LINEARIZED FE APPROXIMATIONS TO A NONLINEAR GRADIENT FLOW*

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Abstract. We study fully discrete linearized Galerkin finite element approximations to a nonlinear gradient flow, applications of which can be found in many areas. Due to the strong nonlinearity of the equation, existing analyses for implicit schemes require certain restrictions on the time step and no analysis has been explored for linearized schemes. This paper focuses on the unconditionally optimal L^2 error estimate of a linearized scheme. The key to our analysis is an iterated sequence of time-discrete elliptic equations and a rigorous analysis of its solution. We prove the $W^{1,\infty}$ boundedness of the solution of the time-discrete system and the corresponding finite element solution, based on a more precise estimate of elliptic PDEs in $W^{2,2+\epsilon_1}$ and $H^{2+\epsilon_2}$ and a physical feature of the gradient-dependent diffusion coefficient. Numerical examples are provided to support our theoretical analysis.

Keywords: finite element, nonlinear diffusion, gradient flow, stability, error estimate

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1. Introduction.

We consider the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (\sigma(|\nabla u|^2)\nabla u) = g \tag{1.1}$$

in a convex polygonal domain Ω in $\mathbb{R}^2$ with the Neumann boundary condition

$$\nabla u \cdot \vec{n} = 0 \qquad \text{on} \quad \partial\Omega \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x) \qquad \text{for} \quad x \in \Omega, \tag{1.3}$$

where g is a given function and

$$\sigma(s^2) = \frac{1}{\sqrt{\lambda^2 + s^2}} \tag{1.4}$$

is a gradient-dependent diffusion coefficient, where λ is a positive constant. The equation has been involved in many applications, such as minimal surface flow [32], prescribed mean curvature flow [16, 24], geometric measure theory [4], and a regularized model in image denoising [11, 13, 14, 19, 25, 34, 35, 38, 40]. A review article for

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the applications in image processing was given in [10]. On the other hand, (1.1) can be viewed as a gradient flow with the energy functional

$$J_\lambda[u] := \int_\Omega \sqrt{|\nabla u|^2 + \lambda^2}\,\mathrm{d}x - \int_\Omega gu\,\mathrm{d}x\,.$$

Mathematical analysis of the nonlinear diffusion equation (1.1) was studied in [21, 24]. In particular, the $W^{1,\infty}$ regularity of the solution was proved in [21], which further implies higher regularity of the solution (by the method of Section 8.3.2 of [18]). Numerical methods and simulations for the nonlinear diffusion equation have been investigated extensively in the last several decades. For examples, see [2, 34, 35, 40] for finite difference methods and [13,16,17,19–22] for finite element methods (FEMs). Explicit schemes may not be efficient due to their strong time-step restrictions. A fully implicit backward Euler–Galerkin FEM was analyzed in [21], where optimal convergence rate was proved under the condition $\tau = O(h^2)$. Suboptimal error estimates for the scheme were presented in [22] under a weaker mesh restriction $\tau = o(h^{1/2})$, and further analysis on the convergence rate of the scheme with respect to the regularization parameter was given in [20]. The implicit backward Euler scheme was also studied in [19] with a lumped mass FEM, where L^∞-boundedness of the numerical solution was proved and no error estimates were presented. In these fully implicit schemes, one has to solve a system of nonlinear equations at each time step and an extra inner iteration is needed. In addition to the implicit schemes, linearized semi-implicit FEMs for the nonlinear diffusion equation have also been investigated by several authors [13,34,37]. In this method, the gradient-dependent diffusion coefficient is calculated with the numerical solution at the last time step and Galerkin FEMs are used to solve the linearized equation. The scheme only requires the solution of a linear system at each time step, which is simple and efficient for implementation [23, 31]. However, theoretical error analysis of the linearized scheme seems very difficult due to the strong nonlinear structure. As far as we know, no optimal error estimates of linearized semi-implicit FEMs are available for the nonlinear diffusion equation. The major difficulty for the analysis of the semi-implicit scheme is due to the nature of the linearization of the scheme, which leads to the arising of the energy-norm errors at two different time levels in the error equation (see (3.23)-(3.26) for the estimates of the error equation).

In this paper, we study linearized backward Euler–Galerkin methods for the nonlinear system (1.1)-(1.3). Our focus is on unconditionally optimal error estimates of numerical methods. The key issue in the analysis is to establish the $W^{1,\infty}$ convergence of the numerical solution. To deal with the strong nonlinearity from the gradient-dependent diffusion coefficient, we introduce an iterated sequence of time-discrete elliptic PDEs as in [28, 29]. Thus the linearized backward Euler–Galerkin method coincides with the corresponding FE approximation to the time-discrete system. We prove the $W^{1,\infty}$ convergence of the solution of the time-discrete system and FE solution, in terms of more precise estimates for elliptic PDEs in $W^{2,2+\epsilon_1}$ and $H^{2+\epsilon_2}$:

$$\begin{aligned}\|u\|_{L^{2+\epsilon_1}} &\le (1+\epsilon_1^*)\|\Delta u\|_{L^{2+\epsilon_1}}\\ \|u\|_{H^{2+\epsilon_2}} &\le (1+\epsilon_2^*)\|\Delta u\|_{H^{\epsilon_2}}\,,\end{aligned}$$

and a physical feature of the gradient flow

$$2|\sigma'(s^2)|s^2 < \sigma(s^2)\,. \tag{1.5}$$

With these a priori estimates, we establish the L^2-norm optimal error estimate without any time-step restrictions.

The rest part of this paper is organized as follows. In Section 2, we introduce some notations and the linearized backward Euler–Galerkin FEM for the nonlinear diffusion equation (1.1)-(1.3), and then we present our main results and our methodology. In Section 3, we prove our main results based on the regularity and $W^{1,\infty}$-convergence of the time-discrete solution, while the rigorous proof of the regularity and $W^{1,\infty}$-convergence of the time-discrete solution is postponed to Section 4. Numerical examples are presented in Section 5, which confirm our theoretical analysis and show clearly that the linearized scheme is efficient and no time-step conditions are needed.

2. Notations and main results. Let Ω be a given convex polygon in $\mathbb{R}^2$. For $1 \le p \le \infty$ and any nonnegative integer k, we denote by $W^{k,p}(\Omega)$ the usual Sobolev space of functions defined on Ω and, to simplify the notations, we set $W^{k,p} := W^{k,p}(\Omega)$, $H^k := W^{k,2}(\Omega)$ and $L^p := W^{0,p}$. For $s \in (0,1)$, we define $H^{k+s} := (H^k, H^{k+1})_{[s]}$ as the complex interpolation space between H^k and H^{k+1}. More detailed discussions for the complex interpolation spaces can be found in literature, *e.g.*, see the classical book [5] by Bergh and Löfström.

For a given quasi-uniform triangulation of Ω into triangles T_j, $j = 1, \cdots, J$, we denote by $h = \max_{1\le j\le J}\{\operatorname{diam} T_j\}$ the mesh size and define a finite element space by

$$V_h^r = \{v_h \in C(\overline{\Omega}) : v_h|_{T_j} \text{ is a polynomial of degree } r\}$$

so that V_h^r is a subspace of $H^1(\Omega)$. Let $\Pi_h : C(\overline{\Omega}) \to V_h^r$ denote the Lagrangian interpolation operator. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0,T]$ with $t_n = n\tau$. For a sequence of functions $\{f^n\}_{n=0}^N$, we define a time-difference operator by

$$D_\tau f^{n+1} = \frac{f^{n+1} - f^n}{\tau}, \quad \text{for } n = 0, 1, \cdots, N-1. \tag{2.1}$$

We define the linearized backward Euler–Galerkin finite element scheme by

$$\big(D_\tau U_h^{n+1}, v\big) + \big(\sigma(|\nabla U_h^n|^2)\nabla U_h^{n+1}, \nabla v\big) = \big(g^{n+1}, v\big), \quad \forall v \in V_h^r, \tag{2.2}$$

with the initial condition $U_h^0 = \Pi_h u_0$ and $r \ge 2$. At each time step, the scheme only requires the solution of a linear system. Also we assume that the solution of (1.1)-(1.3) exists and satisfies

$$\|u_0\|_{H^{r+1}} + \|u\|_{L^\infty((0,T);H^{r+1})} + \|\partial_t u\|_{L^\infty((0,T);H^{r+1})} + \|\partial_{tt} u\|_{L^2((0,T);L^2)} \le M_0, \tag{2.3}$$

where M_0 is some positive constant. For simplicity, we assume that $g = g(x,t)$ in this paper. The analysis presented in this paper can be easily extended to the general case $g = g(u,x,t)$ for the scheme

$$\big(D_\tau U_h^{n+1}, v\big) + \big(\sigma(|\nabla U_h^n|^2)\nabla U_h^{n+1}, \nabla v\big) = \big(g(U_h^n, x, t^n), v\big), \quad \forall v \in V_h^r,$$

if g is a smooth function of u, x and t.

Our main results are given in the following theorem concerning the unconditionally optimal convergence rate of the numerical solution.

THEOREM 2.1. *Suppose that the system* (1.1)-(1.3) *has a unique solution u satisfying the regularity condition* (2.3). *Then there exists a positive constant C_0, which* *is independent of τ and h (but may depend on λ, M_0, Ω and T), such that the finite element system (2.2) admits a unique solution $\{U_h^n\}_{n=1}^N$ satisfying*

$$\|U_h^n - u^n\|_{L^2} \le C_0(\tau + h^{r+1}). \tag{2.4}$$

To prove the above theorem, we introduce an iterated sequence of elliptic PDEs (time-discrete system) as proposed in [28, 29]:

$$D_\tau U^{n+1} - \nabla\cdot(\sigma(|\nabla U^n|^2)\nabla U^{n+1}) = g^{n+1}, \tag{2.5}$$

with the boundary condition $\nabla U^{n+1}\cdot\vec{n} = 0$ on $\partial\Omega$ and the initial condition $U^0 = u_0$. Then the fully discrete solution U_h^{n+1} coincides with the finite element solution of (2.5). In view of this property, we split the error into

$$U_h^n - u^n = (U_h^n - U^n) + (U^n - u^n)$$

and analyze the two error functions separately. The regularity of the solution of the time-discrete system (2.5) is given in the following theorem.

Theorem 2.2. *Under the assumption of Theorem 2.1, there exist positive constants τ_0^*, C_0^*, $p > 2$ and $s_0 > 0$, which are dependent upon λ, M_0, Ω and T and independent of τ and h, such that when $\tau < \tau_0^*$ the time-discrete system (2.5) admits a unique solution $\{U^n\}_{n=0}^N$ satisfying*

$$\max_{0\le n\le N}(\|U^n\|_{W^{2,p}}^2 + \|U^n\|_{H^{2+s_0}}^2) + \sum_{n=1}^N \tau\|D_\tau U^n\|_{H^2}^2 \le C_0^*, \tag{2.6}$$

$$\max_{1\le n\le N}\|e^n\|_{H^1}^2 + \sum_{n=1}^N \tau\|e^n\|_{H^2}^2 + \sum_{n=1}^N \tau\|D_\tau e^n\|_{L^2}^2 \le C_0^*\tau^2, \tag{2.7}$$

$$\max_{1\le n\le N}\|e^n\|_{W^{2,p}} \le C_0^*\tau^{1/3}, \tag{2.8}$$

where $e^n := u^n - U^n$.

The proofs of Theorem 2.1 and Theorem 2.2 will be given in Section 3 and Section 4, respectively. In the rest part of this paper, we denote by C a generic positive constant which is independent of τ, h and n, and by ϵ a generic small positive constant.

3. Proof of Theorem 2.1. In this section, we prove Theorem 2.1 based on the results of Theorem 2.2. The proof of the latter is deferred to Section 4. The following inverse inequalities will be used in this section:

$$\|v\|_{L^p} \le Ch^{2/p-2/q}\|v\|_{L^q}, \qquad \text{for} \quad v\in V_h^r, \quad 1\le q\le p\le\infty, \tag{3.1}$$

$$\|\nabla v\|_{L^p} \le Ch^{-1}\|v\|_{L^p}, \qquad \text{for} \quad v\in V_h^r, \quad 1\le p\le\infty. \tag{3.2}$$

3.1. Preliminaries. Based on Theorem 2.2, we define

$$M = \sup_\tau \max_{1\le n\le N}(\|u^n\|_{W^{1,\infty}} + \|U^n\|_{W^{1,\infty}}) + 1$$

so that

$$\sigma(|\nabla u^n|^2) \ge \sigma_M, \quad |\sigma(|\nabla u^n|^2)| + |\sigma'(|\nabla u^n|^2)| + |\sigma''(|\nabla u^n|^2)| \le C_M,$$
$$\sigma(|\nabla U^n|^2) \ge \sigma_M, \quad |\sigma(|\nabla U^n|^2)| + |\sigma'(|\nabla U^n|^2)| + |\sigma''(|\nabla U^n|^2)| \le C_M,$$

for some positive constants σ_M and C_M.

For any given function $w \in H^1$, we define the following matrix functions:

$$B(\nabla w) = 2\sigma'(|\nabla w|^2)\nabla w(\nabla w)^T\,, \qquad A(\nabla w) = \sigma(|\nabla w|^2)I + B(\nabla w)\,. \tag{3.3}$$

For $n \ge 0$ we define the projection operators $\overline{R}_h^{n+1} : H^1(\Omega) \to V_h^r$ and $R_h^{n+1} : H^1(\Omega) \to V_h^r$ by

$$\big(A(\nabla u^n)\nabla(w - \overline{R}_h^{n+1} w), \nabla v\big) = 0, \quad \forall\ w \in H^1 \text{ and } v \in V_h^r, \tag{3.4}$$

$$\big(A(\nabla U^n)\nabla(w - R_h^{n+1} w), \nabla v\big) = 0, \quad \forall\ w \in H^1 \text{ and } v \in V_h^r, \tag{3.5}$$

where $\int_\Omega \overline{R}_h^{n+1} w \mathrm{d}x = \int_\Omega R_h^{n+1} w \mathrm{d}x = \int_\Omega w \mathrm{d}x$ are enforced for uniqueness, and we set $\overline{R}_h^0 := \overline{R}_h^1$, $R_h^0 := R_h^1$. These two projection operators are well defined since

$$\lambda^2\sigma_M^3|\xi|^2 \le \xi^T A(\nabla u^n)\xi \le 2C_M|\xi|^2, \quad \forall\ \xi \in \mathbb{R}^2\,,$$
$$\lambda^2\sigma_M^3|\xi|^2 \le \xi^T A(\nabla U^n)\xi \le 2C_M|\xi|^2, \quad \forall\ \xi \in \mathbb{R}^2\,.$$

We denote

$$\theta_h^{n+1} = U^{n+1} - R_h^{n+1}U^{n+1}, \quad \text{and} \quad \overline{\theta}_h^{n+1} = u^{n+1} - \overline{R}_h^{n+1}u^{n+1}\,.$$

By the classical theory of finite element methods, with the regularity of U^n given in Theorem 2.2, we have

$$\|u^{n+1} - R_h^{n+1}u^{n+1}\|_{W^{1,\infty}} \le C\|u^{n+1}\|_{H^3} h, \tag{3.6}$$

$$\|\overline{\theta}_h^{n+1}\|_{H^l} \le C\|u^{n+1}\|_{H^{r+1}} h^{r+1-l}, \qquad \text{for } l = 0, 1, \tag{3.7}$$

$$\|R_h^{n+1}U^{n+1}\|_{W^{1,\infty}} + \|\overline{R}_h^{n+1}u^{n+1}\|_{W^{1,\infty}} \le C(\|U^{n+1}\|_{W^{1,\infty}} + \|u^{n+1}\|_{W^{1,\infty}}), \tag{3.8}$$

$$\|\tau D_\tau \nabla U^{n+1}\|_{L^\infty} \le C\|\tau D_\tau \nabla e^{n+1}\|_{L^\infty} + C\|\tau D_\tau \nabla u^{n+1}\|_{L^\infty} \le C\tau^{1/3}, \tag{3.9}$$

$$\|D_\tau A(\nabla U^n)\|_{L^{\bar p}} \le C\|D_\tau \nabla U^n\|_{L^{\bar p}} \le C\|D_\tau U^n\|_{H^2}, \tag{3.10}$$

and

$$\begin{aligned}\|\theta_h^{n+1}\|_{W^{l,q}} &\le \|e^{n+1} - R_h^{n+1}e^{n+1}\|_{W^{l,q}} + \|u^{n+1} - R_h^{n+1}u^{n+1}\|_{W^{l,q}}\\ &\le Ch^{2-l}\|e^{n+1}\|_{W^{2,q}} + Ch^{2-l+2/q}\|u^{n+1}\|_{H^3} \quad \text{for } l = 0, 1 \text{ and } 2 \le q \le p,\end{aligned} \tag{3.11}$$

where p is given in Theorem 2.2 and $1/\bar p + 1/p = 1/2$. The above inequality (3.7) with $l = 0, 1$ is standard L^2 and H^1 error estimate of the finite element method for elliptic equations, respectively. Since $A(\nabla U^n) \in W^{1,p}$ for some $p > 2$, the L^2 error estimate $\|u^{n+1} - R_h^{n+1}u^{n+1}\|_{L^2} \le Ch^3\|u^{n+1}\|_{H^3}$ is also standard. An interpolation error estimate related to (3.6) is

$$\|u^{n+1} - \Pi_h u^{n+1}\|_{W^{1,\infty}} \le Ch\|u^{n+1}\|_{H^3}$$

which can be derived by Bramble-Hilbert Lemma (see page 77 of the book [7]). Then (3.6) can be established by using the above inequality and the standard L^2 error estimate (together with an inverse inequality). Moreover, (3.8) and (3.11) follow from Theorem 8.1.11 and Theorem 8.5.3 of [8], respectively, and (3.9)-(3.10) are consequences of Theorem 2.2. From these inequalities we also derive that

$$\begin{aligned}\|\theta_h^n\|_{W^{1,\infty}} &\le \|e^n - R_h^n e^n\|_{W^{1,\infty}} + \|u^n - R_h^n u^n\|_{W^{1,\infty}}\\ &\le C\|e^n\|_{W^{1,\infty}} + Ch\|u^n\|_{H^3}\\ &\le C(\tau^{1/3} + h).\end{aligned} \tag{3.12}$$

In this section, we shall frequently use the inequalities (3.6)-(3.12) and also, we need the following Lemma.

LEMMA 3.1. *Under the assumptions of Theorem* 2.1, *there exist positive constants* $\widetilde{\tau}_0$ *and* δ_0 *such that when* $\tau \le \widetilde{\tau}_0$,

$$\left(\sum_{n=0}^{N-1} \tau \|D_\tau \theta_h^{n+1}\|_{H^{-1}}^2\right)^{\frac{1}{2}} \le C(\tau^{1/3} + h^{\delta_0})h^2, \tag{3.13}$$

$$\left(\sum_{n=0}^{N-1} \tau \|D_\tau (u^{n+1} - R_h^{n+1} u^{n+1})\|_{H^1}^2\right)^{\frac{1}{2}} \le Ch^r, \tag{3.14}$$

$$\left(\sum_{n=0}^{N-1} \tau \|D_\tau \overline{\theta}_h^{n+1}\|_{L^2}^2\right)^{\frac{1}{2}} \le Ch^{r+1}. \tag{3.15}$$

Proof. Since u^n is smooth enough, (3.14)-(3.15) can be obtained easily. Here we only prove (3.13). Note that

$$\Big(A(\nabla U^n)\nabla(U^{n+1} - R_h^{n+1}U^{n+1}),\, \nabla\phi_h\Big) = 0, \tag{3.16}$$

$$\Big(A(\nabla U^{n-1})\nabla(U^{n+1} - R_h^{n}U^{n+1}),\, \nabla\phi_h\Big) = 0. \tag{3.17}$$

The difference of the above two equations gives

$$\begin{aligned}
&\Big(A(\nabla u^n)\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1}),\, \nabla\phi_h\Big) \\
&+ \Big((A(\nabla U^n) - A(\nabla u^n))\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1}),\, \nabla\phi_h\Big) \\
&+ \Big((A(\nabla U^n) - A(\nabla U^{n-1}))\nabla(U^{n+1} - R_h^n U^{n+1}),\, \nabla\phi_h\Big) = 0,
\end{aligned}$$

which together with Theorem 2.2 implies

$$\begin{aligned}
&\|\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1})\|_{L^2} \\
&\le C\|(A(\nabla U^n) - A(\nabla u^n))\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1})\|_{L^2} \\
&\quad + C\|(A(\nabla U^n) - A(\nabla U^{n-1}))\nabla(U^{n+1} - R_h^n U^{n+1})\|_{L^2} \\
&\le C\|\nabla e^n\|_{L^\infty}\|\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1})\|_{L^2} + C\tau\|D_\tau \nabla U^n\|_{L^{\bar{p}}}\|\nabla(U^{n+1} - R_h^n U^{n+1})\|_{L^p} \\
&\le C\tau^{1/3}\|\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1})\|_{L^2} \\
&\quad + C\tau\|D_\tau U^n\|_{H^2}(\|e^{n+1} - R_h^n e^{n+1}\|_{W^{1,p}} + \|u^{n+1} - R_h^n u^{n+1}\|_{W^{1,p}}) \\
&\le C\tau^{1/3}\|\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1})\|_{L^2} \\
&\quad + C\tau\|D_\tau U^n\|_{H^2}(Ch\|e^{n+1}\|_{W^{2,p}} + Ch^{1+2/p}\|u^{n+1}\|_{H^3}) \\
&\le C\tau^{1/3}\|\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1})\|_{L^2} + C\|D_\tau U^n\|_{H^2}(\tau^{1/3} + h^{2/p})\tau h,
\end{aligned}$$

where we have used (2.8), (3.10) and a similar $W^{1,p}$ estimate as given in (3.11). When $\tau < \widetilde{\tau}_0 := \min(\tau_0^*, (2C)^{-3})$, we get

$$\|\nabla(R_h^n U^{n+1} - R_h^{n+1}U^{n+1})\|_{L^2} \le 2C\|D_\tau U^n\|_{H^2}(\tau^{1/3} + h^{2/p})\tau h. \tag{3.18}$$

To establish the corresponding L^2-norm estimate, for any given $\varphi \in H^1(\Omega)$ we let ψ be the solution of the equation

$$-\nabla \cdot \Big(A(\nabla U^n)\nabla\psi \Big) = \varphi - \frac{1}{|\Omega|}\int_\Omega \varphi \mathrm{d}x$$

with the boundary condition $A(\nabla U^n)\nabla\psi \cdot \vec{n} = 0$ on $\partial\Omega$ and $\int_\Omega \psi \mathrm{d}x = 0$. Due to the structure of the matrix $A(\nabla U^n)$, this boundary condition is equivalent to $\nabla\psi \cdot \vec{n} = 0$ on $\partial\Omega$. Since $A(\nabla U^n)$ is uniformly bounded in $W^{1,p} \cap H^{1+s_0}$, there exists a positive constant $\delta_0 \in (0, \min(2/p, s_0))$ (dependent on the norm $\|\nabla U^n\|_{H^{1+s_0}}$) such that $\|\psi\|_{H^{2+s}} \leq C\|\varphi\|_{H^s}$ for $s \in [0, \delta_0]$ (see Lemma 4.2).

By noting the fact that $\int_\Omega (R_h^n U^{n+1} - R_h^{n+1} U^{n+1})\mathrm{d}x = 0$, we have

$$\begin{aligned}
&\big(R_h^n U^{n+1} - R_h^{n+1} U^{n+1}, \varphi\big) \\
&= \Big(A(\nabla U^n)\nabla(R_h^n U^{n+1} - R_h^{n+1} U^{n+1}),\, \nabla\psi \Big) \\
&= \Big(A(\nabla U^n)\nabla(R_h^n U^{n+1} - R_h^{n+1} U^{n+1}),\, \nabla(\psi - \Pi_h\psi) \Big) \\
&\quad - \Big((A(\nabla U^n) - A(\nabla U^{n-1}))\nabla(U^{n+1} - R_h^n U^{n+1}),\, \nabla(\Pi_h\psi - \psi) \Big) \\
&\quad - \Big((A(\nabla U^n) - A(\nabla U^{n-1})\nabla(U^{n+1} - R_h^n U^{n+1}),\, \nabla\psi \Big) := I_1 + I_2 + I_3
\end{aligned}$$

By (3.11) and (3.18), the first two terms of the right-hand side of the above equation are bounded by

$$\begin{aligned}
|I_1| &\leq C\|D_\tau U^n\|_{H^2}\|\psi\|_{H^2}(\tau^{1/3} + h^{2/p})\tau h^2, \\
|I_2| &\leq C\|D_\tau A(\nabla U^n)\|_{L^{\bar{p}}}\|\nabla(U^{n+1} - R_h^n U^{n+1})\|_{L^p}\|\psi\|_{H^2}\tau h \\
&\leq C\|D_\tau U^n\|_{H^2}\|\psi\|_{H^2}(\tau^{1/3} + h^{2/p})\tau h^2,
\end{aligned}$$

where $1/\bar{p} + 1/p = 1/2$. Again by (2.8), (3.11) and (3.18) and noting the homogeneous boundary condition, with integration by part, we can bound the last term by

$$\begin{aligned}
|I_3| &= \Big|\Big((A(\nabla U^n) - A(\nabla U^{n-1}))\nabla(U^{n+1} - R_h^n U^{n+1}),\, \nabla\psi \Big)\Big| \\
&= \Big|\Big(U^{n+1} - R_h^n U^{n+1}, \nabla \cdot \big[(A(\nabla U^n) - A(\nabla U^{n-1}))\nabla\psi\big] \Big)\Big| \\
&\leq \|U^{n+1} - R_h^n U^{n+1}\|_{L^p}\|\nabla \cdot \big[(A(\nabla U^n) - A(\nabla U^{n-1}))\nabla\psi\big]\|_{L^{p'}} \\
&\leq C(h^2\|e^{n+1}\|_{W^{2,p}} + h^{2+2/p}\|u^{n+1}\|_{H^3})\,\big(\|A(\nabla U^n) - A(\nabla U^{n-1})\|_{H^1}\|\nabla\psi\|_{L^{\tilde{p}}} \\
&\quad + \|A(\nabla U^n) - A(\nabla U^{n-1})\|_{L^{\tilde{p}}}\|\psi\|_{H^2}\big) \\
&\leq C\|D_\tau U^n\|_{H^2}\|\varphi\|_{L^2}(\tau^{1/3} + h^{2/p})\tau h^2,
\end{aligned}$$

where $1/p + 1/p' = 1$ and $1/\widetilde{p} + 1/2 = 1/p'$.

With the above estimates, we obtain

$$\|R_h^n U^{n+1} - R_h^{n+1} U^{n+1}\|_{L^2} \leq C\|D_\tau U^n\|_{H^2}(\tau^{1/3} + h^{2/p})\tau h^2 \quad \text{for} \quad n \geq 1.$$

Since $R_h^0 U^1 = R_h^1 U^1$,

$$\left(\sum_{n=0}^{N-1} \tau\|R_h^n U^{n+1} - R_h^{n+1} U^{n+1}\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C(\tau^{1/3} + h^{2/p})\tau h^2.$$

Finally, we take a standard approach to the H^{-1}-norm estimate (3.13) [8]. Since

$$\begin{aligned}|(\phi - R_h\phi, \varphi)| &= \inf_{\psi_h \in V_h^r} |(A(\nabla U^n)\nabla(\phi - R_h\phi), \nabla(\psi - \psi_h))| \\ &\le C\|\nabla(\phi - R_h\phi)\|_{L^2}\|\psi\|_{H^{2+\delta_0}} h^{1+\delta_0} \\ &\le C\|\phi\|_{H^2}\|\varphi\|_{H^{\delta_0}} h^{2+\delta_0}, \quad \forall\, \varphi \in H^{\delta_0},\end{aligned}$$

we have

$$\|\phi - R_h\phi\|_{H^{-\delta_0}} \le C\|\phi\|_{H^2} h^{2+\delta_0}, \quad \forall\, \phi \in H^2$$

from which, we further derive that

$$\begin{aligned}&\|D_\tau(U^{n+1} - R_h^{n+1}U^{n+1})\|_{H^{-1}} \\ &\le \|D_\tau U^{n+1} - R_h^n D_\tau U^{n+1}\|_{H^{-1}} + \tau^{-1}\|R_h^{n+1}U^{n+1} - R_h^n U^{n+1}\|_{H^{-1}} \\ &\le \|D_\tau U^{n+1} - R_h^n D_\tau U^{n+1}\|_{H^{-\delta_0}} + \tau^{-1}\|R_h^{n+1}U^{n+1} - R_h^n U^{n+1}\|_{L^2} \\ &\le C\|D_\tau e^{n+1}\|_{H^2} h^{2+\delta_0} + C\|D_\tau u^{n+1}\|_{H^3} h^3 + \tau^{-1}\|R_h^{n+1}U^{n+1} - R_h^n U^{n+1}\|_{L^2}.\end{aligned}$$

(3.13) follows immediately. ∎

3.2. Boundedness of the numerical solution. By (2.6) and (3.8), we can re-define

$$\begin{aligned}M = \sup_{\tau,h}\Big(&\max_{0\le n\le N}\|u^n\|_{W^{1,\infty}} + \max_{0\le n\le N}\|\overline{R}_h^n u^n\|_{W^{1,\infty}} \\ &+ \max_{0\le n\le N}\|U^n\|_{W^{1,\infty}} + \max_{0\le n\le N}\|R_h^n U^n\|_{W^{1,\infty}}\Big) + 2.\end{aligned}$$

By the regularity assumptions on σ, there exist σ_M and $C_M > 0$ such that

$$\sigma(s^2) \ge \sigma_M, \qquad \forall\, s \in [-M, M], \tag{3.19}$$

$$|\sigma(s^2)| + |\sigma'(s^2)| + |\sigma''(s^2)| \le C_M, \quad \forall\, s \in [-M, M]. \tag{3.20}$$

Lemma 3.2. *Under the assumptions of Theorem 2.1, there exist positive constants $\widehat{\tau}_0$ and $\widehat{h}_0$ which are independent of n, τ and h, such that the finite element system (2.2) admits a unique solution $\{U_h^n\}_{n=1}^N$ when $\tau < \widehat{\tau}_0$ and $h < \widehat{h}_0$, satisfying*

$$\|U_h^n\|_{L^\infty} + \|\nabla U_h^n\|_{L^\infty} \le M, \tag{3.21}$$

$$\|e_h^n\|_{L^\infty} + \|\nabla e_h^n\|_{L^\infty} < \tau^{1/8} + h^{\delta_0/8}, \tag{3.22}$$

where $e_h^n = R_h^n U^n - U_h^n$ and δ_0 is given in Lemma 3.1.

Proof. By (3.19)-(3.20), the coefficient matrix of the linear system (2.2) is symmetric and positive definite, which implies that (2.2) admits a unique solution $U_h^{n+1} \in V_h^r$ for $0 \le n \le k$.

It is easy to see that the inequalities (3.21)-(3.22) hold for $n = 0$. By mathematical induction, we can assume that (3.21)-(3.22) hold for $0 \le n \le k$ for some $k \ge 0$.

Since the solution U^{n+1} of (2.5) satisfies

$$\big(D_\tau U^{n+1}, v\big) + \big(\sigma(|\nabla U^n|^2)\nabla U^{n+1}, \nabla v\big) = \big(g^{n+1}, v\big), \quad \forall\, v \in V_h^r,$$

the error function e_h^{n+1} satisfies

$$\begin{aligned}
(3.23)\quad & \big(D_\tau e_h^{n+1}, v\big) + \big(\sigma(|\nabla U^n|^2)\nabla e_h^{n+1}, \nabla v\big) \\
& = \Big[-\big(\sigma(|\nabla U^n|^2)\nabla \theta_h^{n+1},\, \nabla v\big) + \big((\sigma(|\nabla U_h^n|^2) - \sigma(|\nabla U^n|^2))\nabla U_h^{n+1}, \nabla v\big)\Big] \\
& \quad - \big(D_\tau \theta_h^{n+1}, v\big) \\
& := J_1(v) + J_2(v), \qquad \forall\, v \in V_h^r.
\end{aligned}$$

By using Taylor's expansion, we see that

$$\begin{aligned}
(3.24)\quad & \\
& (\sigma(|\nabla U_h^n|^2) - \sigma(|\nabla U^n|^2))\nabla U_h^{n+1} \\
& = \big(2\sigma'(|\nabla U^n|^2)\nabla U^n \cdot \nabla(U_h^n - U^n) + \sigma'(|\nabla U^n|^2)|\nabla(U_h^n - U^n)|^2\big)\, \nabla U^{n+1} \\
& \quad + \frac{1}{2}\sigma''(\xi_h^n)|\nabla(U_h^n + U^n)\cdot \nabla(U_h^n - U^n)|^2 \nabla U^{n+1} \\
& \quad + (\sigma(|\nabla U_h^n|^2) - \sigma(|\nabla U^n|^2))\nabla(-e_h^{n+1} - \theta_h^{n+1}) \\
& = -2\sigma'(|\nabla U^n|^2)\nabla U^n \cdot \nabla(e_h^n + \theta_h^{n+1})(\nabla U^n + \tau D_\tau \nabla U^{n+1}) \\
& \quad + 2\sigma'(|\nabla U^n|^2)\nabla U^n \cdot \nabla \tau D_\tau \theta_h^{n+1} \nabla U^{n+1} \\
& \quad + \Big(\sigma'(|\nabla U^n|^2)|\nabla(U_h^n - U^n)|^2 + \frac{1}{2}\sigma''(\xi_h^n)|\nabla(U^n + U_h^n)\cdot \nabla(e_h^n + \theta_h^n)|^2\Big)\nabla U^{n+1} \\
& \quad - (\sigma(|\nabla U_h^n|^2) - \sigma(|\nabla U^n|^2))\nabla(e_h^{n+1} + \theta_h^{n+1})
\end{aligned}$$

where ξ_h^n is some number between $|\nabla U_h^n|^2$ and $|\nabla U^n|^2$. By using the notations in (3.3), we see further that

$$\begin{aligned}
J_1(v) = & -\big(A(\nabla U^n)\nabla \theta_h^{n+1},\, \nabla v\big) \\
& - \big(2\sigma'(|\nabla U^n|^2)\nabla U^n \cdot \nabla \theta_h^{n+1} \tau D_\tau \nabla U^{n+1},\, \nabla v\big) \\
& - \big(2\sigma'(|\nabla U^n|^2)(\nabla U^n \cdot \nabla e_h^n)\nabla U^{n+1},\, \nabla v\big) \\
& + \big(2\sigma'(|\nabla U^n|^2)\nabla U^n \cdot \nabla \tau D_\tau \theta_h^{n+1} \nabla U^{n+1},\, \nabla v\big) \\
& + \big(\sigma'(|\nabla U^n|^2)|\nabla(U_h^n - U^n)|^2 + \frac{1}{2}\sigma''(\xi_h^n)|\nabla(U^n + U_h^n)\cdot \nabla(e_h^n + \theta_h^n)|^2 \nabla U^{n+1},\, \nabla v\big) \\
& - \big((\sigma(|\nabla U_h^n|^2) - \sigma(|\nabla U^n|^2))\nabla(e_h^{n+1} + \theta_h^{n+1}),\, \nabla v\big).
\end{aligned}$$

Let

$$(3.25)\qquad \gamma(|\nabla U^n|^2) = 2|\sigma'(|\nabla U^n|^2)||\nabla U^n|^2 = \sigma(|\nabla U^n|^2) - \lambda^2\sigma(|\nabla U^n|^2)^3.$$

Taking $v = e_h^{n+1}$ in (3.23) and noting the fact $\big(A(\nabla U^n)\nabla\theta_h^{n+1},\, \nabla e_h^{n+1}\big) = 0$, we obtain

$$\begin{aligned}
J_1(e_h^{n+1}) \le & \big(\gamma(|\nabla U^n|^2)|\nabla e_h^{n+1}|, |\nabla e_h^n|\big) \\
& + C\|\tau D_\tau \nabla U^{n+1}\|_{L^\infty}(\|\nabla e_h^n\|_{L^2} + \|\nabla \theta_h^{n+1}\|_{L^2})\|\nabla e_h^{n+1}\|_{L^2} \\
& + C\Big(\sum_{m=n}^{n+1} \|e^m - R_h^m e^m\|_{H^1} + \tau\|D_\tau(u^{n+1} - R_h^{n+1}u^{n+1})\|_{H^1}\Big)\|\nabla e_h^{n+1}\|_{L^2} \\
& + C(\|\nabla e_h^n\|_{L^\infty} + \|\nabla\theta_h^n\|_{L^\infty})(\|\nabla e_h^n\|_{L^2} + \|\nabla\theta_h^n\|_{L^2})\|\nabla e_h^{n+1}\|_{L^2} \\
& + C(\|\nabla e_h^n\|_{L^\infty} + \|\nabla\theta_h^n\|_{L^\infty})(\|\nabla\theta_h^{n+1}\|_{L^2}\|\nabla e_h^{n+1}\|_{L^2} + \|\nabla e_h^{n+1}\|_{L^2}^2).
\end{aligned}$$

From (3.11), (3.12) and (3.22) we have

$$\|\nabla\theta_h^n\|_{L^2} \le Ch\|e^n\|_{H^2} + Ch^2,$$
$$\|\nabla e_h^n\|_{L^\infty} + \|\nabla\theta_h^n\|_{L^\infty} \le C(\tau^{1/8} + h^{\delta_0/8}) < \epsilon$$

when $\tau < \tau_1$ and $h < h_1$ for some positive constants τ_1 and h_1 (which depend on the constant ϵ). With (3.6)-(3.12), the induction assumptions (3.21)-(3.22) and the regularity of U^n given in Theorem 2.2, we derive that,

$$\begin{aligned}
J_1(e_h^{n+1}) \le\ & \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\Big\|_{L^2}^2 + \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n}\Big\|_{L^2}^2 + C\tau^{1/3}\|\nabla e_h^n\|_{L^2}\|\nabla e_h^{n+1}\|_{L^2} \\
& + C(h\|e^{n+1}\|_{H^2} + h\|e^n\|_{H^2} + \tau^{1/3}h^2 + \tau\|D_\tau(u^{n+1} - R_h^{n+1}u^{n+1})\|_{H^1})\|\nabla e_h^{n+1}\|_{L^2} \\
& + \epsilon(\|\nabla e_h^n\|_{L^2}^2 + \|\nabla e_h^{n+1}\|_{L^2}^2) + C\epsilon^{-1}(\|\nabla e_h^n\|_{L^\infty}^2 + \|\nabla\theta_h^n\|_{L^\infty}^2)(\|\nabla\theta_h^n\|_{L^2}^2 + \|\nabla\theta_h^{n+1}\|_{L^2}^2) \\
\le\ & \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\Big\|_{L^2}^2 + \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n}\Big\|_{L^2}^2 \\
& + C\epsilon^{-1}(h^2\|e^{n+1}\|_{H^2}^2 + h^2\|e^n\|_{H^2}^2 + \tau^{2/3}h^4 + \tau^2\|D_\tau(u^{n+1} - R_h^{n+1}u^{n+1})\|_{H^1}^2) \\
& + 2\epsilon(\|\nabla e_h^n\|_{L^2}^2 + \|\nabla e_h^{n+1}\|_{L^2}^2) \\
& + C\epsilon^{-1}(\|\nabla e_h^n\|_{L^\infty}^2 + \tau^{2/3} + h^2)(h^2\|e^n\|_{H^2}^2 + h^2\|e^{n+1}\|_{H^2}^2 + h^4) \\
\le\ & \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\Big\|_{L^2}^2 + \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n}\Big\|_{L^2}^2 \\
& + 3\epsilon(\|\nabla e_h^n\|_{L^2}^2 + \|\nabla e_h^{n+1}\|_{L^2}^2) + C\epsilon^{-1}(h^2\|e^n\|_{H^2}^2 + h^2\|e^{n+1}\|_{H^2}^2) \\
& + C\epsilon^{-1}(\tau^{2/3}h^4 + h^6) + C\epsilon^{-1}\tau^2\|D_\tau(u^{n+1} - R_h^{n+1}u^{n+1})\|_{H^1}^2,
\end{aligned}$$

where we have used the inverse inequality $h^4\|\nabla e_h^n\|_{L^\infty}^2 \le Ch^2\|\nabla e_h^n\|_{L^2}^2 \le \epsilon\|\nabla e_h^n\|_{L^2}^2$.
For $J_2(e_h^{n+1})$, we have the following estimate,

$$\begin{aligned}
J_2(e_h^{n+1}) &\le \|D_\tau\theta_h^{n+1}\|_{H^{-1}}\|e_h^{n+1}\|_{H^1} \\
&\le C\epsilon^{-1}\|D_\tau\theta_h^{n+1}\|_{H^{-1}}^2 + \epsilon\|\nabla e_h^{n+1}\|_{L^2}^2 + \epsilon\|e_h^{n+1}\|_{L^2}^2.
\end{aligned}$$

With the above estimates, (3.23) reduces to

$$\begin{aligned}
& \frac{1}{2}D_\tau\|e_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\Big(\Big\|\sqrt{\sigma(|\nabla U^n|^2)}\nabla e_h^{n+1}\Big\|_{L^2}^2 - \Big\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^n\Big\|_{L^2}^2\Big) \\
& \le 3\epsilon(\|\nabla e^n\|_{L^2}^2 + \|\nabla e^{n+1}\|_{L^2}^2) + C\|D_\tau\theta_h^{n+1}\|_{H^{-1}}^2 \\
& \quad + C\epsilon^{-1}\|e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\tau^2\|D_\tau(u^{n+1} - R_h^{n+1}u^{n+1})\|_{H^1}^2 \\
& \quad + C\epsilon^{-1}(\|e^n\|_{H^2}^2 + \|e^{n+1}\|_{H^2}^2)h^2 + C\epsilon^{-1}(\tau^{2/3}h^4 + h^6).
\end{aligned} \tag{3.26}$$

From (2.8) we derive that

$$\|\tau D_\tau\gamma(|\nabla U^n|^2)\|_{L^\infty} \le C\|\tau D_\tau e^n\|_{W^{1,\infty}} + C\|\tau D_\tau u^n\|_{W^{1,\infty}} \le C\tau^{1/3},$$

which implies

$$
\begin{aligned}
&\|\sqrt{\sigma(|\nabla U^n|^2)}\nabla e_h^{n+1}\|_{L^2}^2 - \|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^n\|_{L^2}^2 \\
&= \|\sqrt{\sigma(|\nabla U^n|^2) - \gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\|_{L^2}^2 + \|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\|_{L^2}^2 - \|\sqrt{\gamma(|\nabla U^{n-1}|^2)}\nabla e_h^n\|_{L^2}^2 \\
&\quad - \big((\gamma(|\nabla U^n|^2) - \gamma(|\nabla U^{n-1}|^2))\nabla e_h^n, \nabla e_h^n\big) \\
&\geq \|\lambda\sigma(|\nabla U^n|^2)^{3/2}\nabla e_h^{n+1}\|_{L^2}^2 + \tau D_\tau\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\|_{L^2}^2 - \tau\|\sqrt{|D_\tau\gamma(|\nabla U^n|^2)|}\nabla e_h^n\|_{L^2}^2 \\
&\geq \lambda^2\sigma_M^3\|\nabla e_h^{n+1}\|_{L^2}^2 + \tau D_\tau\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\|_{L^2}^2 - C\tau^{1/3}\|\nabla e_h^n\|_{L^2}^2.
\end{aligned}
$$

With the above inequality, (3.26) reduces to

$$
\begin{aligned}
&\frac{1}{2}D_\tau\|e_h^{n+1}\|_{L^2}^2 + \frac{\lambda^2\sigma_M^3}{2}\|\nabla e_h^{n+1}\|_{L^2}^2 + \frac{\tau}{2}D_\tau\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\|_{L^2}^2 \\
&\leq 3\epsilon(\|\nabla e^n\|_{L^2}^2 + \|\nabla e^{n+1}\|_{L^2}^2) + C\|D_\tau\theta_h^{n+1}\|_{H^{-1}}^2 \\
&\quad + C\epsilon^{-1}\|e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\tau^2\|D_\tau(u^{n+1} - R_h^{n+1}u^{n+1})\|_{H^1}^2 \\
&\quad + C\epsilon^{-1}(\|e^n\|_{H^2}^2 + \|e^{n+1}\|_{H^2}^2)h^2 + C\epsilon^{-1}(\tau^{2/3}h^4 + h^6).
\end{aligned}
$$

Choosing $\epsilon = \lambda^2\sigma_M^3/72$, by Theorem 2.2, Lemma 3.1 and Gronwall's inequality, we derive that

$$
\|e_h^{k+1}\|_{L^2}^2 + \sum_{m=0}^{k}\tau\|\nabla e_h^{m+1}\|_{L^2}^2 \leq C\tau^2h^2 + C\tau^{2/3}h^4 + Ch^{4+2\delta_0}.
$$

when $\tau < \tau_2 \leq \widetilde{\tau}_0$ and $h < h_2$ for some positive constants τ_2 and h_2.

Applying the inverse inequality, when $\tau \geq h^2$ we have

$$
\begin{aligned}
\|e_h^{k+1}\|_{L^\infty} + \|\nabla e_h^{k+1}\|_{L^\infty} &\leq Ch^{-1}(\|e_h^{k+1}\|_{L^2} + \|\nabla e_h^{k+1}\|_{L^2}) \\
&\leq Ch^{-1}(\tau h^2 + \tau^{-1/3}h^4 + \tau^{-1}h^{4+2\delta_0})^{1/2} \\
&\leq C(\tau^{1/2} + h^{2/3} + h^{\delta_0})
\end{aligned}
$$

and when $\tau \leq h^2$ we have

$$
\begin{aligned}
\|e_h^{k+1}\|_{L^\infty} + \|\nabla e_h^{k+1}\|_{L^\infty} \leq Ch^{-2}\|e_h^{k+1}\|_{L^2} &\leq Ch^{-2}(\tau^2h^2 + \tau^{2/3}h^4 + h^{4+2\delta_0})^{1/2} \\
&\leq C(h + \tau^{1/3} + h^{\delta_0}).
\end{aligned}
$$

In either case, we have

$$
\|e_h^{k+1}\|_{L^\infty} + \|\nabla e_h^{k+1}\|_{L^\infty} \leq \tau^{1/8} + h^{\delta_0/8}, \tag{3.27}
$$

and

$$
\|U_h^{k+1}\|_{L^\infty} + \|\nabla U_h^{k+1}\|_{L^\infty} \leq \|R_h^{k+1}U^{k+1}\|_{L^\infty} + \|\nabla R_h^{k+1}U^{k+1}\|_{L^\infty} + 1 \leq M \tag{3.28}
$$

when $\tau < \tau_3$ and $h < h_3$ for some positive constants τ_3 and h_3. The induction on (3.21)-(3.22) is closed with $\widehat{\tau}_0 = \min\{\tau_0^*, \widetilde{\tau}_0, \tau_1, \tau_2, \tau_3\}$ and $\widehat{h}_0 = \min\{h_1, h_2, h_3\}$.

The proof of Lemma 3.2 is completed. □

3.3. Unconditionally optimal error estimate. Now we turn back to the proof of Theorem 2.1. Let $\bar{e}_h^n = \overline{R}_h^n u^n - U_h^n$. From Lemma 3.2, Theorem 2.2, (3.6) and (3.12), we see that there exist positive constants $\tau_4 < \widehat{\tau}_0$ and $h_4 < \widehat{h}_0$ such that when $\tau < \tau_4$ and $h < h_4$

$$\|U_h^n\|_{L^\infty} + \|\nabla U_h^n\|_{L^\infty} \le M, \qquad \text{for } n = 0, 1, \cdots, N, \tag{3.29}$$

$$\|\bar{e}_h^n\|_{L^\infty} + \|\nabla \bar{e}_h^n\|_{L^\infty} < 2\tau^{1/8} + 2h^{\delta_0/8}, \qquad \text{for } n = 0, 1, \cdots, N. \tag{3.30}$$

Since the exact solution u^n satisfies

$$\big(D_\tau u^{n+1}, v\big) + \big(\sigma(|\nabla u^n|^2)\nabla u^{n+1}, \nabla v\big) = \big(g^{n+1}, v\big) + \big(\mathcal{E}_{\rm tr}^{n+1}, v\big), \quad \forall\, v \in V_h^r,$$

the error function $\bar{e}_h^{n+1}$ satisfies

$$\begin{aligned} &\big(D_\tau \bar{e}_h^{n+1}, v\big) + \big(\sigma(|\nabla u^n|^2)\nabla \bar{e}_h^{n+1}, \nabla v\big) \\ &= \Big[- \big(\sigma(|\nabla u^n|^2)\nabla \bar{\theta}_h^{n+1},\, \nabla v\big) + \big((\sigma(|\nabla U_h^n|^2) - \sigma(|\nabla u^n|^2))\nabla U_h^{n+1}, \nabla v\big)\Big] \\ &\quad - \big(D_\tau \bar{\theta}_h^{n+1}, v\big) + \big(\mathcal{E}_{\rm tr}^{n+1}, v\big) \\ &:= \overline{J}_1(v) + \overline{J}_2(v) + \overline{J}_3(v), \qquad \forall\, v \in V_h^r. \end{aligned} \tag{3.31}$$

To estimate $\overline{J}_i$, $i = 1, 2, 3$, we take the same approach as used for J_1 and J_2 in Section 3.2 and we get

$$\begin{aligned} \overline{J}_1(\bar{e}_h^{n+1}) &= -\big(A(\nabla u^n)\nabla\bar{\theta}_h^{n+1},\, \nabla \bar{e}_h^{n+1}\big) \\ &\quad - \big(2\sigma'(|\nabla u^n|^2)\nabla u^n \cdot \nabla \bar{e}_h^n \nabla u^{n+1},\, \nabla \bar{e}_h^{n+1}\big) \\ &\quad - \big(2\sigma'(|\nabla u^n|^2)\nabla u^n \cdot \nabla \bar{\theta}_h^{n+1} \tau D_\tau \nabla u^{n+1},\, \nabla \bar{e}_h^{n+1}\big) \\ &\quad + \big(2\sigma'(|\nabla u^n|^2)\nabla u^n \cdot \nabla \tau D_\tau \bar{\theta}_h^{n+1} \nabla u^{n+1},\, \nabla \bar{e}_h^{n+1}\big) \\ &\quad + \bigg(\sigma'(|\nabla u^n|^2)|\nabla(u_h^n - u^n)|^2 + \frac{1}{2}\sigma''(\bar{\xi}_h^n)|\nabla(u^n + u_h^n)\cdot\nabla(\bar{e}_h^n + \bar{\theta}_h^n)|^2 \nabla u^{n+1},\, \nabla \bar{e}_h^{n+1}\bigg) \\ &\quad - \big((\sigma(|\nabla U_h^n|^2) - \sigma(|\nabla u^n|^2))\nabla(\bar{e}_h^{n+1} + \bar{\theta}_h^{n+1}),\, \nabla \bar{e}_h^{n+1}\big) \\ &\le \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla \bar{e}_h^{n+1}\Big\|_{L^2}^2 + \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla \bar{e}_h^{n}\Big\|_{L^2}^2 + C\tau\|\nabla \bar{e}_h^n\|_{L^2}\|\nabla \bar{e}_h^{n+1}\|_{L^2} \\ &\quad + (C\tau + C\tau\|D_\tau \bar{\theta}_h^{n+1}\|_{H^1})\|\nabla \bar{e}_h^{n+1}\|_{L^2} \\ &\quad + C(\|\nabla \bar{e}_h^n\|_{L^\infty} + \|\nabla \bar{\theta}_h^n\|_{L^\infty})(\|\nabla \bar{e}_h^n\|_{L^2} + \|\nabla \bar{\theta}_h^n\|_{L^2})\|\nabla \bar{e}_h^{n+1}\|_{L^2} \\ &\quad + C(\|\nabla \bar{e}_h^n\|_{L^\infty} + \|\nabla(u^n - \overline{R}_h^n u^n)\|_{L^\infty})\|\nabla \bar{e}_h^{n+1}\|_{L^2}^2 \\ &\quad + C\|\nabla \bar{\theta}_h^{n+1}\|_{L^\infty}(\|\nabla \bar{e}_h^n\|_{L^2} + \|\nabla \bar{\theta}_h^n\|_{L^2})\|\nabla \bar{e}_h^{n+1}\|_{L^2} \\ &\le \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla \bar{e}_h^{n+1}\Big\|_{L^2}^2 + \frac{1}{2}\Big\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla \bar{e}_h^{n}\Big\|_{L^2}^2 \\ &\quad + \epsilon(\|\nabla \bar{e}_h^n\|_{L^2}^2 + \|\nabla \bar{e}_h^{n+1}\|_{L^2}^2) + C\epsilon^{-1}(1 + \|D_\tau \bar{\theta}_h^{n+1}\|_{H^1}^2)\tau^2 + C\epsilon^{-1}h^{2r+2}, \end{aligned}$$

$$\overline{J}_2(\bar{e}_h^{n+1}) \le C\epsilon^{-1}\|D_\tau \bar{\theta}_h^{n+1}\|_{L^2}^2 + \epsilon\|\bar{e}_h^{n+1}\|_{L^2}^2,$$

and

$$\overline{J}_3(\bar{e}_h^{n+1}) \le \epsilon\|\bar{e}_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}^2$$

when $\tau < \tau_5$ and $h < h_5$ for some positive constants τ_5 and h_5. With the above estimates, (3.31) reduces to

(3.32)
$$\frac{1}{2}D_\tau\|\bar{e}_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\Big(\big\|\sqrt{\sigma(|\nabla U^n|^2)}\nabla\bar{e}_h^{n+1}\big\|_{L^2}^2 - \big\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla\bar{e}_h^{n}\big\|_{L^2}^2\Big)$$
$$\le \epsilon(\|\nabla\bar{e}^n\|_{L^2}^2 + \|\nabla\bar{e}^{n+1}\|_{L^2}^2) + \epsilon\|\bar{e}_h^{n+1}\|_{L^2}^2$$
$$+ C\epsilon^{-1}\tau^2\|D_\tau\bar{\theta}_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}\|D_\tau\bar{\theta}_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}^2 + C\epsilon^{-1}(\tau^2 + h^{2r+2}).$$

Since

$$\|\sqrt{\sigma(|\nabla U^n|^2)}\nabla\bar{e}_h^{n+1}\|_{L^2}^2 - \|\sqrt{\gamma(|\nabla U^n|^2)}\nabla\bar{e}_h^{n}\|_{L^2}^2$$
$$= \|\sqrt{\sigma(|\nabla U^n|^2) - \gamma(|\nabla U^n|^2)}\nabla\bar{e}_h^{n+1}\|_{L^2}^2 + \|\sqrt{\gamma(|\nabla U^n|^2)}\nabla\bar{e}_h^{n+1}\|_{L^2}^2 - \|\sqrt{\gamma(|\nabla U^{n-1}|^2)}\nabla\bar{e}_h^{n}\|_{L^2}^2$$
$$- \big((\gamma(|\nabla U^n|^2) - \gamma(|\nabla U^{n-1}|^2))\nabla\bar{e}_h^n, \nabla\bar{e}_h^n\big)$$
$$\ge \|\lambda\sigma(|\nabla U^n|^2)^{3/2}\nabla\bar{e}_h^{n+1}\|_{L^2}^2 + \tau D_\tau\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla\bar{e}_h^{n+1}\|_{L^2}^2 - \tau\|\sqrt{|D_\tau\gamma(|\nabla U^n|^2)|}\nabla\bar{e}_h^{n}\|_{L^2}^2$$
$$\ge \lambda^2\sigma_M^3\|\nabla\bar{e}_h^{n+1}\|_{L^2}^2 + \tau D_\tau\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla\bar{e}_h^{n+1}\|_{L^2}^2 - C\tau^{1/3}\|\nabla\bar{e}_h^{n}\|_{L^2}^2,$$

the inequality (3.32) reduces to

(3.33)
$$\frac{1}{2}D_\tau\|\bar{e}_h^{n+1}\|_{L^2}^2 + \frac{\lambda^2\sigma_M^3}{2}\|\nabla\bar{e}_h^{n+1}\|_{L^2}^2 + \frac{\tau}{2}D_\tau\|\sqrt{\gamma(|\nabla U^n|^2)}\nabla e_h^{n+1}\|_{L^2}^2$$
$$\le \epsilon(\|\nabla\bar{e}^n\|_{L^2}^2 + \|\nabla\bar{e}^{n+1}\|_{L^2}^2) + \epsilon\|\bar{e}_h^{n+1}\|_{L^2}^2$$
$$+ C\epsilon^{-1}\tau^2\|D_\tau\bar{\theta}_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}\|D_\tau\bar{\theta}_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}^2 + C\epsilon^{-1}(\tau^2 + h^{2r+2})\,.$$

By choosing $\epsilon = \lambda^2\sigma_M^3/24$ and applying Gronwall's inequality, when $\tau < \tau_6$ and $h < h_6$ for some positive constants τ_6 and h_6, we obtain

$$\max_{0\le n\le N}\|\bar{e}_h^n\|_{L^2}^2 + \sum_{n=0}^{N}\tau\|\nabla\bar{e}_h^n\|_{L^2}^2 \le C(\tau^2 + h^{2r+2}). \tag{3.34}$$

So far we have proved Theorem 2.1 for the case $\tau < \tau_7 := \min\{\tau_4, \tau_5, \tau_6\}$ and $h < h_7 := \min\{h_4, h_5, h_6\}$. Now we consider the case that $\tau \ge \tau_7$ or $h \ge h_7$. Substituting $v = U_h^{n+1}$ in (2.2), we get

$$D_\tau\left(\frac{1}{2}\|U_h^{n+1}\|_{L^2}^2\right) \le C\epsilon^{-1}\|g^{n+1}\|_{L^2}^2 + \epsilon\|U_h^{n+1}\|_{L^2}^2,$$

which further implies that (via Gronwall's inequality)

$$\max_{1\le n\le N}\|U_h^n\|_{L^2} \le C. \tag{3.35}$$

Therefore,

(3.36)
$$\max_{1\le n\le N}\|U_h^n - u^n\|_{L^2} \le C \le \frac{C}{\max(\tau, h^{r+1})}(\tau + h^{r+1}) \le \frac{C}{\min(\tau_7, h_7^{r+1})}(\tau + h^{r+1}).$$

Combining (3.7), (3.34) and (3.36), we see that (2.4) holds unconditionally.

The proof of Theorem 2.1 is completed. ∎

4. Proof of Theorem 2.2. First, we consider the Poisson equation

$$\begin{cases} -\Delta v = f - \frac{1}{|\Omega|}\int_\Omega f \mathrm{d}x, & \text{in } \Omega, \\ \partial_{\vec{n}} v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

in a convex polygon, and introduce some lemmas concerning the $W^{2,p}$ and H^{2+s} estimates of its solution.

LEMMA 4.1. *Let v be the solution of (4.1) and $w \in W^{1,3}$ and $w_{\min} \le w(x) \le w_{\max}$, where $w_{\min}$ and $w_{\max}$ are positive constants. If $f \in L^2$, then $v \in H^2$ and for any $\epsilon \in (0, 1/2)$ we have*

$$\|\nabla^2 v\|_{L^2} \le \|f\|_{L^2}, \tag{4.2}$$

(4.3)

$$(1-\epsilon)\int_\Omega \sum_{i,j} |\partial_{ij} v|^2 w \mathrm{d}x \le \int_\Omega \left| f - \frac{1}{|\Omega|}\int_\Omega f \mathrm{d}x \right|^2 w \mathrm{d}x + C_{w_{\min}, w_{\max}, \|w\|_{W^{1,3}}} \epsilon^{-2} \|\nabla v\|_{L^2}^2.$$

Proof. The inequality (4.2) is a consequence of Theorem 3.1.1.1 in [26].

To prove (4.3), we denote by ω_j the interior angle of the corner x_j, $j = 1, 2, \cdots, J$, of the convex polygon Ω and by $\theta_j(x)$ the angle spanned by the two vectors $x_{j+1} - x_j$ and $x - x_j$. If $f \in C_0^\infty(\Omega)$, then the solution v can be decomposed as [26, 33]

$$v = \sum_{j=1}^J \alpha_j \Phi(|x - x_j|)|x - x_j|^{\pi/\omega_j} \cos\left(\frac{\pi}{\omega_j}\theta_j(x)\right) + \widetilde{v}$$

with $\widetilde{v} \in H^3$, where $\Phi(r)$ is a smooth cut-off function which equals 1 in a neighborhood of $r = 0$ and α_j, $j = 1, \cdots, J$, are positive constants. Letting $\omega_{\max} = \max_{1\le j\le J} \omega_j \in (0, \pi)$, from the above expression one can see that $v \in H^{2+s} \cap W^{3,1} \hookrightarrow C^1(\overline{\Omega}) \cap W^{2,1}(\partial\Omega)$ for $s \in (0, \pi/\omega_{\max} - 1)$. Thus the identity

$$\partial_{ii} v \partial_{jj} v = \partial_i(\partial_i v \partial_{jj} v) - \partial_j(\partial_i v \partial_{ij} v) + |\partial_{ij} v|^2$$

holds in $L^1(\Omega)$ and therefore, we derive that

$$\begin{aligned} \int_\Omega \sum_{i,j} |\partial_{ij} v|^2 w \mathrm{d}x = & \int_\Omega |\Delta v|^2 w \mathrm{d}x + \int_\Omega \Big(\Delta v \nabla v \cdot \nabla w - \nabla^2 v \nabla v \cdot \nabla w\Big) \mathrm{d}x \\ & - \int_{\partial\Omega} \Delta v \nabla v \cdot \vec{n}\, w \, \mathrm{d}l + \int_{\partial\Omega} \nabla^2 v \nabla v \cdot \vec{n}\, w \, \mathrm{d}l \end{aligned}$$

By noting the Neumann boundary condition in (4.1), we have $\nabla v\cdot\vec{n}=0$ and $\nabla^2 v\nabla v\cdot\vec{n}=0$ on $\partial\Omega$. Denoting $\widetilde{f}=f-\frac{1}{|\Omega|}\int_\Omega f\mathrm{d}x$, the last equation reduces to

$$\begin{aligned}\int_\Omega\sum_{i,j}|\partial_{ij}v|^2w\mathrm{d}x&=\int_\Omega|\widetilde{f}|^2w\mathrm{d}x+\int_\Omega\Big(-\widetilde{f}\nabla v\cdot\nabla w-\nabla^2v\nabla v\cdot\nabla w\Big)\mathrm{d}x\\&\le(1+\epsilon)\int_\Omega|\widetilde{f}|^2w\mathrm{d}x+\epsilon^{-1}\|w^{-1/2}\nabla w\|_{L^3}^2\|\nabla v\|_{L^6}^2\\&\le(1+\epsilon)\int_\Omega|\widetilde{f}|^2w\mathrm{d}x+C\epsilon^{-1}\|w^{-1/2}\nabla w\|_{L^3}^2(\|\nabla v\|_{L^2}^2+\|\nabla v\|_{L^2}^{4/3}\|\nabla^2v\|_{L^2}^{2/3})\\&\le(1+\epsilon)\int_\Omega|\widetilde{f}|^2w\mathrm{d}x+\epsilon\int_\Omega\sum_{i,j}|\partial_{ij}v|^2w\mathrm{d}x+C\epsilon^{-2}\|w^{-1/2}\nabla w\|_{L^3}^3\|\nabla v\|_{L^2}^2,\end{aligned}$$

which leads to

$$(1-\epsilon)\int_\Omega\sum_{i,j}|\partial_{ij}v|^2w\mathrm{d}x\le(1+\epsilon)\int_\Omega|\widetilde{f}|^2w\mathrm{d}x+C\epsilon^{-2}\|w^{-1/2}\nabla w\|_{L^3}^3\|\nabla v\|_{L^2}^2.$$

Since the above inequality holds for any $f\in C_0^\infty(\Omega)$ and $C_0^\infty(\Omega)$ is dense in L^2, the inequality must hold for all $f\in L^2$. ◻

It can be found in literatures, such as Theorem 4.3.2.3 and Theorem 4.4.3.7 of [26], and (23.3) of [15], that

$$\|\nabla^2v\|_{L^{1+p_*/2}}\le C_*\|f\|_{L^{1+p_*/2}},\tag{4.4}$$

$$\|\nabla^2v\|_{H^{s_*/2}}\le C_*\|f\|_{L^{s_*/2}}\tag{4.5}$$

for some positive constant $C_*\ge 4$, where $p_*=\min(5/2,1/[1-\pi/(2\omega_{\max})])$, $s_*=\pi/\omega_{\max}-1$ and $\omega_{\max}$ denotes the maximal interior angle of the convex polygon Ω. Since the operator from f to ∇^2v defined by (4.1) satisfies (4.2) and (4.4)-(4.5). By applying the complex interpolation (see Theorem 5.6.3 of [5]) to (4.2) and (4.4)-(4.5), we obtain the following lemma.

Lemma 4.2. *Assume that $v\in H^2(\Omega)$ is the solution of the equation* (4.1). *Then*

$$\|\nabla^2v\|_{L^p}\le(1+\varepsilon_p)\|f\|_{L^p}\tag{4.6}$$

$$\|\nabla^2v\|_{H^s}\le(1+\overline{\varepsilon}_s)\|f\|_{H^s}\tag{4.7}$$

for $p\in(2,p_*)$ and $s\in(0,s_*)$, where $\lim_{p\to2}\varepsilon_p=0$ and $\lim_{s\to0}\overline{\varepsilon}_s=0$.

Based on the regularity assumption (2.3), we set

$$K=\|u\|_{L^\infty(\Omega\times(0,T))}+\|\nabla u\|_{L^\infty(\Omega\times(0,T))}+2.$$

Then, by the regularity assumptions on σ, there exist positive constants $0<\sigma_K<1$ and C_K such that for $0\le s\le K$ we have

$$\sigma(s^2)\ge\sigma_K,\quad|\sigma(s^2)|+|\sigma'(s^2)|+|\sigma''(s^2)|\le C_K,\tag{4.8}$$

and we choose p so close to 2 that

$$\varepsilon_p<\lambda^2\sigma_K^2.\tag{4.9}$$

Now we start to prove Theorem 2.2. For the given $U^n \in H^{2+s_n}$, (2.5) can be viewed as a linear elliptic boundary value problem and therefore, it admits a unique solution $U^{n+1} \in H^{2+s_{n+1}}$ for some positive constant $s_{n+1} > 0$ (a qualitative regularity as a consequence of Lemma 4.2). Here we only prove the quantitative estimates (2.6)-(2.8).

Before we study the estimates (2.6)-(2.8), we prove by mathematical induction the following inequalities

$$\|U^n\|_{L^\infty} + \|\nabla U^n\|_{L^\infty} \le K, \tag{4.10}$$

$$\|e^n\|_{W^{2,p}} \le \tau^{1/3} \tag{4.11}$$

assuming $\tau < \tau_0^*$ for some $\tau_0^* > 0$. Since $U^0 = u_0$, the above inequalities hold for $n = 0$. We assume that (4.10)-(4.11) hold for $0 \le n \le k$ for some nonnegative integer k, and prove the inequalities for $n = k + 1$.

From (1.1)-(1.3) and (2.5), we see that e^{n+1} satisfies the equation

$$\begin{aligned} &D_\tau e^{n+1} - \nabla \cdot (\sigma(|\nabla u^n|^2)\nabla e^{n+1}) \\ &= \mathcal{E}_{\rm tr}^{n+1} - \nabla \cdot ((\sigma(|\nabla U^n|^2) - \sigma(|\nabla u^n|^2))\nabla U^{n+1}), \end{aligned} \tag{4.12}$$

with the boundary condition $\nabla e^{n+1} \cdot \vec{n} = 0$ and the initial condition $e^0 = 0$, where

$$\mathcal{E}_{\rm tr}^{n+1} = \partial_t u^{n+1} - D_\tau u^{n+1} + \nabla \cdot [(\sigma(|\nabla u^n|^2) - \sigma(|\nabla u^{n+1}|^2))\nabla u^{n+1}]$$

is the truncation error due to the time discretization. By the regularity assumption (2.3), we have

$$\max_{1\le n\le N} \|\mathcal{E}_{\rm tr}^n\|_{L^2} \le C, \qquad \sum_{n=1}^{N} \tau \|\mathcal{E}_{\rm tr}^n\|_{L^2}^2 \le C\tau^2. \tag{4.13}$$

With a similar approach to (3.24), we can derive that

$$\begin{aligned} &(\sigma(|\nabla U^n|^2) - \sigma(|\nabla u^n|^2))\nabla U^{n+1} \\ &= \big(- 2\sigma'(|\nabla u^n|^2)\nabla u^n \cdot \nabla e^n + \sigma'(|\nabla u^n|^2)|\nabla e^n|^2\big)\nabla U^{n+1} \\ &\quad + \frac{1}{2}\sigma''(\xi^n)|(\nabla u^n + \nabla U^n)\nabla e^n|^2 \nabla U^{n+1} \\ &= -2\sigma'(|\nabla u^n|^2)(\nabla u^n \cdot \nabla e^n)\nabla u^n \\ &\quad - 2\sigma'(|\nabla u^n|^2)(\nabla u^n \cdot \nabla e^n)(\tau D_\tau \nabla u^{n+1} - \nabla e^{n+1}) \\ &\quad + \Big(\sigma'(|\nabla u^n|^2)|\nabla e^n|^2 + \frac{1}{2}\sigma''(\xi^n)|(\nabla u^n + \nabla U^n)\nabla e^n|^2\Big) \cdot \nabla U^{n+1} \\ &\le \gamma(|\nabla u^n|^2)|\nabla e^n| + C\tau|\nabla e^n| + C|\nabla e^n||\nabla e^{n+1}| + C|\nabla e^n|^2, \end{aligned} \tag{4.14}$$

where $\gamma(\cdot)$ is defined in (3.25).

Multiplying (4.12) by e^{n+1} and using (4.14), we get

$$\begin{aligned} &D_\tau \left(\frac{1}{2}\|e^{n+1}\|_{L^2}^2\right) + \|\sqrt{\sigma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2 \\ &\le \frac{1}{2}\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla e^n\|_{L^2}^2 + \frac{1}{2}\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2 + C\tau(\|\nabla e^{n+1}\|_{L^2}^2 + \|\nabla e^n\|_{L^2}^2) \\ &\quad + C\|\nabla e^n\|_{L^\infty}(\|\nabla e^n\|_{L^2}^2 + \|\nabla e^{n+1}\|_{L^2}^2) + \|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}^2 + \|e^{n+1}\|_{L^2}^2, \end{aligned}$$

which implies that

$$D_\tau\left(\frac{1}{2}\|e^{n+1}\|_{L^2}^2\right)+\frac{1}{2}\left(\|\sqrt{\sigma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2-\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla e^n\|_{L^2}^2\right)$$
$$\le C\tau^{1/4}(\|\nabla e^{n+1}\|_{L^2}^2+\|\nabla e^n\|_{L^2}^2)+C\|e^{n+1}\|_{L^2}^2+C\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}^2, \tag{4.15}$$

where we have used (4.11). By noting

$$\begin{aligned}
&\|\sqrt{\sigma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2-\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla e^n\|_{L^2}^2\\
&=\|\sqrt{\sigma(|\nabla u^n|^2)-\gamma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2+\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2-\|\sqrt{\gamma(|\nabla u^{n-1}|^2)}\nabla e^n\|_{L^2}^2\\
&\quad-\big((\gamma(|\nabla u^n|^2)-\gamma(|\nabla u^{n-1}|^2))\nabla e^n,\nabla e^n\big)\\
&\ge\|\lambda\sigma(|\nabla u^n|^2)^{3/2}\nabla e^{n+1}\|_{L^2}^2+\tau D_\tau\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2-\tau\|\sqrt{|D_\tau\gamma(|\nabla u^n|^2)|}\nabla e^n\|_{L^2}^2\\
&\ge\lambda^2\sigma_K^3\|\nabla e^{n+1}\|_{L^2}^2+\tau D_\tau\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2-C\tau\|\nabla e^n\|_{L^2}^2,
\end{aligned}$$

(4.15) reduces to

$$\begin{aligned}
&D_\tau\left(\frac{1}{2}\|e^{n+1}\|_{L^2}^2+\frac{\tau}{2}\|\sqrt{\gamma(|\nabla u^n|^2)}\nabla e^{n+1}\|_{L^2}^2\right)+\frac{\lambda^2\sigma_K^3}{2}\|\nabla e^{n+1}\|_{L^2}^2\\
&\le C\tau^{1/4}(\|\nabla e^{n+1}\|_{L^2}^2+\|\nabla e^n\|_{L^2}^2)+C\|e^{n+1}\|_{L^2}^2+C\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}^2.
\end{aligned}$$

By Gronwall's inequality, when $\tau<\tau_8$ for some positive constant τ_8, we have

$$\max_{0\le n\le k}\|e^{n+1}\|_{L^2}^2+\sum_{n=0}^k\tau\|e^{n+1}\|_{H^1}^2\le C\sum_{n=0}^k\tau\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}^2\le C\tau^2. \tag{4.16}$$

From the above inequality we also see that

$$\|U^{n+1}\|_{L^2}\le\|u^{n+1}\|_{L^2}+\|e^{n+1}\|_{L^2}\le C, \tag{4.17}$$
$$\|D_\tau U^{n+1}\|_{L^2}\le\|D_\tau u^{n+1}\|_{L^2}+\|D_\tau e^{n+1}\|_{L^2}\le C. \tag{4.18}$$

We rewrite (4.12) as

(4.19)

$$\begin{aligned}
&D_\tau e^{n+1}-\sigma(|\nabla u^n|^2)\Delta e^{n+1}\\
&=\mathcal{E}_{\rm tr}^{n+1}+2\sigma'(|\nabla U^n|^2)(\nabla^2U^n\nabla U^n)\cdot\nabla e^{n+1}-(\sigma(|\nabla U^n|^2)-\sigma(|\nabla u^n|^2)\Delta u^{n+1})\\
&\quad-\left[2\sigma'(|\nabla U^n|^2)\nabla^2U^n\nabla U^n-2\sigma'(|\nabla u^n|^2)\nabla^2u^n\nabla u^n\right]\cdot\nabla u^{n+1}\\
&\quad-[\sigma(|\nabla u^n|^2)-\sigma(|\nabla U^n|^2)]\Delta e^{n+1}\\
&=\mathcal{E}_{\rm tr}^{n+1}+2\sigma'(|\nabla U^n|^2)(\nabla^2U^n\nabla U^n)\cdot\nabla e^{n+1}-(\sigma(|\nabla U^n|^2)-\sigma(|\nabla u^n|^2)\Delta u^{n+1})\\
&\quad+2\sigma'(|\nabla u^n|^2)\nabla^2e^n\nabla u^n\cdot\nabla u^n-[2\sigma'(|\nabla U^n|^2)\nabla^2u^n\nabla U^n-2\sigma'(|\nabla u^n|^2)\nabla^2u^n\nabla u^n]\cdot\nabla u^{n+1}\\
&\quad+[2\sigma'(|\nabla U^n|^2)\nabla U^n-2\sigma'(|\nabla u^n|^2)\nabla u^n]\cdot(\nabla^2e^n\nabla u^{n+1})\\
&\quad+2\tau\sigma'(|\nabla u^n|^2)\nabla^2e^n\nabla u^n\cdot\nabla D_\tau u^{n+1}-[\sigma(|\nabla u^n|^2)-\sigma(|\nabla U^n|^2)]\Delta e^{n+1}.
\end{aligned}$$

Multiplying the above equation by $-\Delta e^{n+1}$ leads to

(4.20)
$$
\begin{aligned}
&D_\tau\left(\frac{1}{2}|\nabla e^{n+1}|^2 \mathrm{d}x\right)+\int_\Omega \sigma(|\nabla u^n|^2)|\Delta e^{n+1}|^2 \mathrm{d}x \\
&\le \|\mathcal{E}_{\mathrm{tr}}^{n+1}\|_{L^2}\|\Delta e^{n+1}\|_{L^2} + C\|\nabla^2 U^n\|_{L^p}\|\nabla U^n\|_{L^\infty}\|\nabla e^{n+1}\|_{L^{2p/(p-2)}}\|\Delta e^{n+1}\|_{L^2} \\
&+ C\|\nabla e^{n+1}\|_{L^2}\|\Delta e^{n+1}\|_{L^2} + \int_\Omega \gamma(|\nabla u^n|^2)|\nabla^2 e^n||\Delta e^{n+1}|\mathrm{d}x + C\|\nabla e^n\|_{L^2}\|\Delta e^{n+1}\|_{L^2} \\
&\quad + C\|\nabla e^n\|_{L^\infty}\|\nabla^2 e^n\|_{L^2}\|\Delta e^{n+1}\|_{L^2} + C\tau\|\nabla^2 e^n\|_{L^2}\|\Delta e^{n+1}\|_{L^2} + C\|\nabla e^n\|_{L^\infty}\|\Delta e^{n+1}\|_{L^2}^2 .
\end{aligned}
$$

By the induction assumption (4.10)-(4.11), we have $\|\nabla U^n\|_{L^\infty} \le \|\nabla^2 U^n\|_{L^p} \le C$ and by the Sobolev interpolation inequality, the second term in the hand side of the above inequality is bounded by

$$
\begin{aligned}
\|\nabla^2 U^n\|_{L^p}\|\nabla U^n\|_{L^\infty}\|\nabla e^{n+1}\|_{L^{2p/(p-2)}}\|\Delta e^{n+1}\|_{L^2} &\le C\|e^{n+1}\|_{L^2}^{1-\theta}\|\Delta e^{n+1}\|_{L^2}^{1+\theta} \\
&\le C_\epsilon\|e^{n+1}\|_{L^2}^2 + \frac{\epsilon}{4}\|\Delta e^{n+1}\|_{L^2}^2
\end{aligned}
$$

for some $\theta \in (0,1)$. With (4.11), (3.25) and the above inequality, (4.20) reduces to

$$
\begin{aligned}
&D_\tau\left(\frac{1}{2}|\nabla e^{n+1}|^2 \mathrm{d}x\right)+\frac{1}{2}\int_\Omega \left(\sigma(|\nabla u^{n+1}|^2) + \lambda^2\sigma(|\nabla u^n|^2)^3 - \epsilon\right)|\Delta e^{n+1}|^2 \mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega (\gamma(|\nabla u^n|^2) + \epsilon + C\tau^{1/4})|\nabla^2 e^n|^2 \mathrm{d}x + C_\epsilon(\|\mathcal{E}_{\mathrm{tr}}^{n+1}\|_{L^2}^2 + \|\nabla e^n\|_{L^2}^2 + \|\nabla e^{n+1}\|_{L^2}^2) \\
&\quad + \frac{1}{2}\int_\Omega \left[\sigma(|\nabla u^{n+1}|^2) - \sigma(|\nabla u^n|^2)\right]|\Delta e^{n+1}|^2 \mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega (\gamma(|\nabla u^n|^2) + \epsilon + C\tau^{1/4})|\nabla^2 e^n|^2 \mathrm{d}x + C\tau\int_\Omega |\Delta e^{n+1}|^2 \mathrm{d}x \\
&\quad + C_\epsilon(\|\mathcal{E}_{\mathrm{tr}}^{n+1}\|_{L^2}^2 + \|\nabla e^n\|_{L^2}^2 + \|\nabla e^{n+1}\|_{L^2}^2).
\end{aligned}
$$

Choosing $\epsilon < \lambda^2\sigma_K^3/2$ and $\tau < \tau_9$ for some positive constant τ_9, we get

$$
\begin{aligned}
&D_\tau\left(\frac{1}{2}|\nabla e^{n+1}|^2 \mathrm{d}x\right)+\frac{1}{2}\int_\Omega \left[\sigma(|\nabla u^{n+1}|^2) + \frac{\lambda^2}{2}\sigma(|\nabla u^n|^2)^3\right]|\Delta e^{n+1}|^2 \mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega (\gamma(|\nabla u^n|^2) + 2\epsilon + C\tau^{1/4})|\nabla^2 e^n|^2 \mathrm{d}x + C_\epsilon(\|\mathcal{E}_{\mathrm{tr}}^{n+1}\|_{L^2}^2 + \|\nabla e^n\|_{L^2}^2 + \|\nabla e^{n+1}\|_{L^2}^2).
\end{aligned}
$$

and by applying Lemma 4.1 with $w = \sigma(|\nabla u^{n+1}|^2) + \frac{\lambda^2}{2}\sigma(|\nabla u^n|^2)^3$, we obtain

$$
\begin{aligned}
&D_\tau\left(\frac{1}{2}|\nabla e^{n+1}|^2 \mathrm{d}x\right)+\frac{1}{2}\int_\Omega \left[(1-\epsilon)\sigma(|\nabla u^{n+1}|^2) + \frac{(1-\epsilon)\lambda^2}{2}\sigma(|\nabla u^n|^2)^3\right]|\nabla^2 e^{n+1}|^2 \mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega [\gamma(|\nabla u^n|^2) + \epsilon + C\tau^{1/4}]|\nabla^2 e^n|^2 \mathrm{d}x + C_\epsilon(\|\mathcal{E}_{\mathrm{tr}}^{n+1}\|_{L^2}^2 + \|\nabla e^n\|_{L^2}^2 + \|\nabla e^{n+1}\|_{L^2}^2).
\end{aligned}
$$

By choosing ϵ small enough and when $\tau < \tau_{10}$ for some positive constant τ_{10}, we derive that

$$
\begin{aligned}
&D_\tau\left(\frac{1}{2}|\nabla e^{n+1}|^2 \mathrm{d}x\right)+\frac{1}{2}\int_\Omega \sigma(|\nabla u^{n+1}|^2)|\nabla^2 e^{n+1}|^2 \mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega \left(\sigma(|\nabla u^n|^2) - \lambda^2\sigma_K^3/2\right)|\nabla^2 e^n|^2 \mathrm{d}x + C\left(\|\mathcal{E}_{\mathrm{tr}}^{n+1}\|_{L^2}^2 + \|\nabla e^n\|_{L^2}^2 + \|\nabla e^{n+1}\|_{L^2}^2\right),
\end{aligned}
$$

which in turn shows that (with Gronwall's inequality)

$$\max_{0\le n\le k}\frac{1}{2}\|\nabla e^{n+1}\|_{L^2}^2+\frac{\lambda^2\sigma_K^3}{8}\sum_{n=1}^{k}\tau\|\nabla^2 e^{n+1}\|_{L^2}^2\le C\sum_{n=1}^{k}\tau\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}^2\le C\tau^2. \tag{4.21}$$

From the above inequality we further derive that

$$\max_{0\le n\le k}(\|U^{n+1}\|_{H^1}^2+\|D_\tau U^{n+1}\|_{H^1}^2)+\sum_{n=1}^{k}\tau\|D_\tau U^{n+1}\|_{H^2}^2\le C. \tag{4.22}$$

From (4.19) we see that

$$\|D_\tau e^{n+1}\|_{L^2}\le C\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}+C\|e^n\|_{H^2}+C\|e^{n+1}\|_{H^2},$$

and by using (4.21),

$$\sum_{n=0}^{k}\tau\|D_\tau e^{n+1}\|_{L^2}^2\le C\sum_{n=0}^{k}\tau\|\mathcal{E}_{\rm tr}^{n+1}\|_{L^2}+C\sum_{n=0}^{k}\tau\|e^{n+1}\|_{H^2}\le C\tau^2. \tag{4.23}$$

In particular, the above inequality implies that $\|D_\tau e^{k+1}\|_{L^2}\le C\tau^{1/2}$ and $\|D_\tau e^{k+1}\|_{H^1}\le C$ from (4.22). By an interpolation between L^2 and H^1, we have

$$\|D_\tau e^{k+1}\|_{L^p}\le C\|D_\tau e^{k+1}\|_{L^2}^{2/p}\|D_\tau e^{k+1}\|_{H^1}^{1-2/p}\le C\tau^{1/p}.$$

We rewrite (4.19) by

$$\Delta e^{k+1}=\sigma(|\nabla u^k|^2)^{-1}2\sigma'(|\nabla u^k|^2)\nabla^2 e^k\nabla u^k\cdot\nabla u^k+G, \tag{4.24}$$

where

$$\begin{aligned}\|G\|_{L^p}&\le C\|D_\tau e^{k+1}\|_{L^p}+C\|\mathcal{E}_{\rm tr}^{k+1}\|_{L^p}+(C\|\nabla^2U^k\|_{L^p}+C)\|\nabla e^{k+1}\|_{L^\infty}\\&\quad+C\tau\|\nabla^2e^k\|_{L^p}+C(\|\nabla^2e^k\|_{L^p}+\|\nabla^2e^{k+1}\|_{L^p})\|\nabla e^k\|_{L^\infty}\\&\le C\tau^{1/p}+\epsilon\|\nabla^2e^{k+1}\|_{L^p}+C_{\epsilon^{-1}}\|\nabla e^{k+1}\|_{L^2}+C\tau^{1/4}(\|\nabla^2e^{k+1}\|_{L^p}+\|\nabla^2e^k\|_{L^p})\\&\le C_{\epsilon^{-1}}\tau^{1/p}+\epsilon(\|\nabla^2e^{k+1}\|_{L^p}+\|\nabla^2e^k\|_{L^p}).\end{aligned} \tag{4.25}$$

With (4.9), we apply (4.6) to the elliptic equation (4.24) to get

$$\begin{aligned}\|\nabla^2e^{k+1}\|_{L^p}&\le(1+\lambda^2\sigma_K^2)\|\sigma(|\nabla u^k|^2)^{-1}\gamma(|\nabla u^k|^2)\nabla^2e^k\|_{L^p}+(1+\lambda^2\sigma_K^2)\|G\|_{L^p}\\&\le(1-\lambda^4\sigma_K^4)\|\nabla^2e^k\|_{L^p}+(1+\lambda^2\sigma_K^2)\|G\|_{L^p}.\end{aligned}$$

With $\epsilon=\lambda^4\sigma_K^4/(4+2\lambda^2\sigma_K^2)$ in (4.25), a straightforward calculation gives

$$\|\nabla^2e^{k+1}\|_{L^p}\le(1-\lambda^4\sigma_K^4/2)\|\nabla^2e^k\|_{L^p}+C\tau^{1/p}$$

when $\tau<\tau_{11}$ for some positive constant τ_{11}. By the Sobolev embedding inequality, we have $\|e^k\|_{L^p}+\|\nabla e^k\|_{L^p}\le C\|e^k\|_{H^2}\le C\tau^{1/2}$ and therefore,

$$\|e^{k+1}\|_{W^{2,p}}\le(1-\lambda^4\sigma_K^4/2)\|e^k\|_{W^{2,p}}+C\tau^{1/p} \tag{4.26}$$

which, by noting $1 < p \le 5/2$, leads to

$$\|e^{k+1}\|_{W^{2,p}} \le \tau^{1/3} \tag{4.27}$$

when $\tau < \tau_{12}$ for some positive constant τ_{12}. By using the Sobolev embedding inequality again, we obtain

$$\|e^{k+1}\|_{L^\infty} + \|\nabla e^{k+1}\|_{L^\infty} \le C\|e^{k+1}\|_{W^{2,p}} \le C\tau^{1/3}$$

which further implies that

$$\|e^{k+1}\|_{L^\infty} + \|\nabla e^{k+1}\|_{L^\infty} \le \tau^{1/4}, \tag{4.28}$$

$$\|U^{k+1}\|_{L^\infty} + \|\nabla U^{k+1}\|_{L^\infty} \le K, \tag{4.29}$$

when $\tau < \tau_{13}$ for some positive constant τ_{13}.

The induction on (4.10)-(4.11) is closed, and (4.16) and (4.21)-(4.27) hold for $k = N$ provided $\tau < \tau_0^* := \min\limits_{8\le i\le 13} \tau_i$.

It remains to estimate $\|U^{n+1}\|_{H^{2+s}}$ for some $s > 0$. From (4.27) we see that $\nabla U \in C^\alpha$ for some $\alpha > 0$. Rewrite (2.5) as

$$\begin{aligned}
-\Delta U^{n+1} &= \frac{2\sigma'(|\nabla U^n|^2)}{\sigma(|\nabla U^n|^2)}(\nabla^2 U^n \nabla U^n)\cdot\nabla U^n \\
&\quad + 2\tau\frac{\sigma'(|\nabla U^n|^2)}{\sigma(|\nabla U^n|^2)}(\nabla^2 U^n\nabla U^n)\cdot\nabla D_\tau U^{n+1} + (g^{n+1} - D_\tau U^{n+1})/\sigma(|\nabla U^n|^2) \\
&= l(\nabla^2 U^n) + (g^{n+1} - D_\tau U^{n+1})/\sigma(|\nabla U^n|^2), \qquad (4.30)
\end{aligned}$$

where the linear operator l defined by

$$l(\nabla^2 U^n) = \frac{2\sigma'(|\nabla U^n|^2)}{\sigma(|\nabla U^n|^2)}(\nabla^2 U^n\nabla U^n)\cdot\nabla U^n + 2\tau\frac{\sigma'(|\nabla U^n|^2)}{\sigma(|\nabla U^n|^2)}(\nabla^2 U^n\nabla U^n)\cdot\nabla D_\tau U^{n+1}$$

satisfies that

$$\begin{aligned}
&\|l(\nabla^2 U^n)\|_{L^2} \le \left(\frac{K^2}{\lambda^2 + K^2} + C\tau^{1/3}\right)\|\nabla^2 U^n\|_{L^2} \\
&\|l(\nabla^2 U^n)\|_{H^\alpha} \le C\|\nabla^2 U^n\|_{H^\alpha}.
\end{aligned}$$

By choosing $\tau \le \tau_{14}$ for some $\tau_{14} > 0$ and using the complex interpolation between L^2 and H^α [5] we derive that, there exist positive constants s_K such that

$$\begin{aligned}
\|l(\nabla^2 U^n)\|_{H^s} &\le \left(\frac{K^2}{\lambda^2 + K^2} + C\tau^{1/3}\right)^{1-s_K/\alpha} C^{s_K/\alpha}\|\nabla^2 U^n\|_{H^s} \\
&\le \left(1 - \frac{\lambda^2}{2\lambda^2 + 2K^2}\right)\|\nabla^2 U^n\|_{H^s} \quad \text{for} \quad s \in [0, s_K]. \qquad (4.31)
\end{aligned}$$

Therefore, by applying (4.7) to the equation (4.30) we obtain that

$$\|\nabla^2 U^{n+1}\|_{H^s} \le (1 + \bar{\varepsilon}_s)\left[\left(1 - \frac{\lambda^2}{2\lambda^2 + 2K^2}\right)\|\nabla^2 U^n\|_{H^s} + C\|g^{n+1}\|_{H^s} + C\|D_\tau U^{n+1}\|_{H^s}\right],$$

and choosing s_0 so small that $\bar{\varepsilon}_{s_0} < \lambda^2/(2\lambda^2 + 2K^2)$, we get

$$\|\nabla^2 U^{n+1}\|_{H^{s_0}} \le \left(1 - \frac{\lambda^4}{4(\lambda^2+K^2)^2}\right)\|\nabla^2 U^n\|_{H^{s_0}} + C\|g^{n+1}\|_{H^{s_0}} + C\|D_\tau U^{n+1}\|_{H^{s_0}}.$$

Iterations of the above inequality give

$$\max_{1\le n\le N} \|\nabla^2 U^n\|_{H^{s_0}} \le C\big(\max_{1\le n\le N} \|g^n\|_{H^{s_0}} + \max_{1\le n\le N} \|D_\tau U^n\|_{H^{s_0}}\big) \le C. \tag{4.32}$$

The proof of Theorem 2.2 is completed. □

5. Numerical example. In this section, we present an example to confirm our theoretical analysis. All computations are performed by FreeFEM++ in double precision [27].

We solve (1.1)-(1.3) in the domain $\Omega = [0,1] \times [0,1]$ up to the time $T = 1$, where the diffusion coefficient $\sigma(|\nabla u|^2)$ is given by (1.4), the function g and u_0 are chosen corresponding to the exact solution

$$u(x,y,t) = e^{0.01t}\cos(2\pi x)\cos(2\pi y)/4. \tag{5.1}$$

To test the convergence rate in the spatial direction, a uniform triangulation is generated with $M+1$ points on each side of the rectangular domain with $h = \sqrt{2}/M$, and we choose a very small time step $\tau = 2^{-15}$. In this case, the optimal error estimate given in Theorem 2.1 is, approximately,

$$\|U_h^n - u^n\|_{L^2} = O(h^{r+1}).$$

We present the L^2-norm errors for $\lambda = 1$ in Table 1, where the convergence rate is calculated based on the numerical results corresponding to two finest meshes. We see that the L^2-norm errors are proportional to h^{r+1}, which is consistent with our theoretical error analysis.

TABLE 1
L^2-norm errors of the numerical solution for $\lambda = 1$

M	$\|U_h^N - u^N\|_{L^2}$ for $r = 2$	$\|U_h^N - u^N\|_{L^2}$ for $r = 3$
8	9.0361E-04	3.6292E-04
16	1.1846E-04	7.6558E-05
32	1.4948E-05	4.1758E-07
convergence rate	$O(h^{3.0})$	$O(h^{4.1})$

To test the stability of the numerical solution, we solve (1.1)-(1.3) with several refined meshes for each fixed τ. The L^2-norm errors of the numerical solution are presented in Figure 1 for $r = 2, 3$ and $\lambda = 1$ in the logarithmic scale. We see that, for each fixed τ, the L^2-norm error of the numerical solution tends to a constant which is proportional to τ. Therefore, no restriction on the grid ratio is needed. It has been noted that our theoretical analysis is given under the assumption of λ being a positive constant. Clearly, the numerical accuracy of the linearized scheme depends upon λ and decreases as $\lambda \to 0$. In this example, $|\nabla u| = 0$ at some points and the equation becomes degenerate when $\lambda \to 0$.

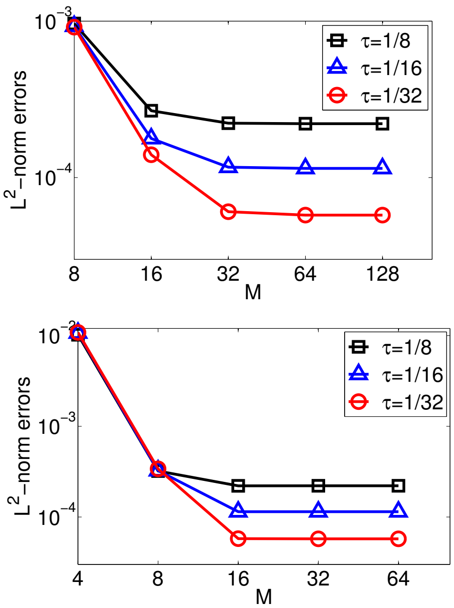

FIG. 1. *L^2-errors of the FEM for the problem with $\lambda = 1$ (left for $r = 2$ and right for $r = 3$)*

6. Conclusion. In this paper, we have presented optimal error estimates for a linearized backward Euler–Galerkin FEM ($r \geq 2$) for a nonlinear and non-degenerate diffusion equation in a convex polygonal domain under certain assumption on the regularity of the exact solution and λ being a positive constant. For this strongly nonlinear equation, no previous works have been devoted to the error analysis for linearized semi-implicit FEMs, and existing analyses for implicit schemes still require certain restrictions on the time-step size. Our analysis shows that the numerical solution of the linearized semi-implicit scheme achieves optimal convergence rate without any time-step condition. The analysis only focuses on the gradient flow with the gradient-dependent diffusion coefficient given in (1.4), while it can be extended easily to the problem with the diffusion coefficient satisfying (1.5).

For $r = 1$, the expected optimal spatial error bound is in the second order and under the assumption $\|U_h^n\|_{W^{1,\infty}} \leq K$, we can derive that

$$\|R_h U^{n+1} - U_h^{n+1}\|_{L^2}^2 + \sum \tau \|\nabla(R_h U^m - U_h^m)\|_{L^2}^2 \leq C_K h^4 \,.$$

However, from this estimate, one may not be able to obtain the uniform boundedness of numerical solution in $W^{1,\infty}$-norm by inverse inequalities. The stability analysis of the lowest order FEM is under investigation. Moreover, in the gradient flow model, λ denotes a regularization parameter. Clearly, the constant C_0 in Theorem 2.1 depends heavily upon λ and therefore, the optimal error estimate given in Theorem 2.1 is not uniform for the parameter λ. There are some applications in which degenerate diffusion equations ($\lambda = 0$) are concerned, such as total variation model [4, 20, 21] and parabolic p-Laplacian [3, 17, 40] without regularization. Numerical analysis for such degenerate equations is extremely difficult. Existing techniques in classical FEMs may not work well. An implicit backward finite element scheme was analyzed in [21]. The uniform convergence to the solution of the degenerate equation as $h, \tau, \lambda \to 0$ was proved and optimal error estimate for the nondegenerate equation was established under the time-step condition $\tau = O(h^2)$. Analysis for linearized schemes was less explored due to the strong nonlinearity of the equations. Developing efficient schemes for the nonlinear degenerate equations with the uniform and optimal convergence is our future work.

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REFERENCES

[1] E. Albrecht and V. Müller, Spectrum of interpolated operators, *Proc. Amer. Math. Soc.*, 129 (2000), pp. 807–814.

[2] A. Araujo, S. Barbeiro and P. Serranho, Stability of finite difference schemes for complex diffusion processes, *SIAM J. Numer. Anal.*, 50 (2012), pp. 1284–1296.

[3] J.W. Barrett and W.B. Liu, Finite element approximation of the parabolic p-Laplacian, *SIAM J. Numer. Anal.*, 31(1994), 413–428.

[4] G. Bellettini and V. Caselles, The total variation flow in $\mathbb{R}^N$, *J. Differential Equations*, 184 (2002), pp. 475–525.

[5] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Springer-Verlag Berlin Heidelberg 1976, Printed in Germany.

[6] C. Bernardi, M. Dauge and Y. Maday, *Polynomials in the Sobolev World*, Preprint IRMAR 07-14, Rennes, March 2007.

[7] D. Braess, Finite Elements, Theory, Fast Solvers, and Applications in Elasticity Theory, Cambridge University Press, New York, 2007.

[8] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Third Edition, Springer Science+Business Media, LLC, 2008.

[9] S.S. Byun and L. Wang, Elliptic equations with measurable coefficients in Reifenberg domains, *Advances in Mathematics*, 225 (2010), pp. 2648–2673.

[10] J. Calder, A. Mansouri and A. Yezzi, Image sharpening via Sobolev gradient flows, *SIAM J. Imaging Sci.*, 3 (2010), pp. 981–1014.

[11] A. Chambolle and P.L. Lions, Image recovery via total variation minimization and related problems, *Numer. Math.*, 76 (1997), pp. 167–188.

[12] T. Chan and J. Shen, On the role of the BV image model in image restoration, *Tech. Report CAM 02-14*, Department of Mathematics, UCLA, 2002.

[13] C. Chen and G. Xu, Gradient-flow-based semi-implicit finite-element method and its convergence analysis for image reconstruction, *Inverse Problems*, 28 (2012), pp. 035006–035024.

[14] N. Chumchob and K. Chen, Improved variational image registration model and a fast algorithm for its numerical approximation, *Numer. Methods Partial Differential Eq.*, 28 (2012), pp. 1966–1995.

[15] M. Dauge, *Elliptic boundary value problems in corner domains*, Springer-Verlag Berlin Heidelberg, 1988.

[16] K. Deckelnick and G. Dziuk, Convergence of a finite element method for non-parametric mean curvature, *Numer. Math.*, 72 (1995), pp. 197–222.

[17] L. Diening, C. Ebmeyer and M. Ruzicka, Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure, *SIAM J. Numer. Anal.*, 45 (2007), pp. 457–472.

[18] L.C. Evans, *Partial differential equations*, Providence, American Mathematical Society, 1998.

[19] C. Ebmeyer and J. Vogelgesang, Finite element approximation of a forward and backward anisotropic diffusion model in image denoising and form generalization, *Numer. Methods Partial Differential Eq.*, 24 (2008), pp. 646–662.

[20] X. Feng and M. von Oehsen and A. Prohl, Rate of convergence of regularization procedures and finite element approximations for the total variation flow, *Numer. Math.*, 100 (2005), pp. 441–456.

[21] X. Feng and A. Prohl, Analysis of total variation flow and its finite element approximations, *M2AN Math. Model. Numer. Anal.*, 37 (2003), pp. 533–556.

[22] X. Feng and A. Prohl, Analysis of gradient flow of a regularized Mumford-Shah functional for image segmentation and image inpainting, *ESAIM: M2AN*, 38 (2004), pp. 291–320.

[23] Y. Hou, B. Li and W. Sun, *Error estimates of splitting Galerkin methods for heat and sweat transport in textile materials*, SIAM J. Numer. Anal. 51 (2013), 88-111.

[24] C. Gerhardt, Boundary value problems for surfaces of prescribed mean curvature, *J. Differential Equations*, 36 (1980), pp. 139–172.

[25] T. Grahs, A. Meister and T. Sonar, Image processing for numerical approximations of conservation laws: nonlinear anisotropic artificial dissipation, *SIAM J. Sci. Comput.*, 23 (2002), pp. 1439–1455.

[26] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, SIAM 2011.

[27] F. Hecht, New development in freefem++, *J. Numer. Math.*, 20 (2012), pp. 251–265.

[28] B. Li and W. Sun, Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media, *SIAM J. Numer. Anal.*, 51 (2013), pp. 1949–1977.

[29] B. Li and W. Sun, Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear parabolic equations, *Int. J. Numer. Anal. & Modeling*, 10 (2013), pp. 622–633.

[30] B. Li, H. Gao and W. Sun, Unconditionally optimal error estimates of a Crank-Nicolson Galerkin method for the nonlinear thermistor equations, *SIAM J. Numer. Anal.*, 52(2014), pp.933954.

[31] B. Li, J. Wang and W. Sun, The stability and convergence of fully discrete Galerkin-Galerkin FEMs for porous medium flows, *Commun. Comput. Phys.*, 15(2014), pp.11411158.

[32] A. Lichnewsky and R. Temam, Pseudoslution of the time-dependent minimal surface problem, *J. Differential Equations*, 30 (1978), pp. 340–364.

[33] R.B. Kellogg, *Corner Singularities and Singular Perturbations*, Annali dell'Università di Ferrara, Sezione VII, Scienze Matematiche, Vol. XLVII pp. 177-206.

[34] Z. Liu and Q. Chang, Numerical analysis of the model of image processing with time-delay regularization, *Appl. Math. Comput.*, 166 (2005), pp. 349–372.

[35] Z. Liu and B. Guo, New numerical algorithms for the nonlinear diffusion model of image denoising and segmentation, *Appl. Math. Comput.*, 178 (2006), pp. 380–389.

[36] Z. Mghazli, Regularity of an elliptic problem with mixed Dirichlet-Robin boundary conditions in a polygonal domain, *Calcolo*, 29 (1992), pp. 241–267.

[37] K. Mikula and F. Sgallari, Semi-implicit finite volume scheme for image processing in 3D cylindrical geometry, *J. Comp. Appl. Math.*, 161 (2003), pp. 119–132.

[38] P. Perona and J. Malik, Scale-space and edge detection using anisotropic diffusion, *IEEE transactions on pattern analysis and machine intelligence*, 12 (1990), pp. 629–639.

[39] R. Rannacher and R. Scott, Some optimal error estimates for piecewise linear finite element approximations, *Math. Comp.*, 38 (1982), pp. 437–445.

[40] L.A. Vese and S.J. Osher, Numerical methods for p-harmonic flows and applications to image processing, *SIAM J. Numer. Anal.*, 40 (2002), pp. 2085–2104.