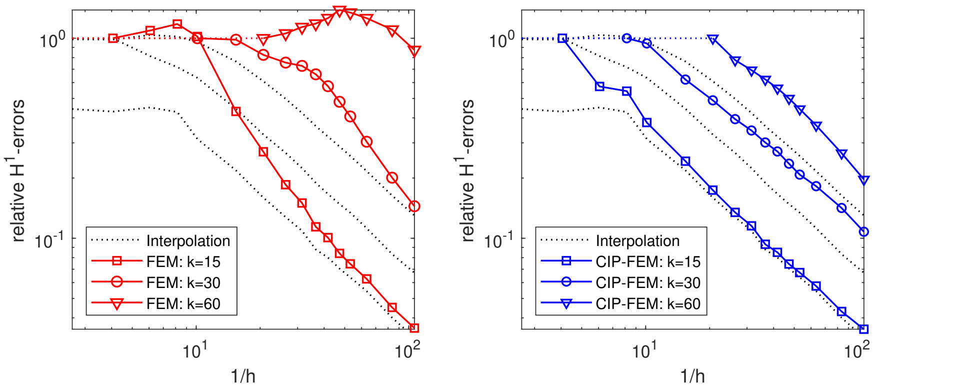

FIG. 5.3. *Relative H^1-errors of the FE solution (left figure) and the CIP-FE solution (right figure), compared with the relative H^1-errors of the Lagrange interpolation (dotted) for $k = 15$, 30, and 60.*

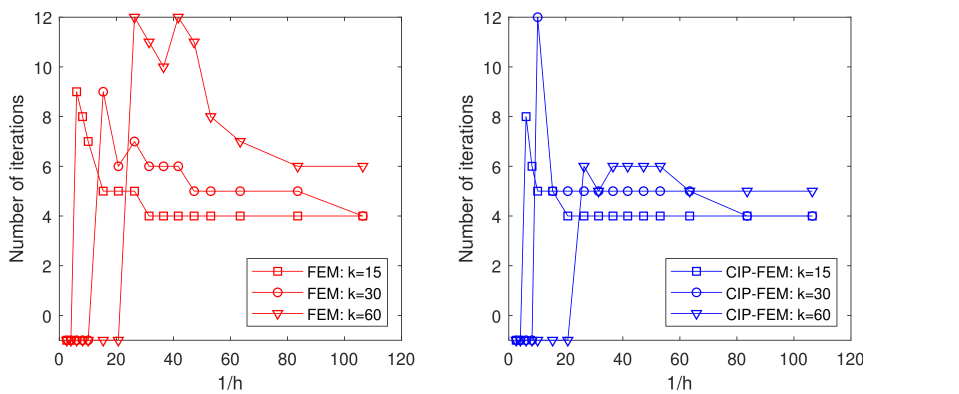

FIG. 5.4. *Number of iterations of FEM and CIP-FEM, where -1 represents the failure of iteration.*

For a given tolerance error 10^{-3}, the number of iterations given by (4.9) is presented in Figure 5.4 for both FEM and CIP-FEM. It is shown that when the mesh size h is small enough the iterative solutions v^l, $l \geq 1$, converge to a stable solution within a few steps.

EXAMPLE 5.3. In this example, we consider a multiple scattering problem with three sources occupying the domain containing three mutually disjoint subdomains, the concerned domain Ω is three disjoint squares surrounding these sources, as illustrated in Figure 5.5 (left). The sources and exact solution are defined in (5.1). The wave number is $k = 10$. All the PML parameters and the interface penalty parameters are the same as those in Example 5.1. The CIP parameters are defined in (5.2). The Figure 5.5 (right) plots the real part of the CIP-FE solution, which shows the three sources clearly. Figure 5.6 gives the relative H^1-errors of CIP-FEM for the coupled PML method and the rectangular PML method, where the horizontal axis represents the DOFs and the elapsed time, respectively. It is shown that the new proposed PML method works well for this multiple sources problem. Moreover, to achieve the same accuracy when the subdomains are well-separated, both the DOFs (which measures the memory cost) and the elapsed time of the proposed coupled PML method are much less than those of the classical rectangular PML method.

6. Conclusion. We have proposed a coupled PML method for solving the Helmholtz equation in a nonconvex computational domain. Rigorous analyses are presented for the well-posedness and the exponential convergence of the coupled PML. An iterative CIP-FEM is proposed for solving the coupled PML system. Compared with the standard PML method (i.e., using one large convex domain to enclose the entire scattering region), the proposed PML method can achieve the same accuracy with much less memory cost by

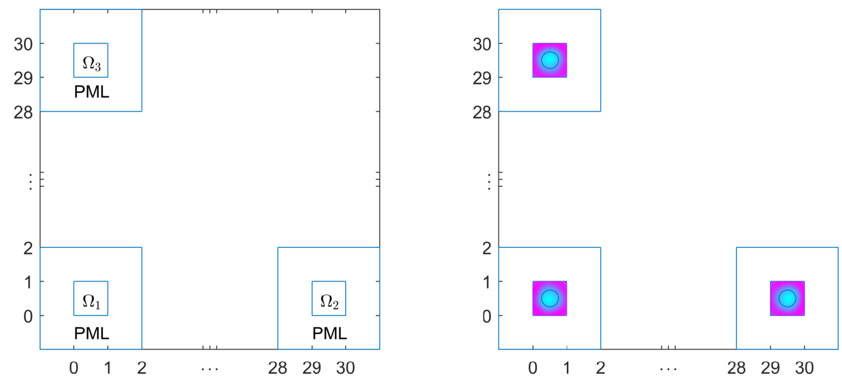

FIG. 5.5. *Multiple scattering problem*

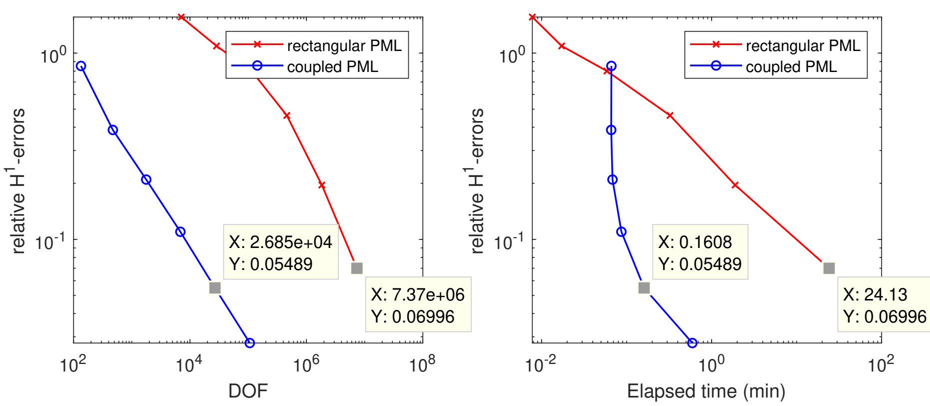

FIG. 5.6. *Relative H^1-errors and elapsed time of the CIP-FEM for the coupled PML method and rectangular PML method.*

using several PMLs to enclose a nonconvex neighborhood of the scattering region. The numerical experiments show that, for the problem with multiple sources, the new PML method requires much less memory cost and CPU time to achieve the same accuracy. For the problem with nonconvex inhomogeneities, the new PML method requires much less memory cost to achieve the same accuracy with the same CPU time.

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